King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 102- Calculus II
Exam II
Term (113)

Tuesday, July 17, 2012 **Time Allowed: 2 hours**

Name: Solution Key ID Number: __________
Section Number: __________ Serial Number: __________
Class Time: __________ Instructor's Name: __________

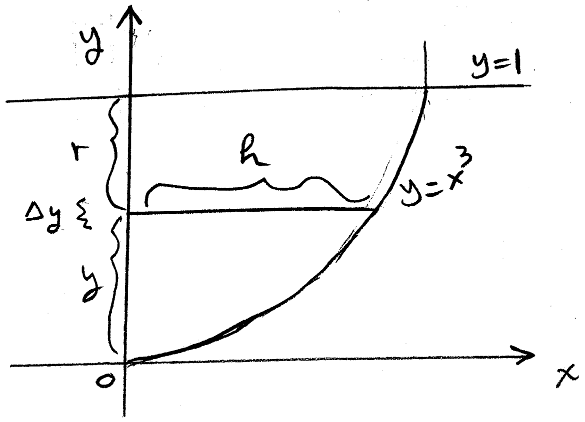
Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 10 pages of problems (Total of 9 Problems)

	Points	Maximum Points
Page 1		10
Page 2		8
Page 3		14
Page 4		8
Page 5		14
Page 6		12
Page 7		15
Page 8		7
Page 9		6
Page 10		6
Total		100

1. Using cylindrical shells, find the volume of the solid generated when the region enclosed by $y = x^3$, $y = 1$, and $x = 0$ is rotated about $y = 1$.

Typical shell:



$$3\text{pts}\left\{\begin{array}{l} \text{thickness} = \Delta y \\ \text{height} = x = \sqrt[3]{y} \\ \text{radius} = 1-y \end{array}\right.$$

2pts + 1pt $\left\{ V = \int_0^1 2\pi(1-y)\, y^{1/3}\, dy = 2\pi \int_0^1 \left(y^{1/3} - y^{4/3}\right) dy \right.$

2 pts $\left\{ = 2\pi \left[\frac{3}{4} y^{4/3} - \frac{3}{7} y^{7/3}\right]_0^1 \right.$

1 pt $\left\{ = 2\pi\left(\frac{3}{4} - \frac{3}{7}\right) = 2\pi\left(\frac{21-12}{28}\right) = \frac{9\pi}{14} \right.$

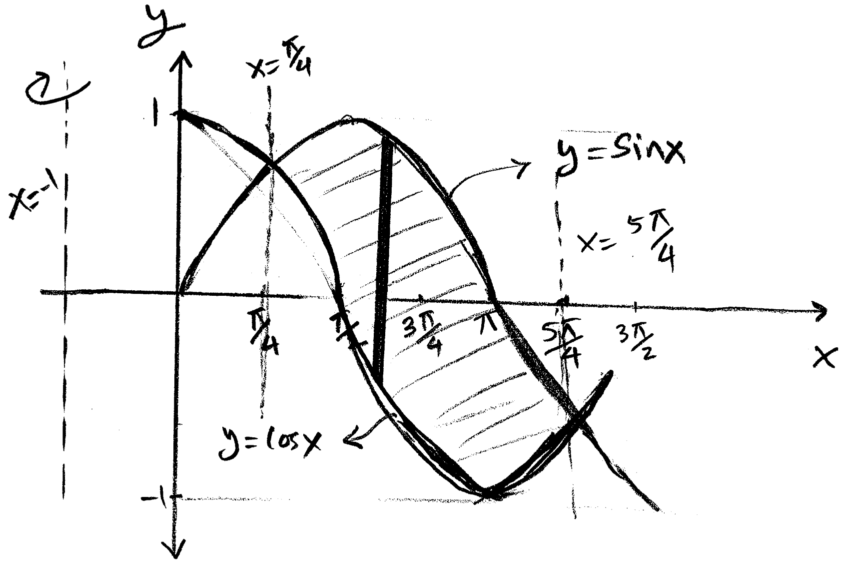
1 pt

2. A solid is obtained when the region enclosed by

$$x = \frac{\pi}{4},\ x = \frac{5\pi}{4},\ y = \cos x,\ \text{and}\ y = \sin x$$

is rotated about $x = -1$.

(a) Sketch the region.



4 pts

(b) Set up, but do not evaluate, an integral for the volume of the solid obtained.

Using cylindrical shells

thickness = Δx

radius = $1 + x$

height = $\sin x - \cos x$

2 pts

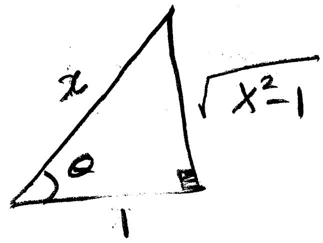
$$V = \int_{\pi/4}^{5\pi/4} 2\pi (1+x)(\sin x - \cos x)\, dx$$

2 pts

14

3. Evaluate the integral $\overbrace{\int \frac{2}{x^3\sqrt{x^2-1}}dx}^{I}$, $x > 1$.

4pts $\left\{ \begin{array}{l} \text{Let } x = \sec\theta, \; 0 \leq \theta < \frac{\pi}{2} \\ \Rightarrow dx = \sec\theta \tan\theta \, d\theta \end{array} \right.$



$$\therefore \; I = \int \frac{2 \cancel{\sec\theta} \tan\theta \, d\theta}{\cancel{\sec^3}^{2}\theta \sqrt{\sec^2\theta - 1}}$$

$$= \int \frac{2 \cancel{\tan\theta} \, d\theta}{\sec^2\theta \, \cancel{\tan\theta}}$$

4 pts

$$= \int 2\cos^2\theta \, d\theta$$

$$= \int (\cos 2\theta + 1) \, d\theta$$

$$= \frac{1}{2}\sin 2\theta + \theta + C$$

$$= \sin\theta\cos\theta + \theta + C$$

4 pts

$$= \frac{\sqrt{x^2-1}}{x^2} + \sec^{-1}x + C$$

2 pts

4. Determine whether the integral $\int_0^1 \frac{dx}{(x-1)^{2/3}}$ is convergent or divergent.

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{t\to 1^-} \int_0^t (x-1)^{-2/3}\,dx \quad \} \text{ 3pts}$$

$$= \lim_{t\to 1^-} 3(x-1)^{1/3}\Big|_0^t$$

$$= \lim_{t\to 1^-} 3(t-1)^{1/3} + 3 \quad \} \text{ 3 pts}$$

$$= 3 \quad \} \text{ 1 pt}$$

Therefore, the integral is convergent. } 1 pt

5. Evaluate the indefinite integral $\int \frac{dx}{\sin x + \tan x} = I$

4pts
$$I = \int \frac{\frac{2\,dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{2t}{1-t^2}} = \int \frac{\frac{2\,dt}{1+t^2}}{\frac{2t(1-t^2)+2t(1+t^2)}{(1+t^2)(1-t^2)}}$$

4pts
$$t = \tan\left(\tfrac{x}{2}\right), \quad -\pi < x < \pi$$
$$dx = \frac{2\,dt}{1+t^2}$$
$$\sin x = \frac{2t}{1+t^2}$$
$$\cos x = \frac{1-t^2}{1+t^2}$$
$$\tan x = \frac{2t}{1-t^2}$$

2pts
$$= \int \frac{\cancel{2}(1-t^2)\,dt}{\cancel{2}t\,(1-t^2+1+t^2)}$$
$$= \int \frac{1-t^2}{2t}\,dt$$

2pts
$$= \frac{1}{2}\int \left(\frac{1}{t} - t\right) dt$$
$$= \frac{1}{2}\ln|t| - \frac{1}{4}t^2 + C$$

2pts
$$= \frac{1}{2}\ln\left|\tan\left(\tfrac{x}{2}\right)\right| - \frac{1}{4}\tan^2\left(\tfrac{x}{2}\right) + C$$

6. Evaluate the integral $\int \frac{9x^3 - 3x + 1}{x^3 - x^2} dx$ $= I$

$$x^3 - x^2 \overline{)9x^3 - 3x + 1}^{\;9}$$
$$\ominus 9x^3 \oplus 9x^2$$
$$9x^2 - 3x + 1$$
1 pt

2pts $\left\{ I = \int 9\,dx + \int \frac{9x^2 - 3x + 1}{x^3 - x^2} dx \right.$

Now, $\frac{9x^2 - 3x + 1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$ } 1 pt

$9x^2 - 3x + 1 = Ax(x-1) + B(x-1) + Cx^2$

Let $x = 1 \Rightarrow \boxed{7 = C}$

$x = 0 \Rightarrow 1 = -B \Rightarrow \boxed{B = -1}$

$x = -1 \Rightarrow 13 = 2A + 2 + 7 \Rightarrow 2A = 4 \Rightarrow \boxed{A = 2}$ } 3pts

$\therefore I = \int 9\,dx + \int \frac{2}{x} dx - \int \frac{dx}{x^2} + \int \frac{7}{x-1} dx$ } 2pt

$= 9x + 2\ln|x| + \frac{1}{x} + 7\ln|x-1| + C$ } 3pts

15

6 7. (a) Evaluate the integral $\int \cos^3 x dx$

3 pts $\left\{ \int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1-\sin^2 x)\cos x \, dx \right.$

3 pts $\left\{ \begin{array}{l} u = \sin x \\ du = \cos x dx \end{array} \quad \int (1-u^2)\, du = u - \frac{1}{3}u^3 + C \right.$

$= \sin x - \frac{1}{3}\sin^3 x + C$

9 (b) Evaluate the integral $\int x \cos^3 x dx$ $= I$

2 pts $\left\{ \text{by parts} \quad \begin{array}{ll} u = x, & dv = \cos^3 x \, dx \\ du = dx & v = \sin x - \frac{1}{3}\sin^3 x \end{array} \right.$

4 pts $\left\{ \therefore I = \int x \cos^3 x \, dx = x\sin x - \frac{1}{3}x\sin^3 x - \int \sin x \, dx + \frac{1}{3}\int \sin^3 x \, dx \right.$

we find $\int \sin^3 x \, dx = \int (1-\cos^2 x)\sin x \, dx \overset{y=\cos x}{\underset{dy=-\sin x dx}{=}} -\int (1-y^2)\, dy$

$= \frac{1}{3}y^3 - y = \frac{1}{3}\cos^3 x - \cos x$.

$\therefore$ 3 pts $\left\{ I = x\sin x - \frac{1}{3}x\sin^3 x + \cos x + \frac{1}{9}\cos^3 x - \frac{1}{3}\cos x + C \right.$

$= x\sin x - \frac{1}{3}x\sin^3 x + \frac{1}{9}\cos^3 x + \frac{2}{3}\cos x + C$

7

8. (a) Evaluate the integral $\int \frac{\tan^{-1} x}{x^2} dx$

Using integration by parts:

1pt. $\begin{cases} u = \tan^{-1}x \quad , \quad dv = \frac{1}{x^2}dx \\ du = \frac{dx}{1+x^2} \quad , \quad v = \frac{-1}{x} \end{cases}$

1pt. $\therefore \int \frac{\tan^{-1}x}{x^2} dx = -\frac{\tan^{-1}x}{x} + \int \frac{dx}{x(1+x^2)}$

Let $I_2 = \int \frac{dx}{x(1+x^2)}$

3pt. The partial fraction decomposition of the integrand has the form

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$

$$\therefore 1 = A(1+x^2) + (Bx+C)x = Ax^2 + A + Bx^2 + Cx$$

$$= (A+B)x^2 + Cx + A$$

$$\Rightarrow \boxed{A=1}, \quad \boxed{C=0}, \quad \boxed{B=-1}$$

$$\therefore I_2 = \int \frac{dx}{x} - \int \frac{x}{1+x^2}dx \quad \text{1 pt.}$$

$$= \ln|x| - \frac{1}{2}\ln(1+x^2) + c'$$

$$\therefore \int \frac{\tan^{-1}x}{x^2} dx = -\frac{\tan^{-1}x}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2) + c \quad \text{1 pt}$$

6 8. (b) Evaluate $\int_1^\infty \frac{\tan^{-1} x}{x^2} dx$ using part (a)

1 pt. $\left\{ \int_1^\infty \frac{\tan^{-1} x}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{\tan^{-1} x}{x^2} dx \right.$

1 pt. $\left\{ = \lim_{t\to\infty} \left[-\frac{\tan^{-1} x}{x^2} + \ln\frac{x}{\sqrt{1+x^2}} \right]_1^t \right.$ (Note that $x > 1$)

1 pt. $\left\{ \begin{aligned} &= \lim_{t\to\infty} \left[\left(-\frac{\tan^{-1} t}{t^2} + \ln\frac{t}{\sqrt{1+t^2}} \right) - \left(-\frac{\pi}{4} + \ln\frac{1}{\sqrt{2}} \right) \right] \\ &= \lim_{t\to\infty} -\frac{\tan^{-1} t}{t^2} + \lim_{t\to\infty} \ln\left(\frac{t}{\sqrt{1+t^2}}\right) + \frac{\pi}{4} + \frac{1}{2}\ln 2 \end{aligned} \right.$

Now,

1 pt. $\left\{ * \lim_{t\to\infty} \frac{-\tan^{-1} t}{t^2} = \lim_{t\to\infty} -\tan^{-1} t \cdot \lim_{t\to\infty} \frac{1}{t^2} = -\frac{\pi}{2} \cdot 0 = 0 \right.$

* To find $\lim_{t\to\infty} \ln\left(\frac{t}{\sqrt{1+t^2}}\right)$, Let $y = \frac{t}{\sqrt{1+t^2}}$, then

1 pt. $\left\{ \begin{aligned} &\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{t}{\sqrt{1+t^2}} = \lim_{t\to\infty} \frac{1}{\sqrt{\frac{1}{t^2}+1}} = 1 \\ &\therefore \lim_{t\to\infty} \ln\left(\frac{t}{\sqrt{1+t^2}}\right) = \lim_{y\to 1} \ln y = 0 \end{aligned} \right.$

<u>Hence,</u> $\left. \begin{aligned} \int_1^\infty \frac{\tan^{-1} x}{x^2} dx &= \frac{\pi}{4} + \frac{1}{2}\ln 2 \\ &= \frac{1}{4}(\pi + \ln 4) \end{aligned} \right\}$ 1 pt.

9. Let f_{ave} be the average value of the function $f(x) = \sin(x^2)$ on $\left[\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2} + h\right]$ where $h > 0$. Find $\lim_{h\to 0} f_{ave}$.

2pts $\left\{ \; f_{ave} = \frac{1}{h} \int_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}+h} \sin(x^2)\,dx \right.$

1pt $\left\{ \; \lim_{h\to 0} f_{ave} = \lim_{h\to 0} \frac{\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\frac{\pi}{2}}+h} \sin(x^2)\,dx}{h} \right.$ $\left(\frac{0}{0}\right)$

1pt $\left\{ \; \stackrel{\text{L'Hospital}}{=} \lim_{h\to 0} \sin\left[\left(\sqrt{\frac{\pi}{2}}+h\right)^2\right] \right.$

1pt $\left\{ \; = \sin\frac{\pi}{4} \right.$

1pt $\left\{ \; = \frac{\sqrt{2}}{2} \right.$