King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 102- Calculus II
Exam II
Term (112)

April 19, 2012 **Net Time Allowed: 2 hours**

Name: Solutions **ID Number:** ______________

Section Number: ______________ **Serial Number:** ______________

Class Time: ______________ **Instructor's Name:** ______________

Instructions:

1. **Calculators and Mobiles are not allowed.**
2. Write neatly and eligibly. You may lose points for messy work.
3. **Show all your work.** No points for answers without justification.
4. Make sure that you have 7 pages of problems (**Total of 13 Problems**)

Page	Points	Maximum Points
Page 1		14
Page 2		16
Page 3		14
Page 4		16
Page 5		12
Page 6		12
Page 7		16
Total		100

1. **(3 Points)** Let $\{a_n\}$ be a sequence such that $a_1 = 1, a_2 = 2$ and $a_{n+1} = a_n + (-1)^{n-1} a_{n-1}$ for $n \geq 2$. Find the value of a_5

$n = 2 \Rightarrow a_3 = a_2 - a_1 = 2 - 1 = 1$ (1 pt)

$n = 3 \Rightarrow a_4 = a_3 + a_2 = 1 + 2 = 3$ (1 pt)

$n = 4 \Rightarrow a_5 = a_4 - a_3 = 3 - 1 = 2$ (1 pt)

2. **(5 Points)** If $K = \int \sec\theta \tan^2\theta \, d\theta$, show that $\int \sqrt{x^2 + 6x} \, dx = 9K$

$\int \sqrt{x^2+6x} \, dx = \int \sqrt{(x+3)^2 - 9} \, dx$ (1 pt)

let $x + 3 = 3\sec\theta \Rightarrow \sqrt{x^2+6x} = 3\tan\theta$ and
$dx = 3\sec\theta \tan\theta \, d\theta$ (2 pts)

$\Rightarrow \int \sqrt{x^2+6x} \, dx = \int (3\tan\theta)(3\sec\theta\tan\theta) \, d\theta$ (1 pt)

$= 9 \int \sec\theta \tan^2\theta \, d\theta = 9K$ (1 pt)

3. **(6 Points)** Evaluate $\int \frac{1}{\sqrt{x+6} + \sqrt{x+1}} dx$

$\int \frac{dx}{\sqrt{x+6} + \sqrt{x+1}} = \int \frac{\sqrt{x+6} - \sqrt{x+1}}{(x+6) - (x+1)} dx$ (2 pts)

$= \frac{1}{5} \int \left(\sqrt{x+6} - \sqrt{x+1} \right) dx$ (1 pt)

$= \frac{1}{5} \int (x+6)^{1/2} dx - \frac{1}{5} \int (x+1)^{1/2} dx$ (1 pt)

$= \frac{2}{15} (x+6)^{3/2} - \frac{2}{15} (x+1)^{3/2} + C$ (2 pts)

4. **(8 Points)** Find the average value of the function $f(x) = x\,e^{-2x}$ on the interval [0,2].

$$f_{ave} = \frac{1}{2-0}\int_0^2 x\,e^{-2x}\,dx \quad \text{(2 pts)}$$

let $u = x$, $dv = e^{-2x} \Rightarrow du = dx$, $v = -\frac{1}{2}e^{-2x}$

$$\Rightarrow f_{ave} = \frac{1}{2}\left[-\frac{1}{2}x\,e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx\right]_0^2 \quad \text{(3 pts)}$$

$$= \frac{1}{4}\left[-xe^{-2x} - \frac{1}{2}e^{-2x}\right]_0^2 \quad \text{(1 pt)}$$

$$= \frac{1}{4}\left[\left(-2e^{-4} - \frac{1}{2}e^{-4}\right) - \left(-\frac{1}{2}\right)\right] \quad \text{(1 pt)}$$

$$= \frac{1}{8} - \frac{5}{8}e^{-4}. \quad \text{(1 pt)}$$

5. **(8 Points)** Evaluate $\int \frac{\sin x}{\csc^2 x - 1}\,dx$

$$\int \frac{\sin x}{\csc^2 x - 1}\,dx = \int \frac{\sin x}{\cot^2 x}\,dx = \int \frac{\sin^3 x}{\cos^2 x}\,dx \quad \text{(2 pts)}$$

$$= \int \frac{1-\cos^2 x}{\cos^2 x}\,\sin x\,dx \quad \text{(2 pts)}$$

Let $u = \cos x \Rightarrow du = -\sin x\,dx \Rightarrow$ (1 pt)

$$\int \frac{\sin x}{\csc^2 x - 1}\,dx = \int \frac{1-u^2}{u^2}(-du) = \int\left(-\frac{1}{u^2} + 1\right)du \quad \text{(1 pt)}$$

$$= \frac{1}{u} + u + C \quad \text{(1 pt)}$$

$$\left.\begin{aligned} &= \frac{1}{\cos x} + \cos x + C \\ \text{OR} \quad &= \sec x + \cos x + C \end{aligned}\right\} \text{(1 pt)}$$

6. **(10 Points)** Evaluate $\int \frac{x^2}{\sqrt{1-4x^2}}\,dx$

Let $2x = \sin\theta \Rightarrow \sqrt{1-4x^2} = \cos\theta$ and

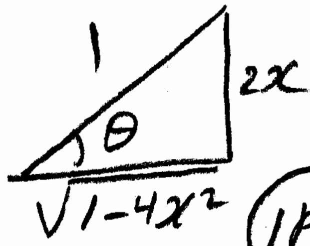
$dx = \frac{1}{2}\cos\theta \, d\theta \Rightarrow$ (2pts)

$$\int \frac{x^2}{\sqrt{1-4x^2}}\,dx = \int \frac{\frac{1}{4}\sin^2\theta}{\cos\theta}\left(\frac{1}{2}\cos\theta\, d\theta\right)$$

$$= \frac{1}{8}\int \sin^2\theta \, d\theta = \frac{1}{16}\int (1-\cos 2\theta)\, d\theta$$ (3pts)

$$= \frac{1}{16}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{16}\left(\theta - \sin\theta\cos\theta\right) + C$$ (2pt)

$$= \frac{1}{16}\left(\sin^{-1}(2x) - 2x\sqrt{1-4x^2}\right) + C$$ (2pts)



(1pt)

7. **(4 Points)** Determine whether the sequence $\left\{\frac{2n+5}{\sqrt{1+2n+9n^2}}\right\}$ converges or diverges. If it converges, find the limit

$$\lim_{n\to\infty} \frac{2n+5}{\sqrt{1+2n+9n^2}} = \lim_{n\to\infty} \frac{2+\frac{5}{n}}{\sqrt{\frac{1}{n^2}+\frac{2}{n}+9}}$$ (2pts)

$$= \frac{2}{\sqrt{9}} = \frac{2}{3}$$ (1pt)

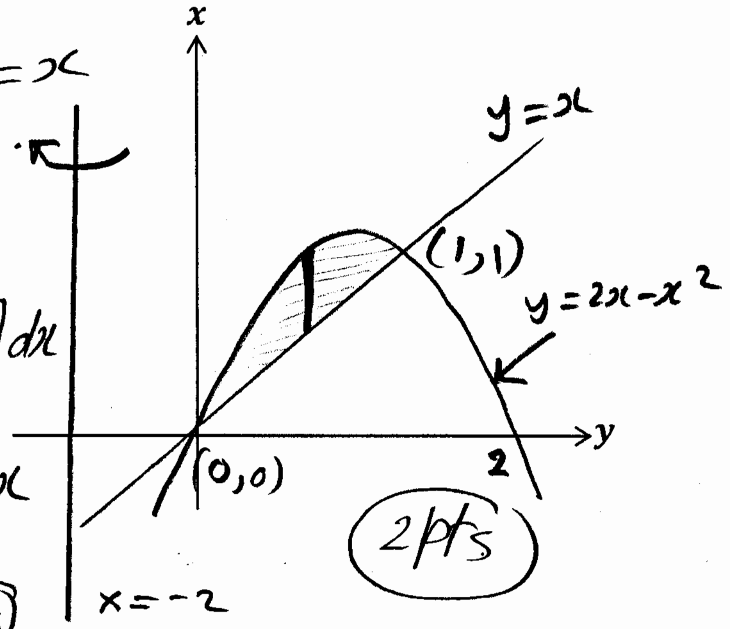
$\Rightarrow$ The sequence Converges to $\frac{2}{3}$ (1pt)

8. **(8 Points)** Use cylindrical shells to set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region R bounded by the parabola $y = 2x - x^2$ and the line $y = x$ about the line $x = -2$. [sketch the region R]

Points of intersection $2x - x^2 = x$

$\Rightarrow x(x-1) = 0 \Rightarrow (0,0), (1,1)$.

(1pt)



$$\text{Volume} = \int_0^1 2\pi(x+2)\left[(2x-x^2)-x\right]dx$$

$$= \int_0^1 2\pi(x+2)(x-x^2)\,dx$$

(1pt) (2pts) (2pts)

9. **(8 Points)** Evaluate $\int_2^{\infty} \frac{dx}{(x-1)[9+\ln(x-1)]^{3/2}}$ if possible

Let $u = 9 + \ln(x-1) \Rightarrow du = \frac{1}{x-1}dx \Rightarrow$

$$\int \frac{dx}{(x-1)[9+\ln(x-1)]^{3/2}} = \int u^{-3/2}\,du = -2u^{-1/2} + c$$

$$= \frac{-2}{\sqrt{9+\ln(x-1)}} + c \quad \text{(3pts)}$$

$$\text{Thus } \int_2^{\infty} \frac{dx}{(x-1)[9+\ln(x-1)]^{3/2}} = \lim_{t\to\infty} \int_2^t \frac{dx}{(x-1)[9+\ln(x-1)]^{3/2}} \quad \text{(2pts)}$$

$$= \lim_{t\to\infty} \left[\frac{-2}{\sqrt{9+\ln(x-1)}}\right]_2^t = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{9+\ln(t-1)}} + \frac{2}{3}\right]$$

(1pt) (1pt)

$$= \frac{2}{3}.$$ (1pt)

10. **(12 Points)** Evaluate $\int \frac{15}{t^3-2t^2+t-2}\,dt$

$$\frac{15}{t^3-2t^2+t-2} = \frac{15}{t^2(t-2)+(t-2)} = \frac{15}{(t-2)(t^2+1)} \quad \text{1pt}$$

$$= \frac{A}{t-2} + \frac{Bt+C}{t^2+1} \quad \text{2pts}$$

$$15 = A(t^2+1) + (Bt+C)(t-2) \quad \text{1pt}$$

$$\boxed{t=2} \Rightarrow 15 = 5A \Rightarrow \boxed{A=3} \quad \text{1pt}$$

$$\boxed{\text{Coef } t^2} \Rightarrow 0 = A+B \Rightarrow \boxed{B=-3} \quad \text{1pt}$$

$$\boxed{\text{Coef } t} \Rightarrow 0 = -2B+C \Rightarrow \boxed{C=-6} \quad \text{1pt}$$

Thus $$\int \frac{15}{t^3-2t^2+t-2}\,dt = \int \frac{3}{t-2}\,dt + \int \frac{-3t-6}{t^2+1}\,dt \quad \text{1pt}$$

$$= 3\ln|t-2| - 3\int \frac{t}{t^2+1}\,dt - 6\int \frac{1}{t^2+1}\,dt \quad \text{2pts}$$

$$= 3\ln|t-2| - \frac{3}{2}\ln(t^2+1) - 6\tan^{-1} t + C. \quad \text{2pts}$$

11.**(12 Points)** Evaluate $\int \sin(\ln x^2)\,dx = I$

Let $u = \sin(\ln x^2)$ and $dv = dx$

$du = \frac{2}{x}\cos(\ln x^2)\,dx$, $v = x$ (2 pts)

Thus $I = x\sin(\ln x^2) - 2\int \cos(\ln x^2)\,dx$ (2 pts)

Let $u = \cos(\ln x^2)$, $dv = dx$

$du = -\frac{2}{x}\sin(\ln x^2)\,dx$, $v = x$ (2 pts)

Thus $I = x\sin(\ln x^2) - 2\Big[x\cos(\ln x^2) + \underbrace{2\int \sin(\ln x^2)\,dx}_{2I}\Big]$ (2 pts)

$\Rightarrow 5I = x\sin(\ln x^2) - 2x\cos(\ln x^2)$ (2 pts)

$\Rightarrow \int \sin(\ln x^2)\,dx = \frac{1}{5}x\sin(\ln x^2) - \frac{2}{5}x\cos(\ln x^2) + C$

(2 pts)

12.**(8 Points)** Determine whether the integral $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \csc x \, dx$ converges or diverges. If it converges, find its value

The integral is improper because $|\csc x| \to \infty$ at $x = \pi$ which is inside the interval of integration (1pt)

we write $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \csc x \, dx = \int_{\frac{\pi}{2}}^{\pi} \csc x \, dx + \int_{\pi}^{\frac{3\pi}{2}} \csc x \, dx$ (2pts)

But $\int_{\frac{\pi}{2}}^{\pi} \csc x \, dx = \lim_{t \to \pi^-} \int_{\frac{\pi}{2}}^{t} \csc x \, dx$ (1pt)

$= \lim_{t \to \pi^-} \left[\ln|\csc x - \cot x| \right]_{\frac{\pi}{2}}^{t} = \lim_{t \to \pi^-} \ln|\csc t - \cot t|$ (2pts)

$= \infty$ (1pt)

So the integral diverges (1pt)

13. **(8 Points)** Determine whether the sequence $\left\{ \frac{n!}{n^n} \right\}$ converges or diverges. If it converges, find the limit

$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n}{n \cdot n \cdot n \cdots n \cdot n}$ (1pt)

$= \frac{1}{n} \left(\frac{2 \cdot 3 \cdots (n-1)\, n}{n \cdot n \cdots n \;\; n} \right) \leqslant \frac{1}{n}$ (2pts)

Thus $0 < a_n \leqslant \frac{1}{n}$ (2pts)

But $\lim_{n \to \infty} \frac{1}{n} = 0$ (1pt)

Therefore $\lim_{n \to \infty} \frac{n!}{n^n} = 0$ by the Squeeze Theorem (1pt).

Thus the sequence converges to 0 (1pt)