

King Fahd University of Petroleum and Minerals  
Department of Mathematics and Statistics

**MATH 102 - Term 152 - Exam II**

Duration: 120 minutes

**KEY**

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Name: \_\_\_\_\_ ID Number: \_\_\_\_\_

Section Number: \_\_\_\_\_ Serial Number: \_\_\_\_\_

ClassTime: \_\_\_\_\_ Instructor's Name: \_\_\_\_\_

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**Instructions:**

1. Calculators and Mobiles are not allowed.
  2. Write neatly and eligibly. You may lose points for messy work.
  3. Show all your work. No points for answers without justification.
  4. Make sure that you have 7 pages of questions ( Total of 10 questions)
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Question Number	Points	Maximum Points
1		8
2		8
3		6
4		8
5		14
6		14
7		12
8		10
9		10
10		10
<b>Total</b>		<b>100</b>

1. (8-points) Find the average value of  $f(x) = \frac{\sin(\pi/x)}{x^2}$  on the interval  $[1, 3]$ .

$$f_{\text{ave}} = \frac{1}{3-1} \int_1^3 \frac{\sin(\pi/x)}{x^2} dx$$

Do substitution  $u = \frac{\pi}{x}$   $du = -\frac{\pi}{x^2} dx$   $x=1 \Rightarrow u=\pi$   $x=3 \Rightarrow u=\frac{\pi}{3}$

$$f_{\text{ave}} = \frac{1}{2\pi} \int_{\pi/3}^{\pi} \sin(u) du$$

$$= \frac{-1}{2\pi} \cos(u) \Big|_{\pi/3}^{\pi} = \frac{-1}{2\pi} \left(-1 - \frac{1}{2}\right) = \frac{3\pi}{4}$$



2. (8-points) Write out, without evaluating the coefficients, the form of the partial fraction decomposition of the function

$$\frac{x^7 + x + 1}{(x+1)^2(x^2+1)(x^4-1)}$$

$$(x+1)^2(x^2+1)(x^4-1) = (x-1)(x+1)^3(x^2+1)^2$$

$$\frac{x^7+x+1}{(x+1)^2(x^2+1)(x^4-1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2} + \frac{A_4}{(x+1)^3} + \frac{A_5x+A_6}{x^2+1} + \frac{A_7x+A_8}{(x^2+1)^2}$$

1pt for each fraction = 6pt

3. (6-points) Evaluate the integral  $\int x \sec^2 x dx$ .

2 pt | Using integration by parts with  $u=x$   $v=\tan x$   
 $du=dx$   $dv=\sec^2 x dx$

$$\int x \sec^2 x dx = \underbrace{x \tan x - \int \tan x dx}_{\underline{2 pt}} = \underbrace{x \tan x - \ln |\sec x| + C}_{\underline{2 pt}}$$

4. (8-points) Evaluate the integral  $\int_0^{\pi/2} \sin^2 x \cos 3x dx$ .

Using the trigonometric identity  $\sin^2 x = \frac{1 - \cos 2x}{2}$  1 pt

$$\int_0^{\pi/2} \sin^2 x \cos 3x dx = \int_0^{\pi/2} \frac{\cos 3x}{2} - \frac{\cos 3x \cos 2x}{2} dx \quad \underline{1 pt}$$

Using the trigonometric identity  $\cos 3x \sin 2x = \frac{1}{2} [\cos 5x + \cos x]$  1 pt

$$\int_0^{\pi/2} \sin^2 x \cos 3x dx = \int_0^{\pi/2} \frac{\cos 3x}{2} - \frac{\cos 5x}{4} - \frac{\cos x}{4} dx \quad \underline{1 pt}$$

$$= \frac{\sin 3x}{6} \Big|_0^{\pi/2} - \frac{\sin 5x}{20} \Big|_0^{\pi/2} - \frac{\sin x}{4} \Big|_0^{\pi/2} \quad \underline{3 pt}$$

$$= \frac{-1}{6} - \frac{1}{20} - \frac{1}{4} = \frac{-7}{15} \quad \underline{1 pt}$$

5. (14-points) Evaluate the integral  $\int \frac{t}{\sqrt{t^2 - 6t + 13}} dt$ .

$$\text{Let } I = \int \frac{t}{\sqrt{t^2 - 6t + 13}} dt.$$

$$I = \int \frac{t}{\sqrt{(t-3)^2 + 4}} dt \quad \underline{1pt}$$

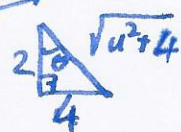
After substitution  $\underline{u = t-3}$ ,  $I = \int \frac{u+3}{\sqrt{u^2+4}} du \quad \underline{1pt}$

Do the trigonometric substitution  $u = 2 \tan \theta$ ,  $-\pi/2 < \theta < \pi/2$   
 $du = 2 \sec^2 \theta d\theta \quad \underline{2pt}$

$$I = \int \frac{2 \tan \theta + 3}{\sqrt{4 \tan^2 \theta + 4}} 2 \sec^2 \theta d\theta = \int \frac{2 \tan \theta + 3}{2 \sec \theta} 2 \sec^2 \theta d\theta \quad \underline{1pt}$$

$$= \int (2 \tan \theta + 3) \sec \theta d\theta = 2 \int \tan \theta \sec \theta + 3 \int \sec \theta d\theta \quad \underline{1pt}$$

$$= 2 \sec \theta + 3 \ln |\sec \theta + \tan \theta| + C \quad \underline{2pt} + \underline{2pt}$$

Substitute back the initial variable using   $\sec \theta = \frac{1}{2} \sqrt{u^2 + 4}$   $\underline{1pt}$

$$I = 2 \cdot \frac{1}{2} \sqrt{u^2 + 4} + 3 \ln \left| \frac{1}{2} \sqrt{u^2 + 4} + \frac{u}{2} \right| + C \quad \underline{1pt}$$

$$I = \sqrt{t^2 - 6t + 13} + 3 \ln \left| \frac{1}{2} \sqrt{t^2 - 6t + 13} + \frac{t}{2} - \frac{3}{2} \right| + C \quad \underline{1pt}$$

or

$$= \sqrt{t^2 - 6t + 13} + 3 \ln \left| \sqrt{t^2 - 6t + 13} + t - 3 \right| + C.$$

6. (14-points) Use substitution  $t = \tan(x/2)$ ,  $-\pi < x < \pi$ , to evaluate the integral

$$\int \frac{1}{\sin x + \cos x} dx.$$

$$t = \tan(x/2) \Rightarrow \cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}, dx = \frac{2}{1+t^2} dt \quad \begin{array}{l} \text{1 pt} \\ \text{1 pt} \\ \text{1 pt} \end{array}$$

$$I = \int \frac{1}{\sin x + \cos x} dx = \int \frac{2}{1+2t-t^2} dt = 2 \int \frac{dt}{2-(t-1)^2} \quad \text{1 pt}$$

$$\text{After substitution } u = t-1, I = 2 \int \frac{du}{2-u^2} \quad \text{1 pt}$$

Decompose  $\frac{1}{2-u^2}$  into partial fractions:

$$\frac{1}{2-u^2} = \frac{1}{(\sqrt{2}-u)(\sqrt{2}+u)} = \frac{A}{\sqrt{2}-u} + \frac{B}{\sqrt{2}+u} \Rightarrow A=B = \frac{1}{2\sqrt{2}} \quad \begin{array}{l} \text{1 pt} \\ \text{1 pt} \end{array}$$

$$I = 2 \cdot \frac{1}{2\sqrt{2}} \left[ \int \frac{1}{\sqrt{2}-u} du + \int \frac{1}{\sqrt{2}+u} du \right] \quad \text{1 pt}$$

$$= \frac{1}{\sqrt{2}} \left[ -\ln|\sqrt{2}-u| + \ln|\sqrt{2}+u| \right] + C \quad \text{2 pt}$$

$$= \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}+u}{\sqrt{2}-u} \right| + C = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}-1+t}{\sqrt{2}+1-t} \right| + C \quad \text{1 pt}$$

$$= \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}-1+\tan(x/2)}{\sqrt{2}+1-\tan(x/2)} \right| + C \quad \text{1 pt}$$

7. (12-points) Evaluate, if possible, the improper integral  $\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}}$ .

Check the improper integrals  $\int_0^1 (x-1)^{-1/3} dx$  and  $\int_1^2 (x-1)^{-1/3} dx$ .

$$\bullet \lim_{t \rightarrow 1^-} \int_0^t (x-1)^{-1/3} dx = \lim_{t \rightarrow 1^-} \left[ \frac{3}{2} (x-1)^{2/3} \Big|_0^t \right] = \lim_{t \rightarrow 1^-} \left( \frac{3}{2} (t-1)^{2/3} - \frac{3}{2} \right) = -\frac{3}{2}$$

1pt      1pt      1pt      1pt

$$\text{Then } \int_0^1 (x-1)^{-1/3} dx = -\frac{3}{2} \quad \underline{\underline{1pt}}$$

$$\bullet \lim_{t \rightarrow 1^+} \int_t^2 (x-1)^{-1/3} dx = \lim_{t \rightarrow 1^+} \left[ \frac{3}{2} (x-1)^{2/3} \Big|_t^2 \right] = \lim_{t \rightarrow 1^+} \left( \frac{3}{2} - \frac{3}{2} (t-1)^{2/3} \right) = \frac{3}{2}$$

1pt      1pt      1pt      1pt

$$\text{Then } \int_1^2 (x-1)^{-1/3} dx = \frac{3}{2} \quad \underline{\underline{1pt}}$$

Since both improper integrals converge, we can write

$$\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}} = - \int_0^1 \frac{dx}{(x-1)^{1/3}} + \int_1^2 \frac{dx}{(x-1)^{1/3}} \quad \underline{\underline{1pt}}$$

$$= - \left( -\frac{3}{2} \right) + \left( \frac{3}{2} \right)$$

$$= 3 \quad \underline{\underline{1pt}}$$

8. (10-points) Find the value of  $a$  so that the length of the arc on the curve  $y = \cosh x$  from the point  $P(0, 1)$  to the point  $Q(\ln a, \frac{a+a^{-1}}{2})$  is 2.

$$L = \int_0^{\ln a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2$$

2pt      1pt

$$\frac{dy}{dx} = \sinh x \quad \underline{1pt}$$


$$1 + \left(\frac{dy}{dx}\right)^2 = \cosh^2 x \quad \underline{1pt}$$

$$\Rightarrow \int_0^{\ln a} \cosh x dx = 2 \quad \underline{1pt}$$

$$\Rightarrow \sinh x \Big|_0^{\ln a} = 2 \quad \underline{1pt}$$

$$\Rightarrow \sinh(\ln a) - \sinh(0) = 2 \Rightarrow \sinh(\ln a) = 2 \quad \underline{1pt}$$

$$\Rightarrow \frac{a-a^{-1}}{2} = 2 \Rightarrow a^2 - 8a - 1 = 0 \Rightarrow a = 4 + \sqrt{17} \quad \underline{1pt}$$

1pt      

9. (10-points) Find the area of the surface obtained by revolving the curve

$y = 4\sqrt{x+1}$ ,  $0 \leq x \leq 4$  about the  $x$ -axis.

$$S = \int_0^4 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

1pt   1pt      1pt

$$\frac{dy}{dx} = \frac{2}{\sqrt{x+1}} \quad \underline{1pt}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{x+5}{x+1} \quad \underline{1pt}$$

$$= \int_0^4 2\pi \cdot 4 \cdot \sqrt{x+1} \sqrt{\frac{x+5}{x+1}} dx \quad \underline{1pt}$$

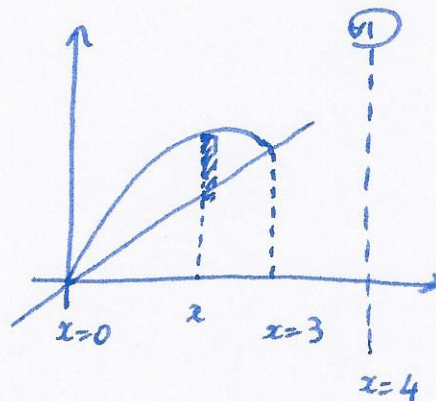
$$= 8\pi \int_0^4 \sqrt{x+5} dx = 8\pi \cdot \frac{2}{3} (x+5)^{3/2} \Big|_0^4 \quad \underline{2pt}$$

$$= \frac{16}{3} \pi (27 - 5^{3/2}) \quad \underline{1pt}$$

10. (10-points) Use the **method of cylindrical shells** to find the volume generated by rotating the region bounded by the line  $y = x$  and the curve  $y = 4x - x^2$  about the line  $x = 4$ .

$$4x - x^2 = x \Rightarrow x^2 - 3x = 0 \Rightarrow x = 0 \text{ or } x = 3 \quad | \quad \underline{\underline{1 \text{ pt}}}$$

$$r = 4 - x \quad \underline{\underline{1 \text{ pt}}} \quad h = (4x - x^2) - x \quad \underline{\underline{1 \text{ pt}}}$$



$$\text{Volume} = \int_0^3 \underbrace{2\pi}_{\underline{\underline{1 \text{ pt}}}} \underbrace{(4-x)}_{\underline{\underline{1 \text{ pt}}}} \underbrace{(3x-x^2)}_{\underline{\underline{1 \text{ pt}}}} dx$$

$$= 2\pi \int_0^3 (12x - 7x^2 + x^3) dx \quad \underline{\underline{1 \text{ pt}}}$$

$$= 2\pi \left( 6x^2 - \frac{7}{3}x^3 + \frac{1}{4}x^4 \right) \Big|_0^3 \quad \underline{\underline{2 \text{ pt}}}$$

$$= 2\pi \left( 54 - 63 + \frac{81}{4} \right)$$

$$= \frac{45\pi}{2} \quad \underline{\underline{1 \text{ pt}}}$$