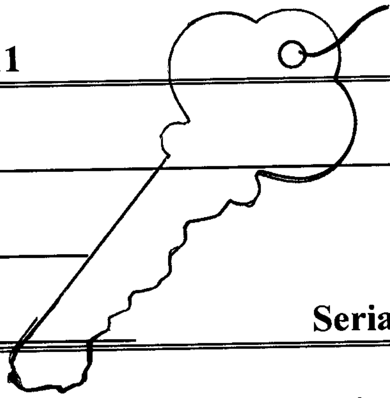
King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 102- Calculus II
Exam II
2011-2012 (Term 111)

Tuesday, Nov. 22, 2011 **Allowed Time: 2 hours**



Name: ______________________

ID Number: ____________

Section Number: ________ **Serial Number:** ________

Instructions:

1. Write neatly and eligibly. You may lose points for messy work.
2. **Show all your work. No points for answers without justification.**
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 9 different problems (6 pages + cover page)

Question #	Grade	Maximum Points
1		7
2		15
3		15
4		10
5		10
6		10
7		8
8		10
9		15
Total		100

(1) **[7 Points]** Find the number c so that $f(c)$ is the average value of the function $f(x) = \sqrt{x}$ over the interval $[0, 2]$.

(2) $f_{ave} = \frac{1}{2-0} \int_0^2 \sqrt{x}\, dx$

(1) $= \frac{1}{2} \cdot \frac{2}{3} x^{3/2} \Big|_0^2$

$= \frac{1}{3} \cdot 2^{3/2}$

(1) $= \frac{2}{3}\sqrt{2}$

(1) $f(c) = f_{ave}$

$\sqrt{c} = \frac{2}{3} \cdot \sqrt{2}$

$\Rightarrow c = \frac{8}{9}$. (2)

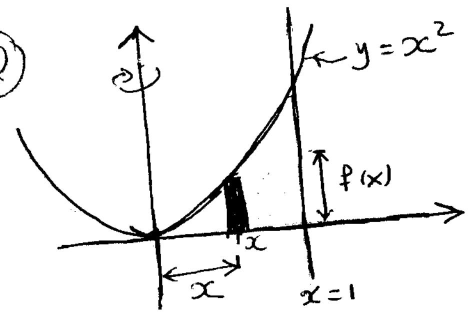
(2) Using the method of cylindrical shells, set up **(but DO NOT EVALUATE)** an integral for the volume of the solid generated by revolving

a) **[7 Points]** The region enclosed by the curves $y = x^2, y = 0, x = 1$ about the $y-$ axis. [Sketch the region and a typical rectangle]

$V = \int_0^1 2\pi \cdot x \cdot x^2 \, dx$

(1) (1) (2) (1)

(2)

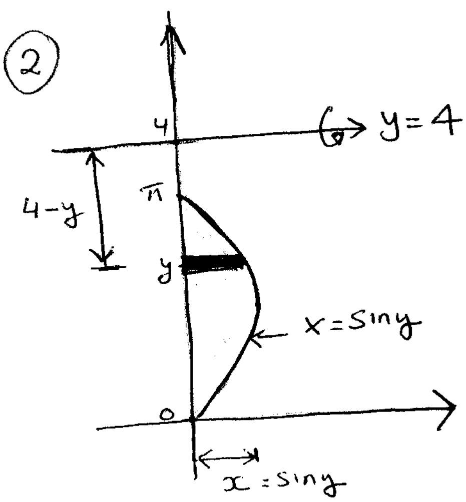


b) **[8 Points]** The region enclosed by the curves $x = \sin y, x = 0, y = 0, y = \pi$ about the line $y = 4$. [Sketch the region and a typical rectangle]

$V = \int_0^\pi 2\pi \cdot (4-y) \cdot \sin y \, dy$

(1) (2) (2) (1)

(2)



(3) a) **[7 Points]** Find $\int (2+\tan x)^2 dx$.

$= \int (4 + 4\tan x + \tan^2 x)\, dx$ 2

$= 4x + 4\ln|\sec x| + \int \tan^2 x\, dx$ (1+1)

$= 4x + 4\ln|\sec x| + \int (\sec^2 x - 1)\, dx$ 1

$= 4x + 4\ln|\sec x| + \tan x - x + C$ 1

$= 3x + 4\ln|\sec x| + \tan x + C$ 1

b) **[8 Points]** Determine whether the integral $\int_6^8 \frac{4}{(x-6)^3}dx$ converges or diverges. If it converges, find its value.

$\int_6^8 \frac{4}{(x-6)^3}\, dx = \lim_{t\to 6^+} \int_t^8 \frac{4}{(x-6)^3}\, dx$ 2

$= \lim_{t\to 6^+} \left. \frac{-2}{(x-6)^2} \right|_t^8$ 2

$= \lim_{t\to 6^+} -\frac{1}{2} + \frac{2}{(t-6)^2}$ 1

$= -\frac{1}{2} + \infty$ 1

$= \infty$ 1

So $\int_6^8 \frac{4}{(x-6)^3}\, dx$ diverges 1

4) **[10 Points]** Find $\int x^2 \sin(2x)dx$.

Use integration by parts: $u = x^2$ $\quad dv = \sin(2x)\,dx$

$du = 2x\,dx$ $\quad v = -\frac{1}{2}\cos(2x)$

$$\int x^2 \sin(2x)\,dx = -\frac{1}{2}x^2\cos(2x) + \int x\cos(2x)\,dx \quad 4$$

by Parts again: $u = x$ $\quad dv = \cos(2x)dx$

$du = dx$ $\quad v = \frac{1}{2}\sin(2x)$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx \quad 4$$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C \quad 2$$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}$$

5) **[10 Points]** Find $\int_{\frac{-\pi}{2}}^{0} \sqrt{\cos x - \cos^3 x}\,dx. = I$

$$I = \int_{-\frac{\pi}{2}}^{0} \sqrt{\cos x(1-\cos^2 x)}\,dx \quad 2$$

$$= \int_{-\frac{\pi}{2}}^{0} \sqrt{\cos x \cdot \sin^2 x}\,dx \quad 1$$

$$= \int_{-\frac{\pi}{2}}^{0} \sqrt{\cos x} \cdot (-\sin x)\,dx \quad 2$$

Since $\sqrt{\sin^2 x} = |\sin x| = -\sin x$ as $-\frac{\pi}{2} < x \leq 0$ 4th quadrant

Let $u = \cos x$. Then $du = -\sin x\,dx$

Note: Deduct 1 point if $\sin x$ is given instead of $-\sin x$.

$$= \int_{0}^{1} \sqrt{u}\,du \quad 3$$

$$= \frac{2}{3}u^{3/2}\Big|_0^1 \quad 1$$

$$= \frac{2}{3} \quad 1$$

6) **[10 Points]** Find $\displaystyle\int \frac{1}{x\sqrt{x^2+4}}\,dx$

Let $x = 2\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. 2

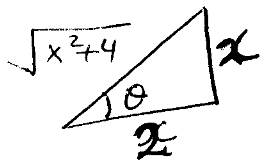
Then $dx = 2\sec^2\theta\,d\theta$ 1

$$\int \frac{1}{x\sqrt{x^2+4}}\,dx = \int \frac{1}{2\tan\theta\cdot\sqrt{4\tan^2\theta+4}}\cdot 2\sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\tan\theta\cdot\sec\theta}\,d\theta \qquad 1$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \frac{1}{2}\int \csc\theta\,d\theta \qquad 2$$



$$= \frac{1}{2}\ln\left|\csc\theta - \cot\theta\right| + C \qquad 2$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}}{x} - \frac{2}{x}\right| + C \qquad 2$$

7) **[8 Points]** Using the substitution $t = \tan\left(\frac{x}{2}\right)$, find the integral $\int \frac{1}{1-3\cos x}dx$.

$t = \tan(\frac{x}{2}) \Rightarrow \boxed{dx = \frac{2}{1+t^2}dt \text{ \& } \cos x = \frac{1-t^2}{1+t^2}}$ (2)

$$\int \frac{1}{1-3\cos x}dx = \int \frac{1}{1-3\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}dt = \int \frac{2}{(1+t^2)-3(1-t^2)}dt = \int \frac{2}{4t^2-2}dt$$

$$= \int \frac{1}{2t^2-1}dt \quad (2)$$

$$= \frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}\,t-1}{\sqrt{2}\,t+1}\right| + C \quad (3)$$

$$= \frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}\tan(\frac{x}{2})-1}{\sqrt{2}\tan(\frac{x}{2})+1}\right| + C \quad (1)$$

8) **[10 Points]** Find $\int (\sin(2x) + 2\cos x)e^{\sin x}dx$.

$$= \int (2\sin x\cos x + 2\cos x)\, e^{\sin x}\, dx \quad (2)$$

$$= 2\int (\sin x + 1) \cos x\, e^{\sin x}\, dx \quad (1)$$

Let $t = \sin x$. Then $dt = \cos x\, dx$ (1)

$$= 2\int (t+1)e^t\, dt \quad (1)$$

$\left[\begin{array}{ll} u = t+1 & dv = e^t dt \\ du = dt & v = e^t \end{array}\right.$

$$= 2\left[(t+1)e^t - \int e^t dt\right] \quad (2)$$

$$= 2\left[(t+1)e^t - e^t\right] + C \quad (1)$$

$$= 2te^t + C \quad (1)$$

$$= 2\sin x\, e^{\sin x} + C. \quad (1)$$

9) **[15 Points]** Determine whether the integral $\displaystyle\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}dx$ converges or diverges. If it converges, find its value.

$$\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}dx = \lim_{t\to\infty} \int_2^{t} \frac{x+3}{(x-1)(x^2+1)}dx \qquad (2)$$

$$\frac{x+3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \qquad (2 = 1+1)$$

$$\Rightarrow x+3 = A(x^2+1) + (x-1)(Bx+C)$$

$$= (A+B)x^2 + (C-B)x + (A-C) \qquad (1)$$

$$\Rightarrow \begin{cases} A+B = 0 \\ C-B = 1 \\ A-C = 3 \end{cases}, \text{ Solving we get } A = 2,\ B = -2,\ C = -1 \qquad (3 = 1+1+1)$$

$$\int_2^{t} \frac{x+3}{(x-1)(x^2+1)}dx = \int_2^{t} \frac{2}{x-1} + \frac{-2x-1}{x^2+1}\,dx$$

$$= \int_2^{t} \frac{2}{x-1} - \frac{2x}{x^2+1} - \frac{1}{x^2+1}\,dx \qquad (3 = 1+1+1)$$

$$= 2\ln|x-1| - \ln|x^2+1| - \tan^{-1}x \Big|_2^t$$

$$= (2\ln|t-1| - \ln|t^2+1| - \tan^{-1}t) - (0 - \ln 5 - \tan^{-1}2)$$

$$\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}dx = \lim_{t\to\infty} \int_2^{t} \frac{x+3}{(x-1)(x^2+1)}dx$$

$$= \lim_{t\to\infty} (2\ln|t-1| - \ln|t^2+1| - \tan^{-1}t) + \ln 5 + \tan^{-1}2$$

$$= \lim_{t\to\infty} \left(\ln\left|\frac{(t-1)^2}{t^2+1}\right| - \tan^{-1}t\right) + \ln 5 + \tan^{-1}2 \qquad (1)$$

$$= \ln|1| - \frac{\pi}{2} + \ln 5 + \tan^{-1}2 \qquad (2 = 1+1)$$

$$= -\frac{\pi}{2} + \ln 5 + \tan^{-1}2$$

So the Improper Integral Converges & its value is $-\frac{\pi}{2} + \ln 5 + \tan^{-1}2$.

(1)