King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 102 - Exam II - Term 142

Duration: 90 minutes

Name: Key

ID Number: __________

Section Number: __________

Serial Number: __________

Class Time: __________

Instructor's Name: __________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 9 Problems)

	Points	Maximum Points
page 1		24
page 2		16
page 3		12
page 4		10
page 5		18
page 6		8
page 7		12
Total		100

1. Evaluate the following integrals

A) (8 points) $\int \frac{2^{\ln x}}{x}\,dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$ (2 pts)

$$\int \frac{2^{\ln x}}{x}\,dx = \int 2^u\,du \quad (2\text{ pts})$$

$$= \frac{2^u}{\ln 2} + C \quad (3\text{ pts})$$

$$= \frac{2^{\ln x}}{\ln 2} + C \quad (1\text{ pt})$$

B) (8 points) $\int \frac{\sin^3 x}{(\cos x)^{3/2}}\,dx$

$$\int \sin^3 x\,(\cos x)^{-\frac{3}{2}}\,dx$$

$$= \int \sin^2 x\,(\cos x)^{-\frac{3}{2}}\sin x\,dx \quad (1\text{pt})$$

$$= \int (1-\cos^2 x)(\cos x)^{-\frac{3}{2}}\sin x\,dx$$

(2 pts) Let $u = \cos x$, $du = -\sin x\,dx$

$$= \int -(1-u^2)\,u^{-\frac{3}{2}}\,du \quad (2\text{ pts})$$

$$= \int \left(u^{\frac{1}{2}} - u^{-\frac{3}{2}}\right)du = \frac{2}{3}u^{\frac{3}{2}} + 2u^{-\frac{1}{2}} + C \quad (2\text{ pts})$$

$$= \frac{2}{3}\cos^{\frac{3}{2}} x + 2\cos^{-\frac{1}{2}} x + C \quad (1\text{ pt})$$

C) (8 points) $\int \tanh^3 x\ \mathrm{sech}^2 x\,dx$

Let $u = \tanh x \Rightarrow du = \mathrm{sech}^2 x\,dx$ (2 pts)

$$\int \tanh^3 x\,\mathrm{sech}^2 x\,dx = \int u^3\,du \quad (2\text{ pts})$$

$$= \frac{u^4}{4} + C \quad (2\text{pts})$$

$$= \frac{\tanh^4 x}{4} + C \quad (2\text{pts})$$

2. (8 points) Determine whether the following sequence converges or diverges. If it converges, find the limit

$$a_n = \frac{(-1)^{n-1}n}{n^2+1}$$

$$\frac{-n}{n^2+1} \le \frac{(-1)^{n-1} n}{n^2+1} \le \frac{n}{n^2+1} \quad (4 \text{ pts})$$

$$\lim_{n\to\infty} \frac{-n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2+1} = 0 \,, \quad (2 \text{ pts})$$

by sequeeze theorem, (1 pt)

$$\lim_{n\to\infty} \frac{(-1)^{n-1} n}{n^2+1} = 0 \,. \quad (1 \text{ pt})$$

3. (8 points) If $\sinh x = -\sqrt{3}$, find $\coth x$

$$\cosh^2 x - \sinh^2 x = 1 \quad (3 \text{ pts})$$

$$\Rightarrow \cosh^2 x = 1 + \sinh^2 x$$

$$= 1 + (-\sqrt{3})^2$$

$$= 4$$

since $\cosh x > 0$ (1 pt)

$\therefore \cosh x = 2$. (1 pt)

$$\left.\begin{aligned} \therefore \coth x = \frac{\cosh x}{\sinh x} &= -\frac{2}{\sqrt{3}} \\ &= \frac{-2\sqrt{3}}{3} \end{aligned}\right\} (2 \text{ pts})$$

(1 pt)

4. (12 points) Evaluate $\int \frac{2x^2+13}{(x-1)(x^2+4)}\,dx$

$$\frac{2x^2+13}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+D}{x^2+4} \qquad (3\text{ pts})$$

$$\Rightarrow \quad 2x^2+13 = A(x^2+4) + (Bx+D)(x-1)$$

$$x=1 \Rightarrow 15 = 5A \Rightarrow A = 3 \qquad (1\text{ pt})$$

Coeff. of x^2 : $A+B=2 \Rightarrow B=-1$ (1 pt)

Constant term : $4A - D = 13$

$$\Rightarrow D = 4A-13 \qquad (1\text{ pt})$$

$$= -1$$

$$\int \frac{2x^2+13}{(x-1)(x^2+4)}\,dx = \int \frac{3}{x-1}\,dx + \int \frac{-x-1}{x^2+4}\,dx$$

$$= \int \frac{3}{x-1}\,dx - \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$= 3\ln|x-1| - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

(1 pt) (2 pts) (2 pts) (1 pt)

5. (10 points) Evaluate $\displaystyle\int_5^{5\sqrt{3}} \frac{1}{x^2\sqrt{x^2+25}}\,dx$

Let $x = 5\tan\theta$, $0 < \theta < \frac{\pi}{2}$ (2 pts)

$\Rightarrow dx = 5\sec^2\theta\, d\theta$, $x^2 = \tan^2\theta$ (1 pt)

if $x = 5 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ (1 pt)

$x = 5\sqrt{3} \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$ (1 pt)

$$\int_5^{5\sqrt{3}} \frac{1}{x^2\sqrt{x^2+25}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{5\sec^2\theta}{25\tan^2\theta\,(5\sec\theta)}\,d\theta \quad \text{(2 pts)}$$

$$= \frac{1}{25}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\theta \cdot \cot^2\theta\, d\theta$$

$$= \frac{1}{25}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta}\, d\theta \quad \text{(1 pt)}$$

$$= -\frac{1}{25\sin\theta}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \left(\sqrt{2} - \frac{2}{\sqrt{3}}\right)\frac{1}{25} \quad \text{(2 pts)}$$

$$= \left(\sqrt{2} - \frac{2\sqrt{3}}{3}\right)\frac{1}{25}$$

$$= \frac{\sqrt{2}}{25} - \frac{2\sqrt{3}}{75}$$

6. (8 points) Write the form of the partial fraction decomposition of $\dfrac{4-2x}{x^2(x^2+4)^3(x^2-4)^2(x^2+x+1)^2}$

$$= \frac{2(2-x)}{x^2(x^2+4)^3(x-2)^2(x+2)^2(x^2+x+1)^2}$$

$$= \frac{-2}{x^2(x^2+4)^3(x-2)(x+2)^2(x^2+x+1)^2} \quad (1\text{ pt})$$

$$= \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B}{x-2} + \frac{C_1x+D_1}{x^2+4} + \frac{C_2x+D_2}{(x^2+4)^2} + \frac{C_3x+D_3}{(x^2+4)^3}$$

$$+ \frac{E_1}{x+2} + \frac{E_2}{(x+2)^2} + \frac{G_1x+H_1}{x^2+x+1} + \frac{G_2x+H_2}{(x^2+x+1)^2}$$

7. (10 points) Evaluate $\displaystyle\int \frac{\sec^2 x}{\tan^2 x + \tan x - 2}\,dx$ $(2+2+1+2)$

let $u = \tan x \Rightarrow du = \sec^2 x\,dx$ (2 pts)

$$\int \frac{\sec^2 x}{\tan^2 x + \tan x - 2}\,dx$$

$$= \int \frac{1}{u^2+u-2}\,du \quad (2\text{ pts})$$

(3pts)

$$= \int \frac{du}{(u+2)(u-1)} = \int \left[\frac{-\frac{1}{3}}{u+2} + \frac{\frac{1}{3}}{u-1}\right] du$$

$$= \frac{1}{3}\ln\left|\frac{u-1}{u+2}\right| + C \quad (2\text{ pts})$$

$$= \frac{1}{3}\ln\left|\frac{\tan x - 1}{\tan x + 2}\right| + C \quad (1\text{ pt})$$

8. (8 points) Evaluate the following improper integral or show that it diverges.

$$\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx$$

$$\lim_{t\to\infty}\int_0^t \frac{16\tan^{-1}x}{1+x^2}\,dx \quad (2\text{pts})$$

$$= \lim_{t\to\infty} 8\left(\tan^{-1}x\right)^2\Big|_0^t \quad (2\text{pts})$$

$$= 8\lim_{t\to\infty}\left(\tan^{-1}t\right)^2 \quad (1\text{pt})$$

$$= 8\left(\frac{\pi}{2}\right)^2 = 2\pi^2 \quad (1\text{pt})$$

$$\therefore \int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx \text{ is Convergent} \quad (1\text{pt})$$

$$\text{and } \int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx = 2\pi^2 \quad (1\text{pt})$$

9. (12 points) Evaluate $\int (\sin^{-1} x)^2 \, dx$

let $u = (\sin^{-1} x)^2 \quad dv = dx \quad$ (1 pt)

$\Rightarrow du = \dfrac{2 \sin^{-1} x}{\sqrt{1-x^2}} dx \quad , \quad v = x \quad$ (2 pts)

$\therefore \int (\sin^{-1} x)^2 dx = x (\sin^{-1} x)^2 - 2 \int \dfrac{x \sin^{-1} x}{\sqrt{1-x^2}} dx \quad$ (3 pts)

Consider $\int \dfrac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

let $u = \sin^{-1} x \qquad dv = \dfrac{x}{\sqrt{1-x^2}} dx \quad$ (1 pt)

$du = \dfrac{1}{\sqrt{1-x^2}} dx \qquad v = -\sqrt{1-x^2} \quad$ (2 pts)

$\therefore \int \dfrac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = -\sin^{-1} x \sqrt{1-x^2} + \int dx$

$= -\sin^{-1} x \sqrt{1-x^2} + x \quad$ (2 pts)

$\Rightarrow \int (\sin^{-1} x)^2 dx$

$= x (\sin^{-1} x)^2 + 2 \sin^{-1} x \sqrt{1-x^2} - 2x + C \quad$ (1 pt)