Exam II. Math 102 Term (092) .Sunday, May 02, 2010. Marks: 100. Time: 2 Hours

Q.1. $(7-\ points)$. Write out the form of partial fraction decomposition of the expression

$$\frac{x^2}{(x+1)(x-2)^3(x^2+1)(x^2+x+1)^2}.$$

[Do not determine the numerical value of the coefficients]

SOLUTION. $\dfrac{x^2}{(x+1)(x-2)^3(x^2+1)(x^2+x+1)^2}$

$$= \overbrace{\frac{A}{x+1}}^{(1-point)} + \overbrace{\frac{B}{(x-2)}}^{(1-point)} + \overbrace{\frac{C}{(x-2)^2}}^{(1-point)} + \overbrace{\frac{D}{(x-2)^3}}^{(1-point)}$$

$$+ \overbrace{\frac{Ex+F}{x^2+1}}^{(1-point)} + \overbrace{\frac{Gx+H}{(x^2+x+1)}}^{(1-point)} + \overbrace{\frac{Jx+K}{(x^2+x+1)^2}}^{(1-point)}$$

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Q.2. $(5-\ points)$. Evaluate $\int \ln(2x)\,dx$.

SOLUTION.
$$\left.\begin{array}{ll} u = \ln(2x) & dv = dx \\ du = \dfrac{2dx}{2x} = \dfrac{dx}{x} & v = x \end{array}\right\} \ldots\ldots \longrightarrow (2-points)$$

$$\int \ln(2x)\,dx = x\ln(2x) - \int dx \ \ldots\ldots. \longrightarrow (2-points)$$

$$= x\ln(2x) - x + C \ \ldots\ldots\ldots \longrightarrow (1-point)$$

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Q.3. $(6-\ points)$. Determine whether the sequence $\left\{\dfrac{e^n+e^{-n}}{e^{2n}-1}\right\}$ converges or diverges.

If it converges, find the limit.

SOLUTION. Let $a_n = \dfrac{e^n+e^{-n}}{e^{2n}-1} = \dfrac{e^{-n}(e^{2n}+1)}{e^{2n}-1}$

$$= e^{-n}\left(\frac{e^{2n}+1}{e^{2n}-1}\right) = e^{-n}\left(\frac{1+\dfrac{1}{e^{2n}}}{1-\dfrac{1}{e^{2n}}}\right) \ \ldots. \longrightarrow (3-points)$$

$$\lim_{n\longrightarrow\infty} a_n = 0\left(\frac{1+0}{1-0}\right) = 0\ldots\ldots\ldots \longrightarrow (2-points)$$

The sequence $\left\{a_n = \dfrac{e^n+e^{-n}}{e^{2n}+1}\right\}$ CONVERGES. .... $\longrightarrow (1-point)$

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Q.4. $(8-\ points)$.$Find$ all possible values of the number b such that the average value of $f(x)=4+8x-3x^2$ on the interval $[0,b]$ is equal to 3.

SOLUTION. $f_{av}=\dfrac{1}{b-0}\displaystyle\int_0^b f(x)\,dx=$

$\dfrac{1}{b}\displaystyle\int_0^b (4+8x-3x^2)\,dx=3......\longrightarrow(2-points)$

$3b=\displaystyle\int_0^b (4+8x-3x^2)\,dx=\left[4x+4x^2-x^3\right]_0^b=4b+4b^2-b^3$

$\Longrightarrow b^3-4b^2-b=0\Longrightarrow b\left(b^2-4b-1\right)=0\ ....\longrightarrow(3-points)$

Since $b\neq 0$,then

$or\ b^2-4b-1=0\Longrightarrow b=\dfrac{4\pm\sqrt{16+4}}{2}=2\pm\sqrt{5}\ .....\longrightarrow(1-point)$

$\Longrightarrow b=2-\sqrt{5}\ .....rejected\ ....\longrightarrow(1-point)$

$\&\ b=2+\sqrt{5}\ ....\ OK\ ....\ (1-point)$.

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Q.5. $(8-\ points)$. Evaluate $\displaystyle\int\frac{\tan^5 x}{\sqrt[3]{\sec x}}dx$.

SOLUTION. $\displaystyle\int\frac{\tan^5 x}{\sqrt[3]{\sec x}}dx=\int\frac{\tan x\sec x\left(\sec^2 x-1\right)^2}{\sec x\sqrt[3]{\sec x}}dx\ .....\longrightarrow(2-points)$

$Let\ u=\sec x\ ,\ du=\sec x\tan x\ dx\ ......\longrightarrow(2-points)$

Then $\displaystyle\int\frac{\tan^5 x}{\sqrt[3]{\sec x}}dx=\int\frac{\left(u^2-1\right)^2}{u^{4/3}}du\ ....\longrightarrow(1-point)$

$=\displaystyle\int\frac{u^4-2u^2+1}{u^{4/3}}du=\int\left(u^{8/3}-2u^{2/3}+u^{-4/3}\right)du\ .\ .....\longrightarrow(1-point)$

$=\dfrac{3}{11}u^{11/3}-\dfrac{6}{5}u^{5/3}-3u^{-1/3}+C\ ......\longrightarrow(1-point)$

$=\dfrac{3}{11}\sec^{11/3}x-\dfrac{6}{5}\sec^{5/3}x-3\sec^{-1/3}x+C\ ......\longrightarrow(1-point)$

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Q.6. $(7-\ points)$. Evaluate $\displaystyle\int \frac{dx}{(16-x^2)^{3/2}}$.

SOLUTION. Let $x = 4\sin\theta$, $dx = 4\cos\theta\ d\theta$ ....$\longrightarrow$ $(2 - points)$

$$\int \frac{dx}{(16-x^2)^{3/2}} = \int \frac{4\cos\theta\ d\theta}{(16\cos^2\theta)^{3/2}} \text{ ...... } \longrightarrow (1-point)$$

$$= \int \frac{1}{16}\sec^2\theta\ d\theta \text{ .... } \longrightarrow (1-point)$$

$$= \frac{1}{16}\tan\theta + C \text{ .... } \longrightarrow (1-point)$$

Use triangle method

$$\text{or } \left[\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{4\sin\theta}{\sqrt{16-16\sin^2\theta}} = \frac{x}{\sqrt{16-x^2}}\right] \text{ .... } \longrightarrow (1-point)$$

$$= \frac{1}{16}\frac{x}{\sqrt{16-x^2}} + C \text{ .... } \longrightarrow (1-point) .$$

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Q.7. $(9-\ points)$. Evaluate $\displaystyle\int \csc^3 x\ dx$.

SOLUTION. $\displaystyle\int \csc^3 x\ dx = \int \csc x \csc^2 x\ dx$ ...... $\longrightarrow (1-point)$

$$\left.\begin{array}{ll} Let\ u = \csc x & dv = \csc^2 x\ dx \\ du = -\csc x\ \cot x\ dx & v = -\cot x \end{array}\right] \text{ .... } \longrightarrow (2-points)$$

$$\int \csc^3 x\ dx = -\csc x\ \cot x - \int \csc x\ \cot^2 x\ dx \text{ .... } \longrightarrow (2-points)$$

$$= -\csc x\ \cot x - \int \csc^3 x\ dx + \int \csc x\ dx \text{ .... } \longrightarrow (1-point)$$

$$2\int \csc^3 x\ dx = -\csc x\ \cot x + \int \csc x\ dx \text{ .... } \longrightarrow (1-point)$$

$$= -\csc x\ \cot x + \ln|\csc x - \cot x| + C \text{ .... } \longrightarrow (1-point)$$

$$\int \csc^3 x\ dx = -\frac{1}{2}\csc x\ \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C \text{ ..... } \longrightarrow (1-point)$$

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Q.8. $(12-\ points)$. Evaluate $\int \frac{4x^2+13x+15}{x^3+4x^2+5x}dx.$

SOLUTION. The form of the partial fraction decomposition is

$$\frac{4x^2+13x+15}{x(x^2+4x+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+5} \ \ldots \longrightarrow (1-point)$$

Then $4x^2+13x+15 = A(x^2+4x+5) + x(Bx+C)$

$= (A+B)x^2 + (4A+C)x + 5A \ \ldots \longrightarrow (1-point)$

If we equate coefficients of x^0, x, and x^2, we get

$$\left.\begin{array}{lcl} 5A=15 & \Longrightarrow & A=3 \\ 4A+C=13 & \Longrightarrow & C=1 \\ A+B=4 & \Longrightarrow & B=1 \end{array}\right\} \ldots\ldots \longrightarrow (3-points)$$

$$\Longrightarrow \int \frac{4x^2+13x+15}{x(x^2+4x+5)}dx = \int \left(\frac{3}{x} + \frac{x+1}{x^2+4x+5}\right) dx \ \ldots \longrightarrow (1-point)$$

$$\int \frac{x+1}{x^2+4x+5}dx = \frac{1}{2}\int \frac{2(x+1)+2-2}{x^2+4x+5}dx\ldots. \longrightarrow (1-point)$$

$$= \frac{1}{2}\int \frac{2x+4-2}{x^2+4x+5}dx$$

$$= \frac{1}{2}\left[\ln\left|x^2+4x+5\right|\right] - \int \frac{2dx}{(x+2)^2+1} \ \ldots.. \longrightarrow (2-points)$$

$$= \frac{1}{2}\left[\ln\left|x^2+4x+5\right|\right] - \tan^{-1}(x+2) + C \ \ldots \longrightarrow (1-point)$$

$$\int \frac{4x^2+13x+15}{x^3+4x^2+5x}dx$$

$$= \underbrace{3\ln|x|}_{(1-point)} + \frac{1}{2}\ln\left|x^2+4x+5\right| - \tan^{-1}(x+2) + C \ \ldots \longrightarrow (1-point)$$

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Q.9. $(6-\ points)$. Determine whether the sequence

$$\left(\frac{2}{4}-\frac{1}{3}\right), \left(\frac{4}{5}-\frac{1}{5}\right), \left(\frac{6}{6}-\frac{1}{7}\right), \left(\frac{8}{7}-\frac{1}{9}\right), \ldots$$

converges or diverges. If it converges, find the limit.

SOLUTION. $a_n = \frac{2n}{n+3} - \frac{1}{2n+1} \ \ldots\ldots \longrightarrow (3-points)$

$$= -\frac{2}{1+3/n} - \frac{1}{2n+1}$$

$$\lim_{n\longrightarrow\infty} a_n = \frac{2}{1+0} - 0 = 2 \ \ldots\ldots. \longrightarrow (2-points)$$

The given sequence CONVERGES. ...... $\longrightarrow (1-point)$

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Q.10. (9 − *points*) . $\int_0^{\pi/3} \sin^3\theta \sec^2\theta \, d\theta.$

<u>SOLUTION</u>. $\int \sin^3\theta \sec^2\theta \, d\theta = \int \sin\theta \frac{(1-\cos^2\theta)}{\cos^2\theta} d\theta$ ..... $\longrightarrow (2 - points)$

Let $u = \cos\theta \Longrightarrow du = -\sin\theta \, d\theta$ .. .. $\longrightarrow (1 - point)$

$\theta = 0 \Longrightarrow u = 1$ and $\theta = \pi/3 \Longrightarrow u = 1/2$ .... $\longrightarrow (2 - points)$

$$\int_0^{\pi/3} \sin^3\theta \sec^2\theta \, d\theta = -\int_1^{1/2} \frac{1-u^2}{u^2} \, du = -\int_1^{1/2} (u^{-2} - 1) \, du \text{ .... } \longrightarrow (2 - points)$$

$= [u^{-1} + u]_1^{1/2}$ ..... $\longrightarrow (1 - point)$

$= 2 + \frac{1}{2} - (1 + 1) = \frac{1}{2}$ .... $\longrightarrow (1 - point)$

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Q.11. (8 − *points*) . Evaluate $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}.$

<u>SOLUTION</u>. *Let* $u = \sqrt[6]{x} \Longrightarrow x = u^6 \Longrightarrow dx = 6u^5 du$ .... $\longrightarrow (2 - points)$

$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \int \frac{6u^5 du}{u^3 + u^2}$ ..... $\longrightarrow (1 - point)$

$= \int \frac{6u^3 du}{u + 1}$ ..... $\longrightarrow (1 - point)$

[Using long or synthetic division we get:]..... $\longrightarrow (1 - point)$

$= 6 \int \left((u^2 - u + 1) - \frac{1}{u+1} \right) du$ .... $\longrightarrow (2 - points)$

$= 6 \left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1| \right] + C$ .... $\longrightarrow (1 - point)$

$= 2u^3 - 3u^2 + 6u - 6\ln|u+1| + C$

$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6\ln\left|x^{1/6} + 1\right| + C$ ...... $\longrightarrow (1 - point)$

$(= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|\sqrt[6]{x} + 1| + C)$

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Q.12. $(7-\ points)$. Determine whether the integral $\int_0^1 \frac{dx}{(x-1)^{2/3}}$ converges or diverges.

[Show your steps and give a reason to your answer]

SOLUTION. The given integral is improper, since $f(x) = \frac{1}{(x-1)^{2/3}}$ has a vertical asymptote at $x = 1$. .... $\longrightarrow (1-point)$

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{t\longrightarrow 1^-} \int_0^t \frac{dx}{(x-1)^{2/3}} \text{ ..... } \longrightarrow (2-points)$$

$$= \lim_{t\longrightarrow 1^-} \left[3(x-1)^{1/3}\right]_0^t \text{ .... } \longrightarrow (1-point)$$

$$= \lim_{t\longrightarrow 1^-} \left[3(t-1)^{1/3} - (-3)\right] \text{ .... } \longrightarrow (1-point)$$

$$= 0 + 3 = 3. \text{ .... } \longrightarrow (1-point)$$

The integral converges. .... $\longrightarrow (1-point)$

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Q.13. $(8-\ points)$. Evaluate $\int \frac{1}{1+\sin x+\cos x}dx$.

[You may use the substitution $t = \tan\frac{x}{2}$]

SOLUTION.

$$\left.\begin{array}{rcl} dx & = & \frac{2dt}{1+t^2} \\ \sin x & = & \frac{2t}{1+t^2} \\ \cos x & = & \frac{1-t^2}{1+t^2} \end{array}\right\} \text{ ..... } \longrightarrow (2-points)$$

$$\int \frac{1}{1+\sin x+\cos x}dx = \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \text{ ..... } \longrightarrow (2-points)$$

$$= \int \frac{2dt}{t^2+1+2t+1-t^2} = \int \frac{2dt}{2t+2} \text{...... } \longrightarrow (2-points)$$

$$= \int \frac{dt}{t+1} = \ln|1+t| + C\text{...... } \longrightarrow (1-point)$$

$$= \ln\left|1+\tan\frac{x}{2}\right| + C\text{...... } \longrightarrow (1-point)$$

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