King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 260 Final Exam
Semester II, 2010- (092)

Name:	**KEY**		
ID:	**KEY**	**Serial no:**	

Section

1	**2**	**3**	**4**
7:00-7:50	**8:00-8:50**	**9:00-9:50**	**10:00-10:50**
Dr. Fairag	**Dr. Fairag**	**Dr. Laradji**	**Dr. Fairag**

Q	Your Points	Points
1		**20**
2		**20**
3		**20**
4		**20**
5		**20**
6		**20**
7		**20**
8		**10**
9		**10**
10		**10**
11		**10**
12		**10**
13		**10**
Total		**200**

☺ **Say Bismillah & Good luck** ☺

(1) Solve the system $X' = AX$

where $A = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}$.

Char. eq. $\lambda^2 - 8\lambda + 17 = 0$

has roots $4 \pm i$

$4+i$ has eigenvector $\begin{bmatrix} 1 \\ 2-i \end{bmatrix}$

put $U = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $V = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Then two lin. indep. solutions are $X_1 = e^{4t}(U\cos t - V\sin t)$ and $X_2 = e^{4t}(U\sin t + V\cos t)$

i.e. $X_1 = e^{4t}\begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix}$ and $X_2 = e^{4t}\begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}$.

Gen. sol. is: $X = C_1X_1 + C_2X_2$.

(2) Solve the system $X' = AX$

where $A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & 4 & 6 \\ -4 & -6 & -8 \end{bmatrix}$.

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 0 & 0 \\ 4 & 4-\lambda & 6 \\ -4 & -6 & -8-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} 4-\lambda & 6 \\ -6 & -8-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\left[\lambda^2 + 4\lambda - 32 + 36\right] = -(\lambda+2)(\lambda^2+4\lambda+4)$$

$$= -(\lambda+2)(\lambda+2)^2 = -(\lambda+2)^3$$

eigenvalues : -2 , -2 , -2 2

$$\left[A-2I \mid 0\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 4 & 6 & 6 & 0 \\ -4 & -6 & -6 & 0 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 4 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad 2k_1 = -3k_2 - 3k_3$$

$\boxed{k_2 = 2,\ k_3 = 0} \longrightarrow V_1 = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$ 2

$\boxed{k_2 = 1,\ k_3 = -1} \longrightarrow V_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ 2

$$\left[A-2I \mid V_2\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 4 & 6 & 6 & 1 \\ -4 & -6 & -6 & -1 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 4 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_2 = 0,\ k_3 = 0 \Rightarrow 4k_1 = 1 \Rightarrow k_1 = 1/4 \qquad V_3 = \begin{bmatrix} 1/4 \\ 0 \\ 0 \end{bmatrix}$ 2

Now, $X_1 = V_1 e^{-2t}$, $X_2 = V_2 e^{-2t}$, $X_3 = (V_2 t + V_3)e^{-2t}$

3 3 3

The general solution

$$X(t) = \left[c_1 V_1 + c_2 V_2 + c_3 (V_2 t + V_3)\right] e^{-2t}$$

3

(3) Solve the initial value problem

$$y'''+4y'=0$$

$$y(0)=-1, \quad y'(0)=2, \quad y''(0)=4$$

Char. equation : $r^3+4r = r(r^2+4)$ 5

roots : $0, \quad \pm 2i$ 5

The general solution : $y = c_1 + c_2 \cos 2t + c_3 \sin 2t$

$y = c_1 + c_2 \cos 2t + c_3 \sin 2t \rightarrow y(0) = c_1 + c_2 = -1$ —— (1)

$y' = -2c_2 \sin 2t + 2c_3 \cos 2t \rightarrow y'(0) = 2c_3 = 2 \rightarrow c_3 = 1$ —— (2)

$y'' = -4c_2 \cos 2t - 4c_3 \sin 2t \rightarrow y''(0) = -4c_2 = 4 \rightarrow c_2 = -1$ —— (3)

5

(1) and (3) $\Rightarrow c_1 = 0$

Hence, the solution for the initial value problem is :

$$\boxed{y(t) = -\cos 2t + \sin 2t}$$ 5

(4) Find a particular solution of

$$y''+2y'-3y = 3-16xe^{x}$$

Sol. Char. equation $r^2+2r-3=(r+3)(r-1)=0$. So complementary function is $y_c = C_1 e^{-3x} + C_2 e^{x}$.

The form of a particular solution is $y_p = A + (Bx+C)xe^{x}$ (where $(Bx+C)e^{x}$ is multiplied by x to eliminate duplication).

We get $y_p' = (Bx^2 + (2B+C)x + C)e^{x}$

& $y_p'' = (Bx^2 + (4B+C)x + 2B+2C)e^{x}$.

$$y_p'' + 2y_p' - 3y_p = \left\{ \begin{array}{l} Bx^2 + (4B+C)x + 2B + 2C \\ + 2Bx^2 + (4B+2C)x + 2C \\ - 3Bx^2 - 3Cx \end{array} \right\} e^{x} - 3A = 3 - 16xe^{x}$$

Comparing coefficients we obtain

$-3A = 3$, $8B = -16$, $2B + 4C = 0$ i.e. $A = -1$, $B = -2$, $C = 1$. So $y_p = -1 + (1-2x)xe^{x}$.

(5) Solve the differential equation

$$\underbrace{(x+2\tan^{-1} y)}_{M}dx+(\underbrace{\frac{2x+y}{1+y^2}}_{N})dy=0 \quad \text{———} (1)$$

$$\frac{\partial M}{\partial y}=\frac{2}{1+y^2}, \quad \frac{\partial N}{\partial x}=\frac{2}{1+y^2} \Rightarrow (1) \text{ is an exact.}$$

4

$$f(x,y)=\int (x+2\tan^{-1}y)\,dx+g(y)$$

$$=\frac{1}{2}x^2+2x\tan^{-1}y+g(y) \quad \text{———} (2)$$

4

$$\frac{\partial f}{\partial y}=0+\frac{2x}{1+y^2}+g'(y)=N=\frac{2x}{1+y^2}+\frac{y}{1+y^2}$$

$$\Rightarrow g'(y)=\frac{y}{1+y^2}$$

4

$$\text{Now, } g(y)=\int \frac{y}{1+y^2}dy=\frac{1}{2}\int \frac{2y}{1+y^2}dy=\frac{1}{2}\ln(1+y^2)$$

(3)

(2) and (3) give : $\boxed{f(x,y)=\frac{1}{2}x^2+2x\tan^{-1}y+\frac{1}{2}\ln(1+y^2)}$

4

Hence, the general solution for (1) is

$$\boxed{\frac{1}{2}x^2+2x\tan^{-1}y+\frac{1}{2}\ln(1+y^2)=C}$$

4

(6) Solve the differential equation

$$y''+(y')^2+1=0$$

Sol. Let $p=y'$ (here y is missing).

then $p'+p^2+1=0$ i.e. $\frac{dp}{p^2+1}=-dx$

Integrate to get : $\tan^{-1} p = C_1 - x$

i.e. $p = y' = \tan(C_1-x)$

This gives: $y=\int \tan(C_1-x)\,dx$

i.e. $y=\ln|\cos(C_1-x)|+C_2$

(7) Verify that $X_1 = e^{-2t}\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$, $X_2 = e^{t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $X_3 = e^{3t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

are linearly independent solutions of the system $X' = AX$

where $A = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}$. Find a particular solution of the system

satisfying the initial conditions $x_1(0) = 1,\ x_2(0) = 2,\ x_3(0) = 3$

Sol. $AX_1 = e^{-2t}\begin{bmatrix} -6 \\ 4 \\ -4 \end{bmatrix} = -2e^{-2t}\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = X_1'$

$AX_2 = e^{t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = X_2'$; $AX_3 = e^{3t}\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = 3e^{3t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = X_3'$.

So X_1, X_2, X_3 are <u>solutions</u> of $X' = AX$.

<u>Wronskian</u> $W(X_1, X_2, X_3) = \begin{vmatrix} 3 & 1 & 1 \\ -2 & -1 & -1 \\ 2 & 1 & 0 \end{vmatrix} e^{-2t} \cdot e^{t} \cdot e^{3t}$

$= \begin{vmatrix} 3 & 1 & -1 \\ 0 & 0 & -1 \\ 2 & 1 & 0 \end{vmatrix} e^{2t} = e^{2t} \neq 0$. So X_1, X_2, X_3 are <u>lin. indep.</u>

<u>Particular Sol</u>. $X_1(0) = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$, $X_1(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $X_2(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

We want a solution $C_1X_1(0) + C_2X_2(0) + C_3X_3(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

So we have the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ -2 & -1 & -1 & 2 \\ 2 & 1 & 0 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 0 & 0 & -1 & 5 \\ 2 & 1 & 0 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -5 \\ 2 & 1 & 0 & 3 \end{array}\right]$$

So $C_3 = -5$, $C_1 = -2 - C_3 = 3$, $C_2 = 3 - 2C_1 = -3$.

Hence particular solution is

$X = 3X_1 - 3X_2 - 5X_3$ (X_1, X_2, X_3 as above).

(8) The rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

equals

$$\xrightarrow{-R_4+R_1} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

10

(a) 1
(b) 2
(c) 3
(d) 4

$$\xrightarrow[R_4 \leftrightarrow R_2]{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# of nonzero row in the echelon form = 2

---

(9) $$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} = \overset{-R_1+R_2}{\underset{-R_3+R_4}{=}} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 9 & 10 & 11 & 12 \\ 4 & 4 & 4 & 4 \end{vmatrix}$$

row 2 and row 4 are identical

$\Rightarrow$ det $= 0$

(a) 4
(b) 3
(c) 2
(d) 1
(e) 0

10

(10) The differential equation $3y^2 y' + y^3 = e^{-x}$ is

10

(a) Seprable (b) Linear (c) Bernoulli (e) None of the above

divid by $3y^2 \rightarrow y' + \frac{1}{3} y = e^{-x} y^{-2}$

it is Bernoulli with $n = -2$

---

(11) The matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ is

eigenvalues ; 2, 2, 2.

$[A - 2I | 0] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ # of free variables $= 1$

$\Rightarrow$ # of linearly indep. eigenvectors $= 1$

(a) diagonalizable (b) not diagonalizable

10

---

(12) The set of vectors $\left\{ \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}, \begin{bmatrix} 2e^{3t} \\ 3e^{3t} \\ -2e^{3t} \end{bmatrix}, \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \\ e^{-4t} \end{bmatrix} \right\}$ is

10

(a) linearly independent (b) linearly dependent

$$W = \begin{vmatrix} 1 & 2e^{3t} & -e^{-4t} \\ 6 & 3e^{3t} & 2e^{-4t} \\ -13 & -2e^{3t} & e^{-4t} \end{vmatrix} \overset{R_3 + R_1}{=} \begin{vmatrix} -12 & 0 & 0 \\ 6 & 3e^{3t} & 2e^{-4t} \\ -13 & -2e^{3t} & e^{-4t} \end{vmatrix}$$

$$= (-12) \begin{vmatrix} 3e^{3t} & 2e^{-4t} \\ -2e^{3t} & e^{-4t} \end{vmatrix} = (-12)\left[3e^{-t} + 4e^{-t}\right] = (-12)(7)e^{-t} \neq 0$$

(13) Let A be an $n \times n$ matrix such that $A^3 = 0$. Show that $\lambda = 0$ is an eigenvalue of A.

Let $A^3 = 0$ and we want to show that $\lambda = 0$ is an eigenvalue of A.

$$A^3 = 0 \Rightarrow |A^3| = 0$$
$$\Rightarrow |A|^3 = 0$$
$$\Rightarrow |A| = 0$$

$\Rightarrow$ A is singular matrix

$\Rightarrow$ $\lambda = 0$ is an eigenvalue

10