King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 102 - Exam II - Term 131

Duration: 120 minutes

Name: Key ID Number:

Section Number: Serial Number:

Class Time: Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 9 pages of problems (Total of 10 Problems)

	Points	Maximum Points
page 1		12
page 2		10
page 3		10
page 4		10
page 5		12
page 6		14
page 7		12
page 8		10
page 9		10
Total		100

1. Evaluate

A) (6 points) $\int \ln(1-x)\,dx$

Let $u = \ln(1-x)$, $dv = dx$

$du = \frac{-dx}{1-x} = \frac{dx}{x-1}$, $v = x$ (2 pts)

$$\int \ln(1-x)\,dx = x\ln(1-x) - \int \frac{x}{x-1}\,dx \quad (2 \text{ pts})$$

$$= x\ln(1-x) - \int \left(1 + \frac{1}{x-1}\right) dx \quad (1 \text{ pt})$$

$$= x\ln(1-x) - x - \ln|x-1| + C \quad (1 \text{ pt})$$

B) (6 points) $\int \left(e^{a\ln x} + e^{x\ln a}\right) dx,\ a > 0 \text{ and } a \neq 1$

$$\int \left(e^{a\ln x} + e^{x\ln a}\right) dx = \int \left(e^{\ln x^a} + e^{\ln a^x}\right) dx \quad (2 \text{ pts})$$

$$= \int \left(x^a + a^x\right) dx \quad (2 \text{ pts})$$

$$= \frac{x^{a+1}}{a+1} + \frac{a^x}{\ln a} + C. \quad (2 \text{ pts})$$

2. (10 points)Evaluate $\int \frac{9x^3+1}{x^3-x^2}\,dx$.

$$x^3 - x^2 \overline{\big) 9x^3 + 1}$$ with quotient 9: $-(9x^3 - 9x^2)$, remainder $9x^2+1$

$$I = \int \frac{9x^3+1}{x^3-x^2}\,dx = \int\left(9 + \frac{9x^2+1}{x^3-x^2}\right)dx \quad (2\text{ pts})$$

by partial fraction:

$$\frac{9x^2+1}{x^3-x^2} = \frac{9x^2+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \quad (2\text{pts})$$

$$\Rightarrow 9x^2+1 = Ax(x-1) + B(x-1) + Cx^2$$

when $x=1 \Rightarrow C=10$ (1 pt)

when $x=0 \Rightarrow -B=1 \Rightarrow B=-1$ (1 pt)

Coef. of $x^2 \Rightarrow A+C=9 \Rightarrow A=9-C=-1$ (1 pt)

So,
$$I = \int\left(9 - \frac{1}{x} - \frac{1}{x^2} + \frac{10}{x-1}\right)dx$$

$$= 9x - \ln|x| + \frac{1}{x} + 10\ln|x-1| + C \quad (3\text{pts})$$

3. Evaluate

A) (5 points) $\int \sin^2 x \tan x \, dx$.

$$I = \int \sin^2 x \tan x \, dx = \int \frac{\sin^3 x}{\cos x} \, dx$$

$$= \int \frac{(1-\cos^2 x)}{\cos x} \sin x \, dx \quad . \quad (1\ pt)$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx \Rightarrow \quad (1\ pt)$

$$I = -\int \frac{1-u^2}{u} \, du = \int \left(u - \frac{1}{u}\right) du \quad (1\ pt)$$

$$= \frac{u^2}{2} - \ln|u| + C = \frac{1}{2}\cos^2 x - \ln|\cos x| + C$$

(1 pt) (1 pt)

B) (5 points) $\int \sec^4 x \, (1 - \tan^2 x) \, dx$.

$$I = \int \sec^4 x \, (1 - \tan^2 x) \, dx$$

$$= \int \sec^2 x \, (1 - \tan^2 x) \sec^2 x \, dx \quad (1\ pt)$$

$$= \int (1 + \tan^2 x)(1 - \tan^2 x) \sec^2 x \, dx$$

$$= \int (1 - \tan^4 x) \sec^2 x \, dx \quad (1\ pt)$$

Let $u = \tan x \Rightarrow du = \sec^2 x \, dx \quad (1\ pt)$

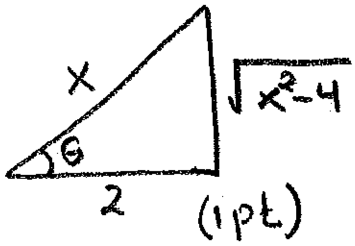
$$\Rightarrow I = \int 1 - u^4 \, du = u - \frac{1}{5}u^5 + C \quad (1\ pt)$$

$$= \tan x - \frac{1}{5}\tan^5 x + C \quad (1\ pt)$$

4. (10 points)Evaluate $\int \frac{16}{x^3\sqrt{x^2-4}}\,dx$, $x > 2$.

Let $x = 2\sec\theta \Rightarrow dx = 2\sec\theta\tan\theta\,d\theta$ (2 pts)

$0 \leq \theta < \frac{\pi}{2}$



(1pt)

$$I = \int \frac{16}{x^3\sqrt{x^2-4}}\,dx$$

$$= \int \frac{(16)(2\sec\theta\tan\theta)}{8\sec^3\theta\,(2\tan\theta)}\,d\theta \quad (1\text{ pt})$$

$$= \int \frac{2}{\sec^2\theta}\,d\theta = \int 2\cos^2\theta\,d\theta \quad (2\text{ pts})$$

$$= \int 2\left(\frac{1+\cos 2\theta}{2}\right)d\theta \quad (1\text{ pt})$$

$$= \int \left(1+\cos(2\theta)\right)d\theta$$

$$= \theta + \frac{1}{2}\sin(2\theta) + C \quad (1\text{ pt})$$

$$= \theta + \sin\theta\cos\theta + C \quad (1\text{ pt})$$

$$= \sec^{-1}\left(\frac{x}{2}\right) + \frac{\sqrt{x^2-4}}{x}\cdot\frac{2}{x} + C$$

$$= \sec^{-1}\left(\frac{x}{2}\right) + \frac{2\sqrt{x^2-4}}{x^2} + C \quad (1\text{ pt})$$

5. A)(5 points) Write out the form of the partial fraction decomposition of

$$F(x) = \frac{x^4+7}{x(x-1)^3(x^2+9)^2}.$$

(Do not determine the numerical values of the coefficients.)

$$\frac{x^4+7}{x(x-1)^3(x^2+9)^2} = \overbrace{\frac{A}{x}}^{(1\,pt)} + \overbrace{\frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}}^{(2\,pts)}$$

$$+ \underbrace{\frac{Ex+F}{x^2+9} + \frac{Gx+H}{(x^2+9)^2}}_{(2\,pts)}.$$

B) (7 points) Evaluate $\int \frac{\sin x}{\cos^2 x - 16}\,dx$.

let $u = \cos x \Rightarrow du = -\sin x\,dx$ (2 pts)

$$I = \int \frac{\sin x}{\cos^2 x - 16}\,dx$$

$$= -\int \frac{1}{u^2-16}\,du \quad (1\,pt)$$

(2 pts)

$$= -\int \frac{1}{(u-4)(u+4)}\,du = -\int\left(\frac{\frac{1}{8}}{u-4} - \frac{\frac{1}{8}}{u+4}\right)du$$

$$= \int\left(\frac{\frac{1}{8}}{u+4} - \frac{\frac{1}{8}}{u-4}\right)du = \frac{1}{8}\ln\left|\frac{u+4}{u-4}\right| + C \quad (1\,pt)$$

$$= \frac{1}{8}\ln\left|\frac{\cos x+4}{\cos x-4}\right| + C \quad (1\,pt)$$

6. (8 points) Evaluate $\int_4^6 (x^2+1)\,\mathrm{sech}(\ln x)\,dx$.

$$I = \int_4^6 (x^2+1)\left(\frac{2}{e^{\ln x}+e^{-\ln x}}\right)dx \quad (2 \text{ pts})$$

$$= \int_4^6 (x^2+1)\left(\frac{2}{x+\frac{1}{x}}\right)dx \quad (2\text{pts})$$

$$= \int_4^6 (x^2+1)\left(\frac{2x}{x^2+1}\right)dx \quad (1 \text{ pt})$$

$$= \int_4^6 2x\,dx = x^2\Big|_4^6 = 36-16$$

$$(1 \text{ pt}) \quad (1 \text{ pt}) \quad = 20 \quad (1 \text{ pt})$$

7. (6 points) Find the slope of the tangent line to the curve $y = \ln(\cosh x)$ at $x = \ln 2$. (Write your answer as fraction $\frac{p}{q}$)

$$y' = \frac{\sinh x}{\cosh x} = \tanh x \quad (2 \text{ pts})$$

$$\text{slope} = y'\Big|_{x=\ln 2}$$

$$\therefore \text{slope} = \tanh(\ln 2)$$

$$= \frac{e^{\ln 2} - e^{-\ln 2}}{e^{\ln 2} + e^{-\ln 2}} \quad (2 \text{ pts})$$

$$= \frac{2-\frac{1}{2}}{2+\frac{1}{2}} = \frac{4-1}{4+1} = \frac{3}{5} \quad (2\text{pts})$$

8. Test the following improper integrals for convergence. If the integral is convergent, then find its value.

A) (6 points) $\int_0^1 \frac{1}{(1+x)\sqrt{x}}\,dx$

(1 pt)

$f(x) = \frac{1}{(1+x)\sqrt{x}}$ is not continuous at $x=0 \in [0,1]$

$I = \int_0^1 \frac{1}{(1+x)\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{(1+x)\sqrt{x}}\,dx$ (1 pt)

let $u=\sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$, $x=u^2$ (1 pt)

$\Rightarrow I = \lim_{t\to 0^+} \int_{\sqrt{t}}^1 \frac{2}{1+u^2}\,du$ (1 pt)

$= \lim_{t\to 0^+} 2\tan^{-1}u \Big|_{\sqrt{t}}^1 = 2 \lim_{t\to 0^+}\left(\frac{\pi}{4} - \tan^{-1}\sqrt{t}\right) = \frac{\pi}{2}.$

(1 pt)

So, the integral converges to $\frac{\pi}{2}$

(1 pt)

B) (6 points) $\int_1^\infty \frac{\ln x}{x}\,dx$

$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx$ (2 pts)

$= \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Big|_1^t$ (2 pts)

$= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$ (1 pt)

So, the integral is divergent. (1 pt)

9. Determine whether each of the following sequences converges or diverges. If a sequence converges, then find its limit.

A) (5 points)The sequence $\frac{(1)(3)}{2}, \frac{(2)(4)}{3}, \frac{(3)(5)}{4}, \frac{(4)(6)}{5}, \ldots$

$$a_n = \frac{n(n+2)}{n+1} \quad (2\text{ pts})$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n}{n+1} = \lim_{n\to\infty} \frac{n^2(1+\frac{2}{n})}{n(1+\frac{1}{n})} \quad (1\text{ pt})$$

$$= \lim_{n\to\infty} \frac{n(1+\frac{2}{n})}{1+\frac{1}{n}} = \infty \quad (1\text{ pt})$$

So, the sequence diverges. (1pt)

B) (5 points) The sequence whose nth term is

$$a_n = \frac{1}{n}\sin\left(\frac{n\pi}{4}\right)$$

$$-\frac{1}{n} \leq \frac{1}{n}\sin\left(\frac{n\pi}{4}\right) \leq \frac{1}{n} \quad (2\text{ pts})$$

$$\text{Since } \lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{+1}{n} \quad (1\text{ pt})$$

$\therefore$ by Sandwich theorem (1 pt)

$$\lim_{n\to\infty} a_n = 0 \quad (1\text{ pt})$$

10. (10 points)Use integration by parts to evaluate $\int_0^1 \left(\int_1^x e^{t^2}\, dt \right) dx$.

Let $u = \int_1^x e^{t^2}\, dt \qquad dv = dx$ (3 Pts)

$du = e^{x^2} dx \qquad v = x$

$$\int_0^1 \left(\int_1^x e^{t^2} dt \right) dx = \left(x \int_1^x e^{t^2} dt \right)\Big|_0^1 - \int_0^1 x e^{x^2} dx \quad \text{(2pts)}$$

$$= \underbrace{(0 - 0)}_{\text{(2pts)}} - \underbrace{\tfrac{1}{2} e^{x^2}\Big|_0^1}_{\text{(1Pt)}}$$

$$= -\tfrac{1}{2}(e - 1)$$

$$= \frac{1-e}{2} = \tfrac{1}{2}(1-e) \quad \text{(2 pts)}$$