King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 102- Calculus II
Exam II
2012-2013 (Term 121)

Thursday, Nov. 15, 2012 **Allowed Time: 2 hours**

Name: ______________________ KEY

ID Number: ____________

Section Number: ___________ **Serial Number:** ___________

Instructions:

1. Write neatly and eligibly. You may lose points for messy work.
2. **Show all your work. No points for answers without justification.**
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 10 different problems (5 pages + cover page)

Question #	Grade	Maximum Points
1		7
2		14
3		16
4		7
5		10
6		10
7		8
8		10
9		10
10		8
Total		100

1. **[7 points]** Write out the form of the partial fraction decomposition of the expression

$$\frac{x^2}{(x-\sqrt{3})(x-3)^3(x^2+4)(x^2+x+2)^2}$$

DO NOT EVALUATE the numerical values of the coefficients.

$$\frac{A}{x-\sqrt{3}} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D}{(x-3)^3} + \frac{Ex+F}{x^2+4} + \frac{Gx+H}{x^2+x+2} + \frac{Ix+F}{(x^2+x+2)^2}$$

1 point for each fraction

2. **[7+7 points]** Evaluate the following integrals:

(a) $\displaystyle\int \frac{2x^2+1}{2x+1}dx$

$$\begin{array}{r|l} & x-\frac{1}{2} \\ \hline 2x+1 & 2x^2+1 \\ & 2x^2+x \\ \hline & -x+1 \\ & -x-\frac{1}{2} \\ \hline & \frac{3}{2} \end{array}$$

$= \displaystyle\int \left[x - \frac{1}{2} + \frac{\frac{3}{2}}{2x+1}\right] dx$ 3

$= \frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{2}\cdot\frac{1}{2}\ln|2x+1| + C$

$= \frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{4}\ln|2x+1| + C$

1 1 2

(b) $\displaystyle\int_0^{\sqrt[4]{\pi/4}} t^3\sec^4(t^4)\tan^4(t^4)dt = I$

Let $x = t^4$. Then $dx = 4t^3 dt$; $t=0 \Rightarrow x=0$; $t=\sqrt[4]{\pi/4} \Rightarrow x = \pi/4$

$I = \frac{1}{4}\displaystyle\int_0^{\pi/4} \sec^4 x \tan^4 x \, dx$ 2

$= \frac{1}{4}\displaystyle\int_0^{\pi/4} \sec^2 x \cdot \tan^4 x \cdot \sec^2 x \, dx$ 1

$= \frac{1}{4}\displaystyle\int_0^{\pi/4} (1+\tan^2 x)\cdot \tan^4 x \cdot \sec^2 x \, dx$ 1

Let $u = \tan x$. Then $du = \sec^2 x \, dx$; $x=0 \Rightarrow u=0$; $x=\frac{\pi}{4} \Rightarrow u=1$

$= \frac{1}{4}\displaystyle\int_0^1 (1+u^2)u^4 \, du$ 2

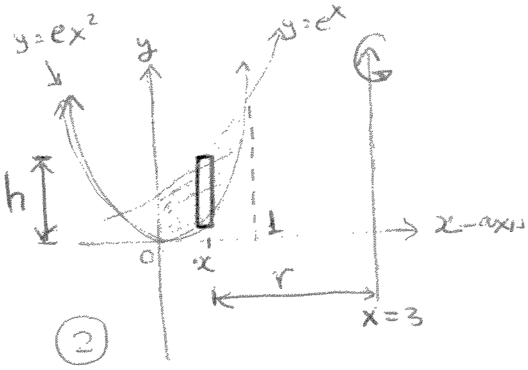
$= \dfrac{3}{35}$ 1

3. [**8+8 points**] Using the method of cylindrical shells, **Set up, but DO NOT EVALUATE**, an integral for the volume of the solid obtained by rotating

(a) The region in the **first quadrant** bounded by the curves $y = e^x$ and $y = ex^2$ about the line $x = 3$.
[Sketch the region and a typical rectangle]

$$V = \int_0^1 2\pi \cdot (3-x) \cdot (e^x - ex^2)\, dx$$

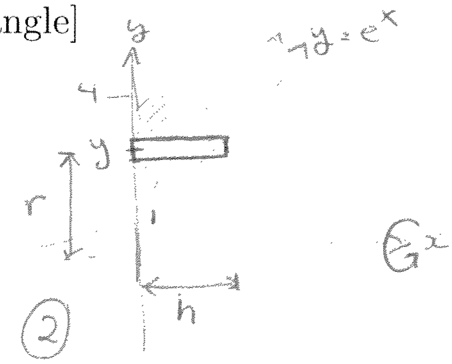
2 2 2



(b) The region bounded by the curves $y = e^x, y = 4$, and $x = 0$ about the x-axis.[Sketch the region and a typical rectangle]

$$V = \int_1^4 2\pi \cdot y \cdot \ln y \cdot dy$$

2 2 2



4. [**7 points**] Let $f(x) = 3x^2 - 2ax + b$, where $a \neq 1$. Find the value of b if the average value of f over the interval $[1, a]$ is 4.

$$f_{ave} = \frac{1}{a-1}\int_1^a (3x^2 - 2ax + b)\, dx \quad 2$$

$$= \frac{1}{a-1} \cdot x^3 - ax^2 + bx \Big|_1^a \quad 1$$

$$= \frac{1}{a-1}(ab - 1 + a - b) \quad 1$$

$$= \frac{1}{a-1}[b(a-1) + (a-1)]$$

$$= \frac{1}{a-1} \cdot (a-1)(b+1)$$

$$4 = b+1 \quad (a \neq 1) \implies b = 3 \quad 3$$

5. **[10 points]** Evaluate $\int (x \ln x)^2 dx$

$\int (x \ln x)^2 dx = \int x^2 (\ln x)^2 dx$ (2) $u = (\ln x)^2$ $dv = x^2 dx$

$du = \frac{2 \ln x}{x} dx$ $v = \frac{1}{3}x^3$

$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int x^2 \ln x \, dx$ (4) (2) $u = \ln x$ $dv = x^2 dx$

$du = \frac{1}{x} dx$ $v = \frac{1}{3}x^3$

(2)

$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left[\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx\right]$ (4)

$= \frac{1}{3}x^3(\ln)^2 - \frac{2}{3}\left[\frac{1}{3}x^3 \ln x - \frac{1}{3}\cdot\frac{1}{3}x^3\right] + C$ (1)

$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C$ (1)

6. **[10 points]** Evaluate $\int \frac{x^2}{(4 - x^2)^{3/2}} dx$

Let $x = 2\sin\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. (2)

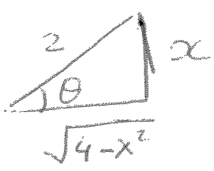
Then $\cdot\ dx = 2\cos\theta \, d\theta$ (1)

$\cdot\ (4 - x^2)^{3/2} = (4 - 4\sin^2\theta)^{3/2} = (4\cos^2\theta)^{3/2} = 8\cos^3\theta$ (1) $\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$\int \frac{x^2}{(4-x^2)^{3/2}} dx = \int \frac{4\sin^2\theta}{8\cos^3\theta} \cdot 2\cos\theta \, d\theta$

$= \int \tan^2\theta \, d\theta$ (1)

$= \int (\sec^2\theta - 1) \, d\theta$ (1)



$= \tan\theta - \theta + C$ (2)

$= \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$ (2)

7. **[8 points]** Evaluate $\int 8\cos^4 t\,dt$

$$8\cos^4 t = 8\left[\frac{1+\cos(2t)}{2}\right]^2 \quad 1$$
$$= 2\left[1+2\cos(2t)+\cos^2(2t)\right] \quad 1$$
$$= 2\left[1+2\cos(2t)+\frac{1+\cos(4t)}{2}\right] \quad 2$$
$$= 3+4\cos(2t)+\cos(4t) \quad 1$$

$$\int 8\cos^4 t\,dt = \int[3+4\cos(2t)+\cos(4t)]\,dt$$
$$= \underset{1}{3t} + \underset{1}{2\sin(2t)} + \underset{1}{\tfrac{1}{4}\sin(4t)} + C$$

8. **[10 points]** Evaluate $\displaystyle\int \frac{15}{(x-2)(x^2+1)}\,dx = I$

$$\frac{15}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} \quad 2$$

$$\Rightarrow 15 = A(x^2+1) + (Bx+C)(x-2)$$
$$= (A+B)x^2 + (C-2B)x + (A-2C)$$

$$\Rightarrow \begin{cases} A+B=0 \\ C-2B=0 \\ A-2C=15 \end{cases}$$ Solving we get $\underset{1}{A=3}$, $\underset{1}{B=-3}$, $\underset{1}{C=-6}$

$$I = \int \frac{3}{x-2} + \frac{-3x-6}{x^2+1}\,dx$$
$$= \int \frac{3}{x-2} - \frac{3x}{x^2+1} - \frac{6}{x^2+1}\,dx \quad 2$$
$$= \underset{1}{3\ln|x-2|} - \underset{1}{\tfrac{3}{2}\ln(x^2+1)} - \underset{1}{6\tan^{-1}x} + C$$

9. **[10 points]** Use the substitution $t = \tan(x/2)$, $-\pi < x < \pi$, to evaluate the integral $\displaystyle\int \frac{\sqrt{3}}{4 - 2\cos x}dx$

$t = \tan\left(\frac{x}{2}\right) \Longrightarrow \cos t = \frac{1-t^2}{1+t^2}$ (2) & $dx = \frac{2}{1+t^2}\,dt$ (2)

$$\int \frac{\sqrt{3}}{4-2\cos x}\,dx = \int \frac{\sqrt{3}}{4 - 2\cdot\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2\sqrt{3}}{4(1+t^2)-2(1-t^2)}\,dt \quad (1)$$

$$= \int \frac{2\sqrt{3}}{2+6t^2}\,dt$$

$$= \int \frac{\sqrt{3}}{1+3t^2}\,dt \quad (1) \qquad , \ u = \sqrt{3}\,t \Rightarrow du = \sqrt{3}\,dt \quad (1)$$

$$= \int \frac{1}{1+u^2}\,du \quad (1)$$

$$= \tan^{-1} u + C \qquad = \tan^{-1}(\sqrt{3}\,t) + C \quad (1)$$

$$= \tan^{-1}\left(\sqrt{3}\tan\left(\tfrac{x}{2}\right)\right) + C \quad (1)$$

10. **[8 points]** Evaluate $\displaystyle\int \sqrt{\frac{1}{x} + \frac{1}{\sqrt{x}}}\,dx$

Let $t = \sqrt{x}$. (2) Then $x = t^2$ (1) & $dx = 2t\,dt$ (1)

$$\int \sqrt{\frac{1}{x}+\frac{1}{\sqrt{x}}}\,dx = \int \sqrt{\frac{1}{t^2}+\frac{1}{t}} \cdot 2t\,dt$$

$$= \int \sqrt{\frac{1+t}{t^2}} \cdot 2t\,dt \quad (1)$$

$$= \int \frac{\sqrt{1+t}}{t} \cdot 2t\,dt$$

$$= 2 \cdot \int \sqrt{1+t}\,dt \quad (1)$$

$$= 2 \cdot \frac{2}{3}(1+t)^{3/2} + C \quad (1)$$

$$= \frac{4}{3}(1+\sqrt{x})^{3/2} + C \quad (1)$$