1. (~~6~~ 7 points) Find all positive numbers b such that the average value of the function $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

• The average value is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

• So $$f_{ave} = \frac{1}{b-0}\int_0^b (2+6x-3x^2)\,dx$$ 2 points

$$= \frac{1}{b}\left[2x+3x^2-x^3\right]_0^b$$

$$= \frac{1}{b}\left[2b+3b^2-b^3\right]$$

$$= 2+3b-b^2.$$ 2 points

• Now, $f_{ave} = 3 \Longrightarrow 2+3b-b^2 = 3$. 1 point

So $b^2 - 3b + 1 = 0$

Then $b = \frac{3+\sqrt{5}}{2}$ or $b = \frac{3-\sqrt{5}}{2}$.

1 point 1 point

2. (a) (~~6~~ 7 points) Evaluate $\int x(x+1)\,e^x\,dx$.

- $\int x(x+1)e^x dx = \int (x^2+x)e^x dx$. 1 point
- use Integration by part:

$u = x^2+x$, $dv = e^x dx$ $\Longrightarrow$ $du = (2x+1)\,dx$, $V = e^x$.

- $\int x(x+1)e^x dx = (x^2+x)e^x - \int (2x+1)e^x dx$. 2 points
- Again use the integration by part to evaluate $\int (2x+1)e^x dx$

$u = 2x+1$, $dv = e^x dx$ $\Longrightarrow$ $du = 2$, $V = e^x$ 2 points

- $\int (2x+1)e^x dx = (2x+1)e^x - \int 2e^x dx = (2x+1)e^x - 2e^x + C$

Thus $\int x(x+1)e^x dx = (x^2+x)e^x - (2x+1)e^x + 2e^x - C$ 1 point

$= (x^2 - x + 1)e^x + C$. 1 point

(b) (~~6~~ 7 points) Evaluate $\int x\tan^{-1} x\,dx$.

- use integration by part.

$u = \tan^{-1}x$, $dv = x\,dx$ $\Longrightarrow$ $\left.\begin{array}{l} du = \dfrac{dx}{1+x^2} \\ V = \dfrac{1}{2}x^2 \end{array}\right\}$ 1 point

- $\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \int \frac{1}{2}x^2 \cdot \frac{dx}{1+x^2}$ 2 points

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}dx$

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx$ 1 point

$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$ 1 point

$= \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x + C$ 1 point

3. (~~12~~ 13 points) Evaluate $\int \sin^2 x \cos^4 x \, dx$.

• use the identities:
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$.

2 points 2 points

• $\int \sin^2 x \cos^4 x \, dx = \int \frac{1}{2}(1-\cos 2x)\left(\frac{1}{2}(1+\cos 2x)\right)^2 dx$

$= \frac{1}{8}\int \left(1+\cos 2x - \cos^2 2x - \cos^3 2x\right) dx$ 2 points

$= \frac{1}{8}\int \left(1+\cos 2x - \frac{1}{2}(1+\cos 4x) - (1-\sin^2 2x)\cos 2x \, dx\right.$ 2 points

$= \frac{1}{8}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x + \sin^2 2x \cos 2x\right) dx$ 3 points

$= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right] + C$ 2 points

Second Method:

$\int \sin^2 x \cos^4 x \, dx = \int (\sin x \cos x)^2 \cdot \cos^2 x \, dx$

$= \int \frac{1}{4}\sin^2 2x \cdot \frac{1}{2}(1+\cos 2x) \, dx.$ 2 points

$= \frac{1}{8}\left[\int (\sin^2 2x + \sin^2 2x \cos 2x) \, dx\right]$ 2 points

$= \frac{1}{8}\left[\int \sin^2 2x \, dx + \int \sin^2 2x \cos 2x \, dx\right]$ 2 points

$= \frac{1}{8}\left[\int \frac{1}{2}(1-\cos 4x) \, dx + \int \sin^2 2x \cos 2x \, dx\right]$ 3 points

$= \frac{1}{8}\left[\frac{x}{2} - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right] + C$ 2 points

$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$ 2 points

4. 9 ~~(8~~ points) Evaluate $\int \frac{\sin^3\theta\, d\theta}{\cos^6\theta}$.

- $\int \frac{\sin^3\theta\, d\theta}{\cos^6\theta} = \int \frac{\sin^3\theta\, d\theta}{\cos^3\theta \cos^3\theta}$

$= \int \tan^3\theta \sec^3\theta\, d\theta$ 2 points

- Save one $\tan\theta\sec\theta$ factor and use the substitution $u = \sec\theta$ and the identity $\tan^2\theta = \sec^2\theta - 1 = u^2 - 1$. 2 points

So: $\int \tan^3\theta \sec^3\theta\, d\theta = \int \tan^2\theta \sec^2\theta \tan\theta\sec\theta\, d\theta$ 1 point

$= \int (u^2-1)u^2\, du$. 1 point

$= \int (u^4 - u^2)\, du$. 1 point

$= \frac{u^5}{5} - \frac{u^3}{3} + C$. 1 point

$= \frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3} + C$. 1 point

5. (~~8~~ 9 points) Evaluate $\int \frac{dx}{(4-x^2)^{3/2}}$.

1 point

• Set $x = 2\sin\theta$. Then $dx = 2\cos\theta\, d\theta$.

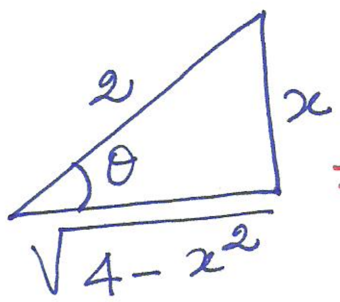
So $\int \frac{dx}{(4-x^2)^{3/2}} = \int \frac{2\cos\theta\, d\theta}{(4-4\sin^2\theta)^{3/2}}$ 2 points

$= \frac{2}{8}\int \frac{\cos\theta\, d\theta}{\cos^3\theta}$ 1 point

$= \frac{1}{4}\int \sec^2\theta\, d\theta$. 1 point

$= \frac{1}{4}\tan\theta + C$ 1 point

$= \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C$. 2 points



1 point

6. (~~9~~ 10 points) Evaluate $\int \frac{x^5+2}{x^2-1}\,dx$.

- <u>Use partial fraction Method:</u>

$$\frac{x^5+2}{x^2-1} \overset{\text{2 points}}{=} x^3+x+\frac{x+2}{x^2-1} \quad \text{(by Long Division)}$$

- $$\frac{x+2}{x^2-1} = \frac{x+2}{(x-1)(x+1)} = \frac{A}{x-1}+\frac{B}{x+1}. \quad \text{1 point}$$

- Multiply both sides by $(x-1)(x+1)$:

$$x+2 = A(x+1)+B(x-1)$$

$$x+2 = Ax+A+Bx-B$$

$$= (A+B)x+(A-B).$$

1 point

- $$\left.\begin{array}{ll}(1) & 1=A+B \\ (2) & 2=A-B\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(1)+(2):\ 3=2A.\ \text{So } \boxed{A=\frac{3}{2}} \\ (1) \Rightarrow B=1-\frac{3}{2}.\ \text{So } \boxed{B=-\frac{1}{2}}\end{array}\right.$$

1 point

- $$\frac{x^5+2}{x^2-1} = x^3+x+\frac{3}{2(x-1)}-\frac{1}{2(x+1)} \quad \text{1 point}$$

- Then $$\int \frac{x^5+2}{x^2-1}\,dx \overset{\text{2 points}}{=} \int\left(x^3+x+\frac{3}{2(x-1)}-\frac{1}{2(x+1)}\right)dx$$

$$= \frac{x^4}{4}+\frac{1}{2}x^2+\frac{3}{2}\ln|x-1|-\frac{1}{2}\ln|x+1| + C$$

$$\boxed{\int \frac{x^5+2}{x^2-1}\,dx = \frac{x^4}{4}+\frac{x^2}{2}+\frac{3}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C}$$

2 points

7. (~~14~~ 15 points) Evaluate $\int \frac{dx}{x^3+1}$.

- $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ 2 points

- Multiply both sides by x^3+1 and get:

$$1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$1 = (A+B)x^2 + (B+C-A)x + (A+C).$$

- Then $\left\{\begin{array}{ll} (1) & A+B=0 \\ (2) & B+C-A=0 \\ (3) & A+C=1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} A = 1/3 \quad 1 \\ B = -1/3 \quad 1 \\ C = 2/3. \quad 1 \end{array}\right\}$ 3 points

- $\int \frac{dx}{x^3+1} = \frac{1}{3}\left[\int \frac{dx}{x+1} - \int \frac{x-2}{x^2-x+1}\,dx\right]$ 1 point

- $\int \frac{dx}{x+1} = \operatorname{Ln}|x+1| + C.$ 1 point

- $\int \frac{x-2}{x^2-x+1}\,dx = \frac{1}{2}\int \frac{2x-4}{x^2-x+1} = \frac{1}{2}\int \frac{2x-1-3}{x^2-x+1}\,dx$

$$= \frac{1}{2}\int \left(\frac{2x-1}{x^2-x+1} - \frac{3}{x^2-x+1}\right)dx \quad \text{2 points}$$

$$= \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{3}{2}\int \frac{dx}{x^2-x+1}.$$

$$= \frac{1}{2}\operatorname{Ln}|x^2-x+1| - \frac{3}{2}\int \frac{dx}{x^2-x+1}. \quad \text{1 point}$$

- $\int \frac{dx}{x^2-x+1} = \int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{4}{3}\int \frac{dx}{\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)^2+1}$ 2 points

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right) + C. \quad \text{2 points}$$

Then:

- $\int \frac{dx}{x^3+1} = \frac{1}{3}\operatorname{Ln}|x+1| - \frac{1}{6}\operatorname{Ln}(x^2-x+1) + \sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right) + C$

1 point

8. (~~12~~ 13 points) Evaluate $\int \frac{\sin 2x}{1+\cos^4 x}\,dx$.

- $\int \frac{\sin 2x\,dx}{1+\cos^4 x} = \int \frac{2\sin x \cos x\,dx}{1+\cos^4 x}$

$\Rightarrow$ Let $u = \cos x$. Then $du = -\sin x\,dx$. 2 points

So $\int \frac{\sin 2x\,dx}{1+\cos^4 x} = \int \frac{2\sin x\cos x}{1+\cos^4 x}\,dx$. 2 points

$= \int \frac{-2u\,du}{1+u^4}$ 2 points

$= -\int \frac{2u\,du}{1+u^4}$.

$\Rightarrow$ Set $t = u^2$. Then $dt = 2u\,du$. 2 points

So $\int \frac{\sin 2x\,dx}{1+\cos^4 x} = -\int \frac{2u\,du}{1+u^4} = -\int \frac{dt}{1+t^2}$ 1 point

$= -\tan^{-1} t + C$. 1 point

$= -\tan^{-1} u^2 + C$. 1 point

$= -\tan^{-1}(\cos^2 x) + C$. 2 points

Second Method.

Set $u = \cos 2x$. Then $du = -2\sin 2x\,dx$ 2points

So $\int \frac{\sin 2x\,dx}{1+\cos^4 x} = \int \frac{\sin 2x}{1+\frac{\cos^2 2x - 2\cos 2x + 1}{4}}$ 2points $= -2\int \frac{du}{u^2-2u+5}$ 2points

$= -2\int \frac{du}{(u-1)^2+4}$ 3points $= -\tan^{-1}\left(\frac{u-1}{2}\right) + C$ 2points $= -\tan^{-1}(\cos^2 x) + C$ 2points

9. 10 ~~(9~~ points) Determine whether the integral $\int_1^3 \frac{1}{\sqrt{x-1}}\,dx$ is convergent or divergent.

- The function $f(x) = \frac{1}{\sqrt{x-1}}$ is not defined at 1.

So $\int_1^3 \frac{dx}{\sqrt{x-1}} = \lim_{t\to 1^+} \int_t^3 \frac{dx}{\sqrt{x-1}}$. 2 points

- $\int_t^3 \frac{dx}{\sqrt{x-1}} = \int_t^3 (x-1)^{-1/2}\,dx$.

$$= \left[2(x-1)^{1/2}\right]_t^3 \quad \text{2 points}$$

$$= 2\left(\sqrt{2} - \sqrt{t-1}\right). \quad \text{2 points}$$

- So $\int_1^3 \frac{dx}{\sqrt{x-1}} = \lim_{t\to 1^+} 2\left(\sqrt{2} - \sqrt{t-1}\right)$ 2 points

$$= 2\left(\sqrt{2} - 0\right)$$

$$= 2\sqrt{2}. \quad \text{1 point}$$

- The improper integral is Convergent and its Value is $2\sqrt{2}$. 1 point.