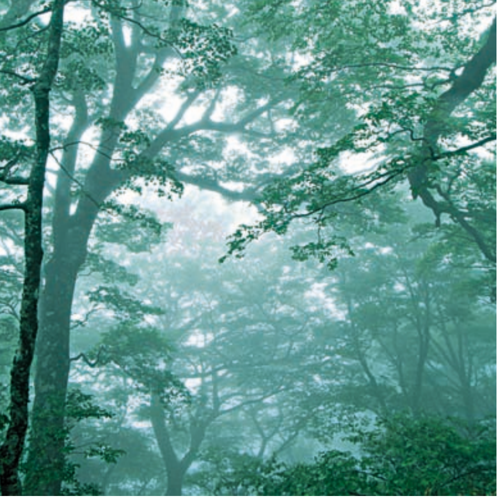


# 3

# DIFFERENTIATION

**OVERVIEW** In the beginning of Chapter 2 we discussed how to determine the slope of a curve at a point and how to measure the rate at which a function changes. Now that we have studied limits, we can define these ideas precisely and see that both are interpretations of the *derivative* of a function at a point. We then extend this concept from a single point to the *derivative function*, and we develop rules for finding this derivative function easily, without having to calculate any limits directly. These rules are used to find derivatives of most of the common functions reviewed in Chapter 1, as well as various combinations of them. The derivative is one of the key ideas in calculus, and we use it to solve a wide range of problems involving tangents and rates of change.

## 3.1 Tangents and the Derivative at a Point

In this section we define the slope and tangent to a curve at a point, and the derivative of a function at a point. Later in the chapter we interpret the derivative as the instantaneous rate of change of a function, and apply this interpretation to the study of certain types of motion.

### Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant through $P$ and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 3.1). If the limit exists, we call it the slope of the curve at $P$ and define the tangent at $P$ to be the line through $P$ having this slope.

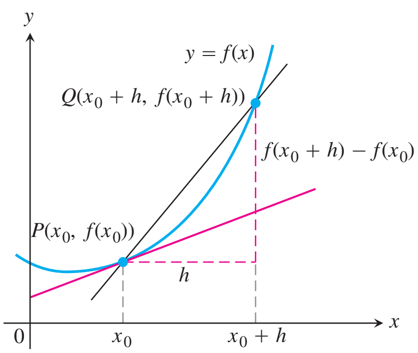



**FIGURE 3.1** The slope of the tangent line at $P$ is $\lim_{h\to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$.

> **DEFINITIONS** The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number
>
> $$m = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$
>
> The **tangent line** to the curve at $P$ is the line through $P$ with this slope.

In Section 2.1, Example 3, we applied these definitions to find the slope of the parabola $f(x) = x^2$ at the point $P(2, 4)$ and the tangent line to the parabola at $P$. Let's look at another example.

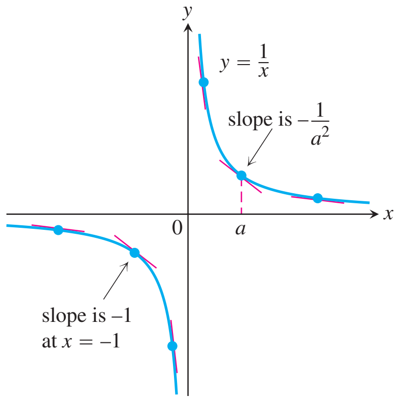



**FIGURE 3.2** The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away (Example 1).

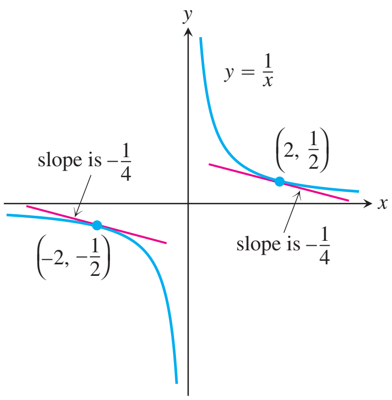



**FIGURE 3.3** The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 1).

## EXAMPLE 1

**(a)** Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

**(b)** Where does the slope equal $-1/4$?

**(c)** What happens to the tangent to the curve at the point $(a, 1/a)$ as $a$ changes?

### Solution

**(a)** Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h \to 0} \frac{-h}{ha(a+h)} = \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h \to 0}$" before each fraction until the stage where we could evaluate the limit by substituting $h = 0$. The number $a$ may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 3.2).

**(b)** The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 3.3).

**(c)** The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 3.2). We see this situation again as $a \to 0^-$. As $a$ moves away from the origin in either direction, the slope approaches 0 and the tangent levels off to become horizontal. ■

## Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of $f$ at $x_0$ with increment $h$**. If the difference quotient has a limit as $h$ approaches zero, that limit is given a special name and notation.

**DEFINITION** The **derivative of a function $f$ at a point $x_0$**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

If we interpret the difference quotient as the slope of a secant line, then the derivative gives the slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$. Exercise 31 shows that the derivative of the linear function $f(x) = mx + b$ at any point $x_0$ is simply the slope of the line, so

$$f'(x_0) = m,$$

which is consistent with our definition of slope.

If we interpret the difference quotient as an average rate of change (Section 2.1), the derivative gives the function's instantaneous rate of change with respect to $x$ at the point $x = x_0$. We study this interpretation in Section 3.4.

**EXAMPLE 2** In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first $t$ sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's *exact* speed at this time?

**Solution** We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds, for $h > 0$, was found to be

$$\frac{f(1 + h) - f(1)}{h} = \frac{16(1 + h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's speed at the instant $t = 1$ is then

$$\lim_{h \to 0} 16(h + 2) = 16(0 + 2) = 32 \text{ ft/sec}.$$

Our original estimate of 32 ft/sec in Section 2.1 was right. ■

## Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, and the derivative of a function at a point. All of these ideas refer to the same limit.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to $x$ at $x = x_0$
4. The derivative $f'(x_0)$ at a point

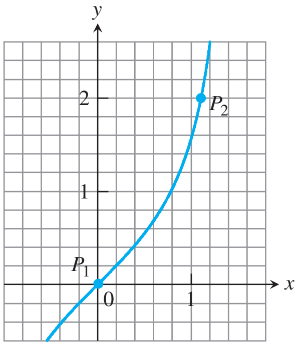
In the next sections, we allow the point $x_0$ to vary across the domain of the function $f$.

# Exercises 3.1

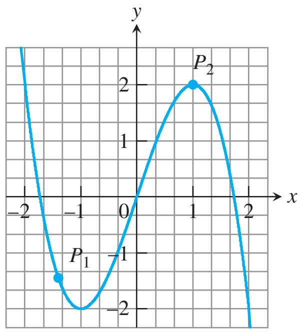
## Slopes and Tangent Lines

In Exercises 1–4, use the grid and a straight edge to make a rough estimate of the slope of the curve (in $y$-units per $x$-unit) at the points $P_1$ and $P_2$.

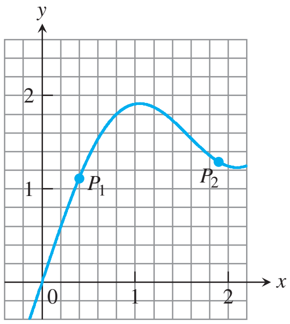
**1.**



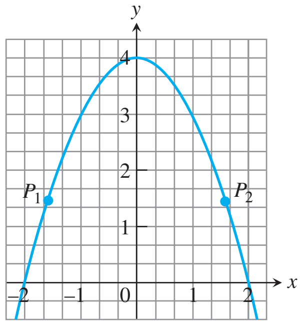

**2.**




**3.**




**4.**




In Exercises 5–10, find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together.

**5.** $y = 4 - x^2, \quad (-1, 3)$

**6.** $y = (x - 1)^2 + 1, \quad (1, 1)$

**7.** $y = 2\sqrt{x}, \quad (1, 2)$

**8.** $y = \dfrac{1}{x^2}, \quad (-1, 1)$

**9.** $y = x^3, \quad (-2, -8)$

**10.** $y = \dfrac{1}{x^3}, \quad \left(-2, -\dfrac{1}{8}\right)$

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

**11.** $f(x) = x^2 + 1, \quad (2, 5)$

**12.** $f(x) = x - 2x^2, \quad (1, -1)$

**13.** $g(x) = \dfrac{x}{x - 2}, \quad (3, 3)$

**14.** $g(x) = \dfrac{8}{x^2}, \quad (2, 2)$

**15.** $h(t) = t^3, \quad (2, 8)$

**16.** $h(t) = t^3 + 3t, \quad (1, 4)$

**17.** $f(x) = \sqrt{x}, \quad (4, 2)$

**18.** $f(x) = \sqrt{x + 1}, \quad (8, 3)$

In Exercises 19–22, find the slope of the curve at the point indicated.

**19.** $y = 5x^2, \quad x = -1$

**20.** $y = 1 - x^2, \quad x = 2$

**21.** $y = \dfrac{1}{x - 1}, \quad x = 3$

**22.** $y = \dfrac{x - 1}{x + 1}, \quad x = 0$

## Tangent Lines with Specified Slopes

At what points do the graphs of the functions in Exercises 23 and 24 have horizontal tangents?

**23.** $f(x) = x^2 + 4x - 1$

**24.** $g(x) = x^3 - 3x$

**25.** Find equations of all lines having slope $-1$ that are tangent to the curve $y = 1/(x - 1)$.

**26.** Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

## Rates of Change

**27. Object dropped from a tower** An object is dropped from the top of a 100-m-high tower. Its height above ground after $t$ sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

**28. Speed of a rocket** At $t$ sec after liftoff, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing 10 sec after liftoff?

**29. Circle's changing area** What is the rate of change of the area of a circle $(A = \pi r^2)$ with respect to the radius when the radius is $r = 3$?

**30. Ball's changing volume** What is the rate of change of the volume of a ball $(V = (4/3)\pi r^3)$ with respect to the radius when the radius is $r = 2$?

**31.** Show that the line $y = mx + b$ is its own tangent line at any point $(x_0, mx_0 + b)$.

**32.** Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = 4$.

## Testing for Tangents

**33.** Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

**34.** Does the graph of

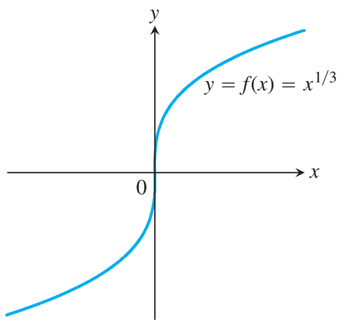
$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

## Vertical Tangents

We say that a continuous curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if $\lim_{h\to 0} (f(x_0 + h) - f(x_0))/h = \infty$ or $-\infty$. For example, $y = x^{1/3}$ has a vertical tangent at $x = 0$ (see accompanying figure):

$$\lim_{h\to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{1/3} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty.$$

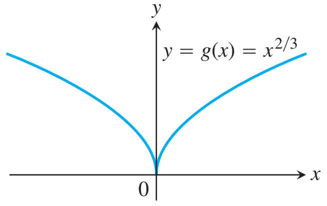



VERTICAL TANGENT AT ORIGIN

However, $y = x^{2/3}$ has *no* vertical tangent at $x = 0$ (see next figure):

$$\lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \lim_{h\to 0}\frac{h^{2/3}-0}{h} = \lim_{h\to 0}\frac{1}{h^{1/3}}$$

does not exist, because the limit is $\infty$ from the right and $-\infty$ from the left.




NO VERTICAL TANGENT AT ORIGIN

**35.** Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent at the origin? Give reasons for your answer.

**36.** Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases}$$

have a vertical tangent at the point (0, 1)? Give reasons for your answer.

**T** Graph the curves in Exercises 37–46.

**a.** Where do the graphs appear to have vertical tangents?

**b.** Confirm your findings in part (a) with limit calculations. But before you do, read the introduction to Exercises 35 and 36.

**37.** $y = x^{2/5}$

**38.** $y = x^{4/5}$

**39.** $y = x^{1/5}$

**40.** $y = x^{3/5}$

**41.** $y = 4x^{2/5} - 2x$

**42.** $y = x^{5/3} - 5x^{2/3}$

**43.** $y = x^{2/3} - (x-1)^{1/3}$

**44.** $y = x^{1/3} + (x-1)^{1/3}$

**45.** $y = \begin{cases} -\sqrt{|x|}, & x \le 0 \\ \sqrt{x}, & x > 0 \end{cases}$

**46.** $y = \sqrt{|4-x|}$

## COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 47–50:

**a.** Plot $y = f(x)$ over the interval $(x_0 - 1/2) \le x \le (x_0 + 3)$.

**b.** Holding $x_0$ fixed, the difference quotient

$$q(h) = \frac{f(x_0+h)-f(x_0)}{h}$$

at $x_0$ becomes a function of the step size $h$. Enter this function into your CAS workspace.

**c.** Find the limit of $q$ as $h \to 0$.

**d.** Define the secant lines $y = f(x_0) + q \cdot (x - x_0)$ for $h = 3, 2,$ and 1. Graph them together with $f$ and the tangent line over the interval in part (a).

**47.** $f(x) = x^3 + 2x, \quad x_0 = 0$

**48.** $f(x) = x + \dfrac{5}{x}, \quad x_0 = 1$

**49.** $f(x) = x + \sin(2x), \quad x_0 = \pi/2$

**50.** $f(x) = \cos x + 4\sin(2x), \quad x_0 = \pi$

# 3.2 The Derivative as a Function

HISTORICAL ESSAY

The Derivative

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from $f$ by considering the limit at each point $x$ in the domain of $f$.

**DEFINITION** The **derivative** of the function $f(x)$ with respect to the variable $x$ is the function $f'$ whose value at $x$ is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

provided the limit exists.

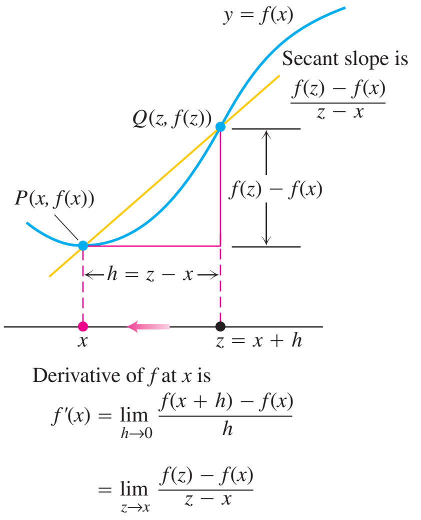



**FIGURE 3.4** Two forms for the difference quotient.

We use the notation $f(x)$ in the definition to emphasize the independent variable $x$ with respect to which the derivative function $f'(x)$ is being defined. The domain of $f'$ is the set of points in the domain of $f$ for which the limit exists, which means that the domain may be the same as or smaller than the domain of $f$. If $f'$ exists at a particular $x$, we say that $f$ is **differentiable (has a derivative) at $x$**. If $f'$ exists at every point in the domain of $f$, we call $f$ **differentiable**.

If we write $z = x + h$, then $h = z - x$ and $h$ approaches 0 if and only if $z$ approaches $x$. Therefore, an equivalent definition of the derivative is as follows (see Figure 3.4). This formula is sometimes more convenient to use when finding a derivative function.

**Alternative Formula for the Derivative**

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

## Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$. Example 1 of Section 3.1 illustrated the differentiation process for the function $y = 1/x$ when $x = a$. For $x$ representing any point in the domain, we get the formula

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

**Derivative of the Reciprocal Function**

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad x \neq 0$$

Here are two more examples in which we allow $x$ to be any point in the domain of $f$.

**EXAMPLE 1** Differentiate $f(x) = \dfrac{x}{x - 1}$.

**Solution** We use the definition of derivative, which requires us to calculate $f(x + h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} && \text{Definition} \\
&= \lim_{h \to 0} \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} && \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd} \\
&= \lim_{h \to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} && \text{Simplify.} \\
&= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. && \text{Cancel } h \neq 0.
\end{aligned}$$

■

**Derivative of the Square Root Function**

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$

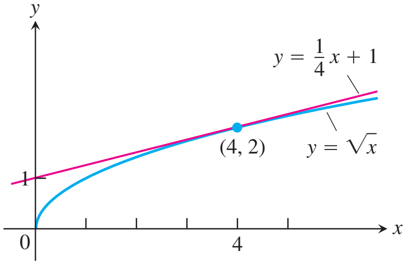



**FIGURE 3.5** The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

## EXAMPLE 2

**(a)** Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.
**(b)** Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

**Solution**

**(a)** We use the alternative formula to calculate $f'$:

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{\left(\sqrt{z} - \sqrt{x}\right)\left(\sqrt{z} + \sqrt{x}\right)} \\ &= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

**(b)** The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope 1/4 (Figure 3.5):

$$y = 2 + \frac{1}{4}(x - 4)$$
$$y = \frac{1}{4}x + 1.$$ ■

## Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is $x$ and the dependent variable is $y$. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

The symbols $d/dx$ and $D$ indicate the operation of differentiation. We read $dy/dx$ as "the derivative of $y$ with respect to $x$," and $df/dx$ and $(d/dx)f(x)$ as "the derivative of $f$ with respect to $x$." The "prime" notations $y'$ and $f'$ come from notations that Newton used for derivatives. The $d/dx$ notations are similar to those used by Leibniz. The symbol $dy/dx$ should not be regarded as a ratio (until we introduce the idea of "differentials" in Section 3.11).

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

For instance, in Example 2

$$f'(4) = \left.\frac{d}{dx}\sqrt{x}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

## Graphing the Derivative

We can often make a reasonable plot of the derivative of $y = f(x)$ by estimating the slopes on the graph of $f$. That is, we plot the points $(x, f'(x))$ in the $xy$-plane and connect them with a smooth curve, which represents $y = f'(x)$.

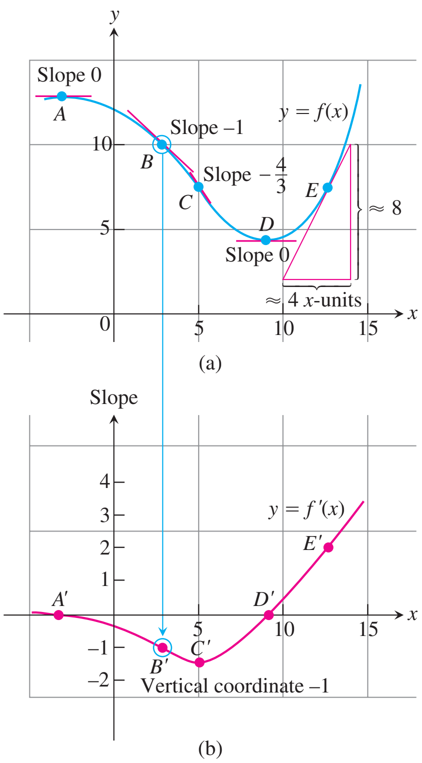



**FIGURE 3.6** We made the graph of $y = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of $B'$ is the slope at $B$ and so on. The slope at $E$ is approximately $8/4 = 2$. In (b) we see that the rate of change of $f$ is negative for $x$ between $A'$ and $D'$; the rate of change is positive for $x$ to the right of $D'$.

**EXAMPLE 3** Graph the derivative of the function $y = f(x)$ in Figure 3.6a.

**Solution** We sketch the tangents to the graph of $f$ at frequent intervals and use their slopes to estimate the values of $f'(x)$ at these points. We plot the corresponding $(x, f'(x))$ pairs and connect them with a smooth curve as sketched in Figure 3.6b. ■

What can we learn from the graph of $y = f'(x)$? At a glance we can see

1. where the rate of change of $f$ is positive, negative, or zero;
2. the rough size of the growth rate at any $x$ and its size in relation to the size of $f(x)$;
3. where the rate of change itself is increasing or decreasing.

## Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior $(a, b)$ and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } a$$

$$\lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } b$$

exist at the endpoints (Figure 3.7).

Right-hand and left-hand derivatives may be defined at any point of a function's domain. Because of Theorem 6, Section 2.4, a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

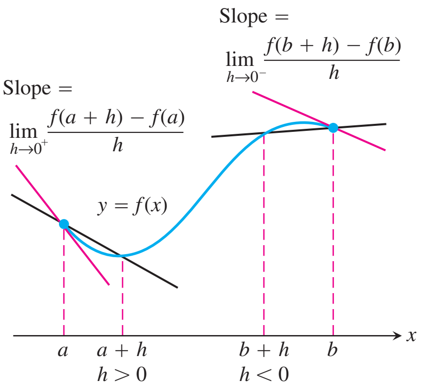



**FIGURE 3.7** Derivatives at endpoints are one-sided limits.

**EXAMPLE 4** Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

**Solution** From Section 3.1, the derivative of $y = mx + b$ is the slope $m$. Thus, to the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \qquad \frac{d}{dx}(mx + b) = m,\ |x| = x$$

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \qquad |x| = -x$$

(Figure 3.8). There is no derivative at the origin because the one-sided derivatives differ there:

$$\text{Right-hand derivative of } |x| \text{ at zero} = \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h}$$

$$= \lim_{h\to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0$$

$$= \lim_{h\to 0^+} 1 = 1$$

$$\text{Left-hand derivative of } |x| \text{ at zero} = \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h}$$

$$= \lim_{h\to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0$$

$$= \lim_{h\to 0^-} -1 = -1.$$

■

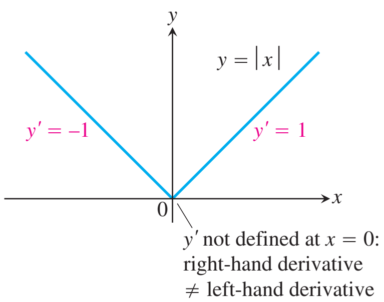



**FIGURE 3.8** The function $y = |x|$ is not differentiable at the origin where the graph has a "corner" (Example 4).

**EXAMPLE 5** In Example 2 we found that for $x > 0$,

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

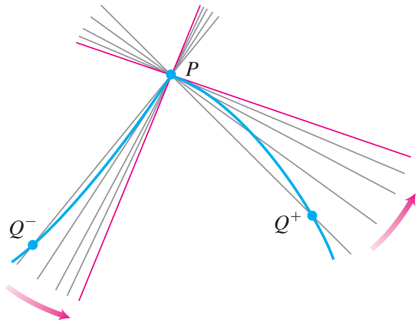
We apply the definition to examine if the derivative exists at $x = 0$:

$$\lim_{h\to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty.$$

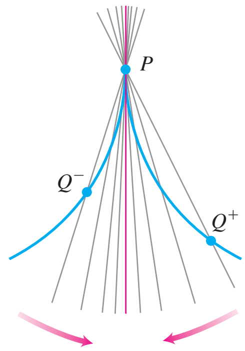
Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points $(h, \sqrt{h})$ on a graph of $y = \sqrt{x}$ approach $\infty$, the graph has a *vertical tangent* at the origin. (See Figure 1.17 on page 9). ■

## When Does a Function *Not* Have a Derivative at a Point?

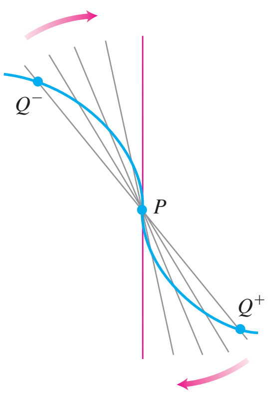
A function has a derivative at a point $x_0$ if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point $Q$ on the graph approach a finite limit as $Q$ approaches $P$. Whenever the secants fail to take up a limiting position or become vertical as $Q$ approaches $P$, the derivative does not exist. Thus differentiability is a "smoothness" condition on the graph of $f$. A function can fail to have a derivative at a point for many reasons, including the existence of points where the graph has



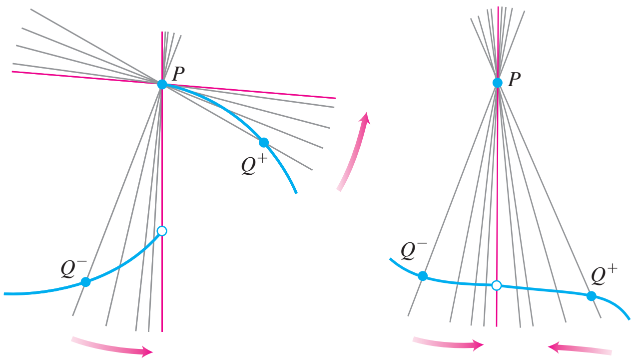

**1.** a *corner*, where the one-sided derivatives differ.




**2.** a *cusp*, where the slope of $PQ$ approaches $\infty$ from one side and $-\infty$ from the other.




**3.** a *vertical tangent*, where the slope of $PQ$ approaches $\infty$ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).




**4.** a *discontinuity* (two examples shown).

Another case in which the derivative may fail to exist occurs when the function's slope is oscillating rapidly near $P$, as with $f(x) = \sin(1/x)$ near the origin, where it is discontinuous (see Figure 2.31).

## Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

> **THEOREM 1—Differentiability Implies Continuity** If $f$ has a derivative at $x = c$, then $f$ is continuous at $x = c$.

**Proof** Given that $f'(c)$ exists, we must show that $\lim_{x \to c} f(x) = f(c)$, or equivalently, that $\lim_{h \to 0} f(c + h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c+h) &= f(c) + (f(c+h) - f(c)) \\ &= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h. \end{aligned}$$

Now take limits as $h \to 0$. By Theorem 1 of Section 2.2,

$$\begin{aligned} \lim_{h \to 0} f(c+h) &= \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h \to 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) + 0 \\ &= f(c). \end{aligned}$$

■

Similar arguments with one-sided limits show that if $f$ has a derivative from one side (right or left) at $x = c$ then $f$ is continuous from that side at $x = c$.

Theorem 1 says that if a function has a discontinuity at a point (for instance, a jump discontinuity), then it cannot be differentiable there. The greatest integer function $y = \lfloor x \rfloor$ fails to be differentiable at every integer $x = n$ (Example 4, Section 2.5).

Caution The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw in Example 4.

# Exercises 3.2

### Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2$; $f'(-3), f'(0), f'(1)$
2. $F(x) = (x - 1)^2 + 1$; $F'(-1), F'(0), F'(2)$
3. $g(t) = \frac{1}{t^2}$; $g'(-1), g'(2), g'(\sqrt{3})$
4. $k(z) = \frac{1 - z}{2z}$; $k'(-1), k'(1), k'(\sqrt{2})$
5. $p(\theta) = \sqrt{3\theta}$; $p'(1), p'(3), p'(2/3)$
6. $r(s) = \sqrt{2s + 1}$; $r'(0), r'(1), r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

7. $\frac{dy}{dx}$ if $y = 2x^3$
8. $\frac{dr}{ds}$ if $r = s^3 - 2s^2 + 3$
9. $\frac{ds}{dt}$ if $s = \frac{t}{2t + 1}$
10. $\frac{dv}{dt}$ if $v = t - \frac{1}{t}$
11. $\frac{dp}{dq}$ if $p = \frac{1}{\sqrt{q + 1}}$
12. $\frac{dz}{dw}$ if $z = \frac{1}{\sqrt{3w - 2}}$

## Slopes and Tangent Lines

In Exercises 13–16, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

**13.** $f(x) = x + \dfrac{9}{x}, \quad x = -3$

**14.** $k(x) = \dfrac{1}{2 + x}, \quad x = 2$

**15.** $s = t^3 - t^2, \quad t = -1$

**16.** $y = \dfrac{x + 3}{1 - x}, \quad x = -2$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

**17.** $y = f(x) = \dfrac{8}{\sqrt{x - 2}}, \quad (x, y) = (6, 4)$

**18.** $w = g(z) = 1 + \sqrt{4 - z}, \quad (z, w) = (3, 2)$

In Exercises 19–22, find the values of the derivatives.

**19.** $\left.\dfrac{ds}{dt}\right|_{t=-1}$ if $s = 1 - 3t^2$

**20.** $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{3}}$ if $y = 1 - \dfrac{1}{x}$

**21.** $\left.\dfrac{dr}{d\theta}\right|_{\theta=0}$ if $r = \dfrac{2}{\sqrt{4 - \theta}}$

**22.** $\left.\dfrac{dw}{dz}\right|_{z=4}$ if $w = z + \sqrt{z}$

## Using the Alternative Formula for Derivatives

Use the formula

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

to find the derivative of the functions in Exercises 23–26.

**23.** $f(x) = \dfrac{1}{x + 2}$

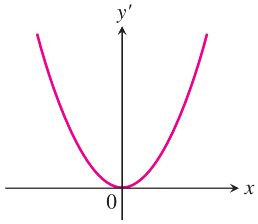
**24.** $f(x) = x^2 - 3x + 4$

**25.** $g(x) = \dfrac{x}{x - 1}$

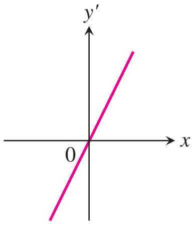
**26.** $g(x) = 1 + \sqrt{x}$

## Graphs

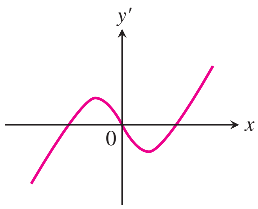
Match the functions graphed in Exercises 27–30 with the derivatives graphed in the accompanying figures (a)–(d).



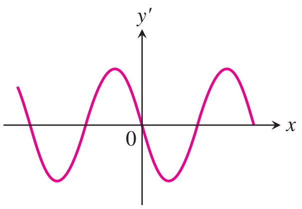
(a)



(b)

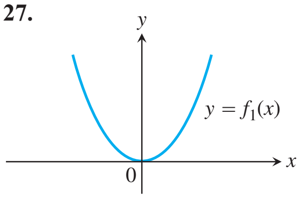


(c)

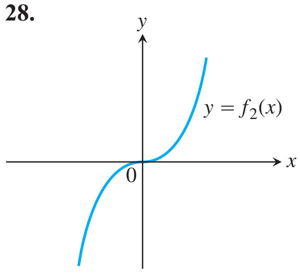


(d)

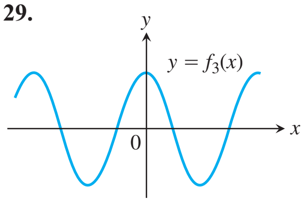
**27.**



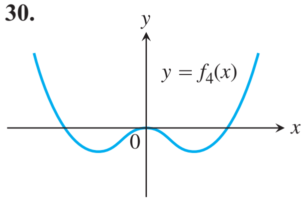

**28.**



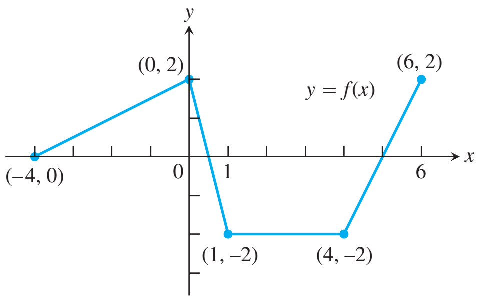

**29.**




**30.**




**31. a.** The graph in the accompanying figure is made of line segments joined end to end. At which points of the interval $[-4, 6]$ is $f'$ not defined? Give reasons for your answer.




**b.** Graph the derivative of $f$.

The graph should show a step function.

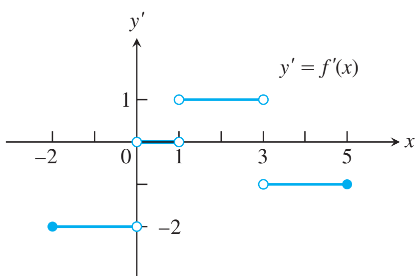
**32. Recovering a function from its derivative**

**a.** Use the following information to graph the function $f$ over the closed interval $[-2, 5]$.

**i)** The graph of $f$ is made of closed line segments joined end to end.

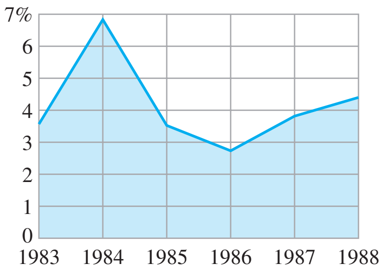
**ii)** The graph starts at the point $(-2, 3)$.

**iii)** The derivative of $f$ is the step function in the figure shown here.



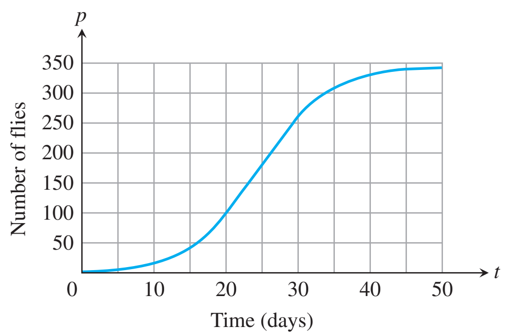

**b.** Repeat part (a) assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

**33. Growth in the economy** The graph in the accompanying figure shows the average annual percentage change $y = f(t)$ in the U.S. gross national product (GNP) for the years 1983–1988. Graph $dy/dt$ (where defined).



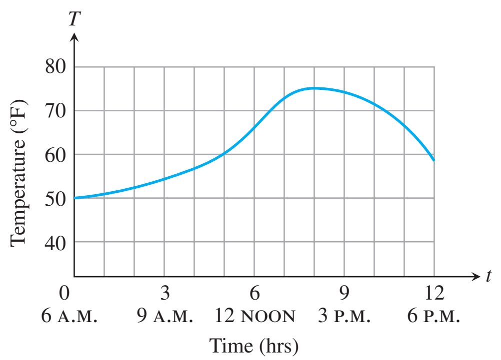

**34. Fruit flies** (*Continuation of Example 4, Section 2.1.*) Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

**a.** Use the graphical technique of Example 3 to graph the derivative of the fruit fly population. The graph of the population is reproduced here.




**b.** During what days does the population seem to be increasing fastest? Slowest?

**35. Temperature** The given graph shows the temperature $T$ in °F at Davis, CA, on April 18, 2008, between 6 A.M. and 6 P.M.



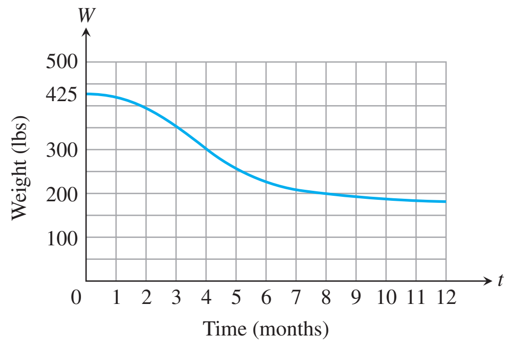

**a.** Estimate the rate of temperature change at the times

**i)** 7 A.M. **ii)** 9 A.M. **iii)** 2 P.M. **iv)** 4 P.M.

**b.** At what time does the temperature increase most rapidly? Decrease most rapidly? What is the rate for each of those times?

**c.** Use the graphical technique of Example 3 to graph the derivative of temperature $T$ versus time $t$.

**36. Weight loss** Jared Fogle, also known as the "Subway Sandwich Guy," weighed 425 lb in 1997 before losing more than 240 lb in 12 months (http://en.wikipedia.org/wiki/Jared_Fogle). A chart showing his possible dramatic weight loss is given in the accompanying figure.




**a.** Estimate Jared's rate of weight loss when

**i)** $t = 1$ **ii)** $t = 4$ **iii)** $t = 11$

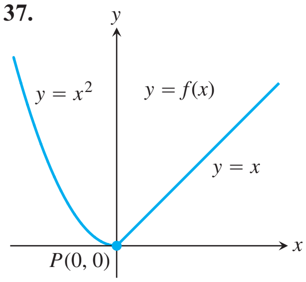
**b.** When does Jared lose weight most rapidly and what is this rate of weight loss?

**c.** Use the graphical technique of Example 3 to graph the derivative of weight $W$.

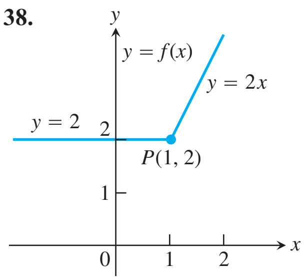
## One-Sided Derivatives

Compute the right-hand and left-hand derivatives as limits to show that the functions in Exercises 37–40 are not differentiable at the point $P$.

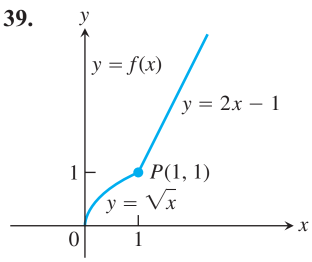
**37.**



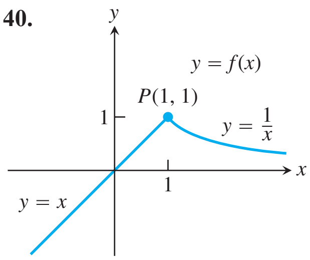

**38.**




**39.**




**40.**




In Exercises 41 and 42, determine if the piecewise defined function is differentiable at the origin.

**41.** $f(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$

**42.** $g(x) = \begin{cases} x^{2/3}, & x \geq 0 \\ x^{1/3}, & x < 0 \end{cases}$

## Differentiability and Continuity on an Interval

Each figure in Exercises 43–48 shows the graph of a function over a closed interval $D$. At what domain points does the function appear to be

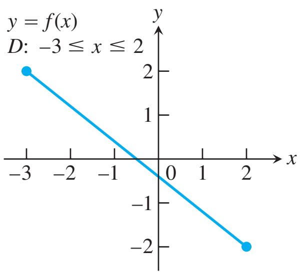
**a.** differentiable?

**b.** continuous but not differentiable?

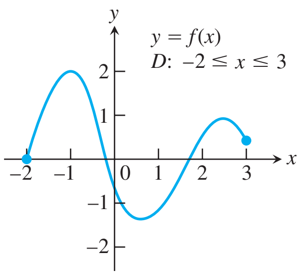
**c.** neither continuous nor differentiable?

Give reasons for your answers.

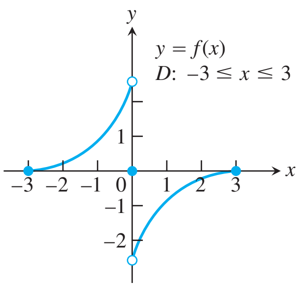
**43.**



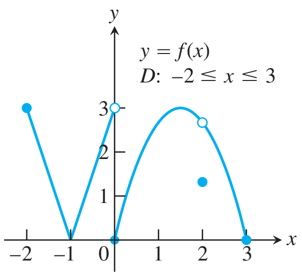

**44.**



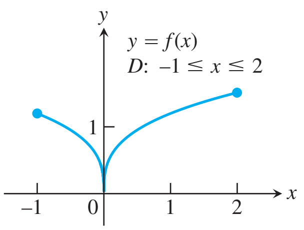

**45.**



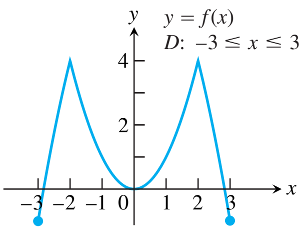

**46.**




**47.**




**48.**




## Theory and Examples

In Exercises 49–52,

**a.** Find the derivative $f'(x)$ of the given function $y = f(x)$.

**b.** Graph $y = f(x)$ and $y = f'(x)$ side by side using separate sets of coordinate axes, and answer the following questions.

**c.** For what values of $x$, if any, is $f'$ positive? Zero? Negative?

**d.** Over what intervals of $x$-values, if any, does the function $y = f(x)$ increase as $x$ increases? Decrease as $x$ increases? How is this related to what you found in part (c)? (We will say more about this relationship in Section 4.3.)

**49.** $y = -x^2$

**50.** $y = -1/x$

**51.** $y = x^3/3$

**52.** $y = x^4/4$

**53. Tangent to a parabola** Does the parabola $y = 2x^2 - 13x + 5$ have a tangent whose slope is $-1$? If so, find an equation for the line and the point of tangency. If not, why not?

**54. Tangent to $y = \sqrt{x}$** Does any tangent to the curve $y = \sqrt{x}$ cross the $x$-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

**55. Derivative of $-f$** Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.

**56. Derivative of multiples** Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

**57. Limit of a quotient** Suppose that functions $g(t)$ and $h(t)$ are defined for all values of $t$ and $g(0) = h(0) = 0$. Can $\lim_{t\to 0}(g(t))/(h(t))$ exist? If it does exist, must it equal zero? Give reasons for your answers.

**58. a.** Let $f(x)$ be a function satisfying $|f(x)| \le x^2$ for $-1 \le x \le 1$. Show that $f$ is differentiable at $x = 0$ and find $f'(0)$.

**b.** Show that

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

T **59.** Graph $y = 1/(2\sqrt{x})$ in a window that has $0 \le x \le 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

T **60.** Graph $y = 3x^2$ in a window that has $-2 \le x \le 2$, $0 \le y \le 3$. Then, on the same screen, graph

$$y = \frac{(x+h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

**61. Derivative of $y = |x|$** Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

T **62. Weierstrass's nowhere differentiable continuous function** The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty} (2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + (2/3)^1 \cos(9\pi x) + (2/3)^2 \cos(9^2\pi x) + (2/3)^3 \cos(9^3\pi x) + \cdots + (2/3)^7 \cos(9^7\pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

## COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 63–68.

**a.** Plot $y = f(x)$ to see that function's global behavior.

**b.** Define the difference quotient $q$ at a general point $x$, with general step size $h$.

**c.** Take the limit as $h \to 0$. What formula does this give?

**d.** Substitute the value $x = x_0$ and plot the function $y = f(x)$ together with its tangent line at that point.

**e.** Substitute various values for $x$ larger and smaller than $x_0$ into the formula obtained in part (c). Do the numbers make sense with your picture?

**f.** Graph the formula obtained in part (c). What does it mean when its values are negative? Zero? Positive? Does this make sense with your plot from part (a)? Give reasons for your answer.

**63.** $f(x) = x^3 + x^2 - x, \quad x_0 = 1$

**64.** $f(x) = x^{1/3} + x^{2/3}, \quad x_0 = 1$

**65.** $f(x) = \dfrac{4x}{x^2 + 1}, \quad x_0 = 2$

**66.** $f(x) = \dfrac{x - 1}{3x^2 + 1}, \quad x_0 = -1$

**67.** $f(x) = \sin 2x, \quad x_0 = \pi/2$

**68.** $f(x) = x^2 \cos x, \quad x_0 = \pi/4$

# 3.3 Differentiation Rules

This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, exponential functions, rational functions, and certain combinations of them, simply and directly, without having to take limits each time.

## Powers, Multiples, Sums, and Differences

A simple rule of differentiation is that the derivative of every constant function is zero.

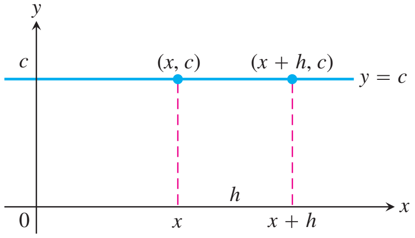



**FIGURE 3.9** The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

**Derivative of a Constant Function**

If $f$ has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

**Proof** We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value $c$ (Figure 3.9). At every value of $x$, we find that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0. \qquad \blacksquare$$

From Section 3.1, we know that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \text{or} \quad \frac{d}{dx}\left(x^{-1}\right) = -x^{-2}.$$

From Example 2 of the last section we also know that

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}, \quad \text{or} \quad \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

These two examples illustrate a general rule for differentiating a power $x^n$. We first prove the rule when $n$ is a positive integer.

**Power Rule for Positive Integers:**

If $n$ is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

HISTORICAL BIOGRAPHY

Richard Courant
(1888–1972)

**Proof of the Positive Integer Power Rule** The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then from the alternative formula for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z\to x} \frac{f(z) - f(x)}{z - x} = \lim_{z\to x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z\to x}(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) \qquad n \text{ terms} \\ &= nx^{n-1}. \end{aligned}$$

■

The Power Rule is actually valid for all real numbers $n$. We have seen examples for a negative integer and fractional power, but $n$ could be an irrational number as well. To apply the Power Rule, we subtract 1 from the original exponent $n$ and multiply the result by $n$. Here we state the general version of the rule, but postpone its proof until Section 3.8.

**Power Rule (General Version)**

If $n$ is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all $x$ where the powers $x^n$ and $x^{n-1}$ are defined.

**EXAMPLE 1** Differentiate the following powers of $x$.

**(a)** $x^3$ **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\frac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

**Solution**

**(a)** $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$ **(b)** $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

**(c)** $\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$ **(d)** $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

**(e)** $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

**(f)** $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$ ■

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

**Derivative Constant Multiple Rule**

If $u$ is a differentiable function of $x$, and $c$ is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if $n$ is any real number, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}.$$

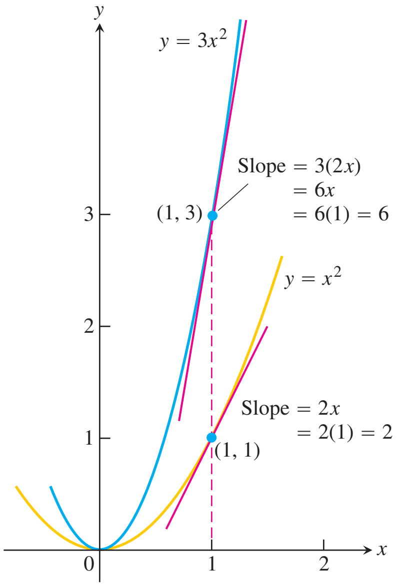



**FIGURE 3.10** The graphs of $y = x^2$ and $y = 3x^2$. Tripling the $y$-coordinate triples the slope (Example 2).

**Denoting Functions by $u$ and $v$**
The functions we are working with when we need a differentiation formula are likely to be denoted by letters like $f$ and $g$. We do not want to use these same letters when stating general differentiation rules, so we use letters like $u$ and $v$ instead that are not likely to be already in use.

**Proof**

$$\frac{d}{dx}cu = \lim_{h\to 0}\frac{cu(x+h) - cu(x)}{h} \qquad \text{Derivative definition with } f(x) = cu(x)$$

$$= c\lim_{h\to 0}\frac{u(x+h) - u(x)}{h} \qquad \text{Constant Multiple Limit Property}$$

$$= c\frac{du}{dx} \qquad u \text{ is differentiable.}$$ ■

## EXAMPLE 2

**(a)** The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each $y$-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.10).

**(b) Negative of a function**

The derivative of the negative of a differentiable function $u$ is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$ ■

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

> **Derivative Sum Rule**
> If $u$ and $v$ are differentiable functions of $x$, then their sum $u + v$ is differentiable at every point where $u$ and $v$ are both differentiable. At such points,
> $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

For example, if $y = x^4 + 12x$, then $y$ is the sum of $u(x) = x^4$ and $v(x) = 12x$. We then have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) = 4x^3 + 12.$$

**Proof** We apply the definition of the derivative to $f(x) = u(x) + v(x)$:

$$\frac{d}{dx}[u(x) + v(x)] = \lim_{h\to 0}\frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h}$$

$$= \lim_{h\to 0}\left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right]$$

$$= \lim_{h\to 0}\frac{u(x+h) - u(x)}{h} + \lim_{h\to 0}\frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.$$ ■

Combining the Sum Rule with the Constant Multiple Rule gives the **Difference Rule**, which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives:

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

The Sum Rule also extends to finite sums of more than two functions. If $u_1, u_2, \ldots, u_n$ are differentiable at $x$, then so is $u_1 + u_2 + \cdots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$

For instance, to see that the rule holds for three functions we compute

$$\frac{d}{dx}(u_1 + u_2 + u_3) = \frac{d}{dx}((u_1 + u_2) + u_3) = \frac{d}{dx}(u_1 + u_2) + \frac{du_3}{dx} = \frac{du_1}{dx} + \frac{du_2}{dx} + \frac{du_3}{dx}.$$

A proof by mathematical induction for any finite number of terms is given in Appendix 2.

**EXAMPLE 3** Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

**Solution**

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \qquad \text{Sum and Difference Rules}$$

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

■

We can differentiate any polynomial term by term, the way we differentiated the polynomial in Example 3. All polynomials are differentiable at all values of $x$.

**EXAMPLE 4** Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

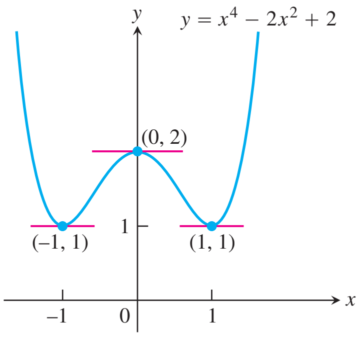



**FIGURE 3.11** The curve in Example 4 and its horizontal tangents.

**Solution** The horizontal tangents, if any, occur where the slope $dy/dx$ is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for $x$:

$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1. \end{aligned}$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1$, and $-1$. The corresponding points on the curve are $(0, 2)$, $(1, 1)$ and $(-1, 1)$. See Figure 3.11. We will see in Chapter 4 that finding the values of $x$ where the derivative of a function is equal to zero is an important and useful procedure. ■

## Derivatives of Exponential Functions

We briefly reviewed exponential functions in Section 1.5. When we apply the definition of the derivative to $f(x) = a^x$, we get

$$\begin{aligned}
\frac{d}{dx}(a^x) &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} && \text{Derivative definition} \\
&= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h} && a^{x+h} = a^x \cdot a^h \\
&= \lim_{h \to 0} a^x \cdot \frac{a^h - 1}{h} && \text{Factoring out } a^x \\
&= a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} && a^x \text{ is constant as } h \to 0. \\
&= \underbrace{\left(\lim_{h \to 0} \frac{a^h - 1}{h}\right)}_{\text{a fixed number } L} \cdot a^x. && (1)
\end{aligned}$$

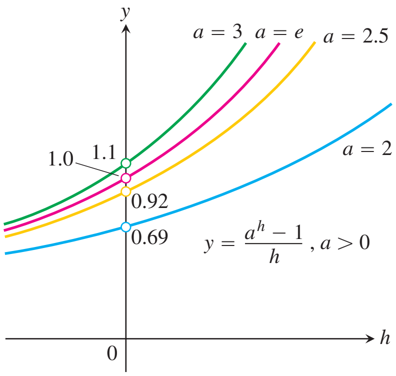



**FIGURE 3.12** The position of the curve $y = (a^h - 1)/h$, $a > 0$, varies continuously with $a$.

Thus we see that the derivative of $a^x$ is a constant multiple $L$ of $a^x$. The constant $L$ is a limit unlike any we have encountered before. Note, however, that it equals the derivative of $f(x) = a^x$ at $x = 0$:

$$f'(0) = \lim_{h \to 0} \frac{a^h - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h} = L.$$

The limit $L$ is therefore the slope of the graph of $f(x) = a^x$ where it crosses the $y$-axis. In Chapter 7, where we carefully develop the logarithmic and exponential functions, we prove that the limit $L$ exists and has the value ln $a$. For now we investigate values of $L$ by graphing the function $y = (a^h - 1)/h$ and studying its behavior as $h$ approaches 0.

Figure 3.12 shows the graphs of $y = (a^h - 1)/h$ for four different values of $a$. The limit $L$ is approximately 0.69 if $a = 2$, about 0.92 if $a = 2.5$, and about 1.1 if $a = 3$. It appears that the value of $L$ is 1 at some number $a$ chosen between 2.5 and 3. That number is given by $a = e \approx 2.718281828$. With this choice of base we obtain the natural exponential function $f(x) = e^x$ as in Section 1.5, and see that it satisfies the property

$$f'(0) = \lim_{h \to 0} \frac{e^h - 1}{h} = 1. \tag{2}$$

That the limit is 1 implies an important relationship between the natural exponential function $e^x$ and its derivative:

$$\frac{d}{dx}(e^x) = \lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) \cdot e^x \qquad \text{Eq. (1) with } a = e$$

$$= 1 \cdot e^x = e^x. \qquad \text{Eq. (2)}$$

Therefore the natural exponential function is its own derivative.

**Derivative of the Natural Exponential Function**

$$\frac{d}{dx}(e^x) = e^x$$

**EXAMPLE 5** Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

**Solution** Since the line passes through the origin, its equation is of the form $y = mx$, where $m$ is the slope. If it is tangent to the graph at the point $(a, e^a)$, the slope is $m = (e^a - 0)/(a - 0)$. The slope of the natural exponential at $x = a$ is $e^a$. Because these slopes are the same, we then have that $e^a = e^a/a$. It follows that $a = 1$ and $m = e$, so the equation of the tangent line is $y = ex$. See Figure 3.13. ■

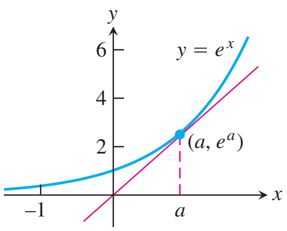



**FIGURE 3.13** The line through the origin is tangent to the graph of $y = e^x$ when $a = 1$ (Example 5).

We might ask if there are functions *other* than the natural exponential function that are their own derivatives. The answer is that the only functions that satisfy the property that $f'(x) = f(x)$ are functions that are constant multiples of the natural exponential function, $f(x) = c \cdot e^x$, $c$ any constant. We prove this fact in Section 7.2. Note from the Constant Multiple Rule that indeed

$$\frac{d}{dx}(c \cdot e^x) = c \cdot \frac{d}{dx}(e^x) = c \cdot e^x.$$

## Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while} \quad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

> **Derivative Product Rule**
>
> If $u$ and $v$ are differentiable at $x$, then so is their product $uv$, and
>
> $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The derivative of the product $uv$ is $u$ times the derivative of $v$ plus $v$ times the derivative of $u$. In *prime notation*, $(uv)' = uv' + vu'$. In function notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

**EXAMPLE 6** Find the derivative of **(a)** $y = \frac{1}{x}\left(x^2 + e^x\right)$, **(b)** $y = e^{2x}$.

**Solution**

**(a)** We apply the Product Rule with $u = 1/x$ and $v = x^2 + e^x$:

$$\frac{d}{dx}\left[\frac{1}{x}\left(x^2 + e^x\right)\right] = \frac{1}{x}\left(2x + e^x\right) + \left(x^2 + e^x\right)\left(-\frac{1}{x^2}\right)$$

$$= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2}$$

$$= 1 + (x - 1)\frac{e^x}{x^2}.$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, and $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

**(b)** $$\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(e^x) = 2e^x \cdot e^x = 2e^{2x}$$ ■

### Proof of the Derivative Product Rule

$$\frac{d}{dx}(uv) = \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of $u$ and $v$, we subtract and add $u(x + h)v(x)$ in the numerator:

$$\frac{d}{dx}(uv) = \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$= \lim_{h \to 0}\left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}\right]$$

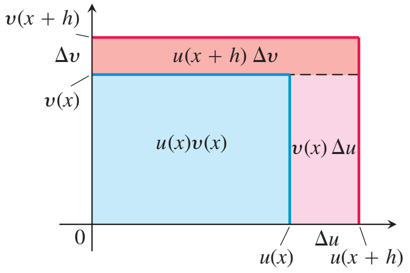
$$= \lim_{h \to 0} u(x+h) \cdot \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}.$$

As $h$ approaches zero, $u(x + h)$ approaches $u(x)$ because $u$, being differentiable at $x$, is continuous at $x$. The two fractions approach the values of $dv/dx$ at $x$ and $du/dx$ at $x$. In short,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$ ■

**Picturing the Product Rule**

Suppose $u(x)$ and $v(x)$ are positive and increase when $x$ increases, and $h > 0$.




Then the change in the product $uv$ is the difference in areas of the larger and smaller "squares," which is the sum of the upper and right-hand reddish-shaded rectangles. That is,

$$\Delta(uv) = u(x+h)v(x+h) - u(x)v(x)$$
$$= u(x+h)\Delta v + v(x)\Delta u.$$

Division by $h$ gives

$$\frac{\Delta(uv)}{h} = u(x+h)\frac{\Delta v}{h} + v(x)\frac{\Delta u}{h}.$$

The limit as $h \to 0^+$ gives the Product Rule.

**EXAMPLE 7** Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

**Solution**

**(a)** From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned}\frac{d}{dx}\left[(x^2+1)(x^3+3)\right] &= (x^2+1)(3x^2) + (x^3+3)(2x) \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\\ &= 3x^4 + 3x^2 + 2x^4 + 6x\\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

**(b)** This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for $y$ and differentiating the resulting polynomial:

$$\begin{aligned}y &= (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3\\ \frac{dy}{dx} &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

This is in agreement with our first calculation. ■

The derivative of the quotient of two functions is given by the Quotient Rule.

**Derivative Quotient Rule**

If $u$ and $v$ are differentiable at $x$ and if $v(x) \neq 0$, then the quotient $u/v$ is differentiable at $x$, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

**EXAMPLE 8** Find the derivative of **(a)** $y = \dfrac{t^2 - 1}{t^3 + 1}$, **(b)** $y = e^{-x}$.

**Solution**

**(a)** We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{(t^3+1)\cdot 2t - (t^2-1)\cdot 3t^2}{(t^3+1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}\\ &= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3+1)^2}\\ &= \frac{-t^4 + 3t^2 + 2t}{(t^3+1)^2}.\end{aligned}$$

**(b)** $\dfrac{d}{dx}(e^{-x}) = \dfrac{d}{dx}\left(\dfrac{1}{e^x}\right) = \dfrac{e^x\cdot 0 - 1\cdot e^x}{(e^x)^2} = \dfrac{-1}{e^x} = -e^{-x}$ ■

**Proof of the Derivative Quotient Rule**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h \to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}$$

$$= \lim_{h \to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of $u$ and $v$, we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h \to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)}$$

$$= \lim_{h \to 0} \frac{v(x)\dfrac{u(x+h) - u(x)}{h} - u(x)\dfrac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}.$$

Taking the limits in the numerator and denominator now gives the Quotient Rule. ■

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

**EXAMPLE 9** Rather than using the Quotient Rule to find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4},$$

expand the numerator and divide by $x^4$:

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$ ■

## Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If $f'$ is also differentiable, then we can differentiate $f'$ to get a new function of $x$ denoted by $f''$. So $f'' = (f')'$. The function $f''$ is called the **second derivative** of $f$ because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^{\,2} f(x).$$

The symbol $D^2$ means the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

Thus $D^2(x^6) = 30x^4$.

**How to Read the Symbols for Derivatives**

| | |
|---|---|
| $y'$ | "$y$ prime" |
| $y''$ | "$y$ double prime" |
| $\dfrac{d^2y}{dx^2}$ | "$d$ squared $y$ $dx$ squared" |
| $y'''$ | "$y$ triple prime" |
| $y^{(n)}$ | "$y$ super $n$" |
| $\dfrac{d^ny}{dx^n}$ | "$d$ to the $n$ of $y$ by $dx$ to the $n$" |
| $D^n$ | "$D$ to the $n$" |

If $y''$ is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of $y$ with respect to $x$. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting the ***n*th derivative** of $y$ with respect to $x$ for any positive integer $n$.

We can interpret the second derivative as the rate of change of the slope of the tangent to the graph of $y = f(x)$ at each point. You will see in the next chapter that the second derivative reveals whether the graph bends upward or downward from the tangent line as we move off the point of tangency. In the next section, we interpret both the second and third derivatives in terms of motion along a straight line.

**EXAMPLE 10** The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

The function has derivatives of all orders, the fifth and later derivatives all being zero. ■

## Exercises 3.3

### Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $s = 5t^3 - 3t^5$
4. $w = 3z^7 - 7z^3 + 21z^2$
5. $y = \dfrac{4x^3}{3} - x + 2e^x$
6. $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + \dfrac{x}{4}$
7. $w = 3z^{-2} - \dfrac{1}{z}$
8. $s = -2t^{-1} + \dfrac{4}{t^2}$
9. $y = 6x^2 - 10x - 5x^{-2}$
10. $y = 4 - 2x - x^{-3}$
11. $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$
12. $r = \dfrac{12}{\theta} - \dfrac{4}{\theta^3} + \dfrac{1}{\theta^4}$

In Exercises 13–16, find $y'$ **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$
14. $y = (2x + 3)(5x^2 - 4x)$
15. $y = (x^2 + 1)\left(x + 5 + \dfrac{1}{x}\right)$
16. $y = (1 + x^2)(x^{3/4} - x^{-3})$

Find the derivatives of the functions in Exercises 17–40.

17. $y = \dfrac{2x + 5}{3x - 2}$
18. $z = \dfrac{4 - 3x}{3x^2 + x}$
19. $g(x) = \dfrac{x^2 - 4}{x + 0.5}$
20. $f(t) = \dfrac{t^2 - 1}{t^2 + t - 2}$
21. $v = (1 - t)(1 + t^2)^{-1}$
22. $w = (2x - 7)^{-1}(x + 5)$
23. $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$
24. $u = \dfrac{5x + 1}{2\sqrt{x}}$
25. $v = \dfrac{1 + x - 4\sqrt{x}}{x}$
26. $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$
27. $y = \dfrac{1}{(x^2 - 1)(x^2 + x + 1)}$
28. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$
29. $y = 2e^{-x} + e^{3x}$
30. $y = \dfrac{x^2 + 3e^x}{2e^x - x}$
31. $y = x^3e^x$
32. $w = re^{-r}$
33. $y = x^{9/4} + e^{-2x}$
34. $y = x^{-3/5} + \pi^{3/2}$
35. $s = 2t^{3/2} + 3e^2$
36. $w = \dfrac{1}{z^{1.4}} + \dfrac{\pi}{\sqrt{z}}$
37. $y = \sqrt[7]{x^2} - x^e$
38. $y = \sqrt[3]{x^{9.6}} + 2e^{1.3}$
39. $r = \dfrac{e^s}{s}$
40. $r = e^\theta\left(\dfrac{1}{\theta^2} + \theta^{-\pi/2}\right)$

Find the derivatives of all orders of the functions in Exercises 41–44.

41. $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$
42. $y = \dfrac{x^5}{120}$
43. $y = (x - 1)(x^2 + 3x - 5)$
44. $y = (4x^3 + 3x)(2 - x)$

Find the first and second derivatives of the functions in Exercises 45–52.

45. $y = \dfrac{x^3 + 7}{x}$
46. $s = \dfrac{t^2 + 5t - 1}{t^2}$
47. $r = \dfrac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$
48. $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$
49. $w = \left(\dfrac{1 + 3z}{3z}\right)(3 - z)$
50. $p = \dfrac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$
51. $w = 3z^2e^{2z}$
52. $w = e^z(z - 1)(z^2 + 1)$

**53.** Suppose $u$ and $v$ are functions of $x$ that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

**a.** $\dfrac{d}{dx}(uv)$ **b.** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c.** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d.** $\dfrac{d}{dx}(7v - 2u)$

**54.** Suppose $u$ and $v$ are differentiable functions of $x$ and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

**a.** $\dfrac{d}{dx}(uv)$ **b.** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c.** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d.** $\dfrac{d}{dx}(7v - 2u)$

## Slopes and Tangents

**55. a. Normal to a curve** Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point (2, 1).

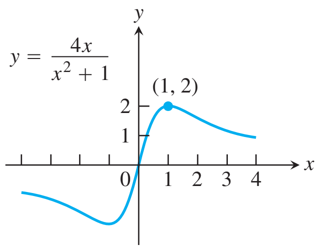
**b. Smallest slope** What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

**c. Tangents having specified slope** Find equations for the tangents to the curve at the points where the slope of the curve is 8.

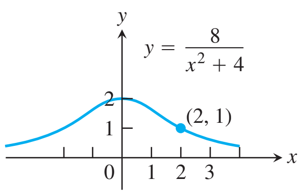
**56. a. Horizontal tangents** Find equations for the horizontal tangents to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the points of tangency.

**b. Smallest slope** What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent at this point.

**57.** Find the tangents to *Newton's serpentine* (graphed here) at the origin and the point (1, 2).




**58.** Find the tangent to the *Witch of Agnesi* (graphed here) at the point (2, 1).




**59. Quadratic tangent to identity function** The curve $y = ax^2 + bx + c$ passes through the point (1, 2) and is tangent to the line $y = x$ at the origin. Find $a$, $b$, and $c$.

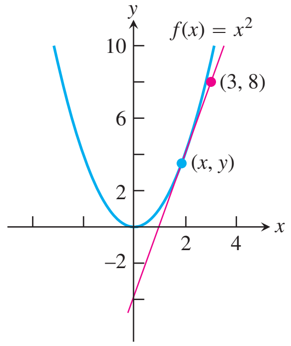
**60. Quadratics having a common tangent** The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point (1, 0). Find $a$, $b$, and $c$.

**61.** Find all points $(x, y)$ on the graph of $f(x) = 3x^2 - 4x$ with tangent lines parallel to the line $y = 8x + 5$.

**62.** Find all points $(x, y)$ on the graph of $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 1$ with tangent lines parallel to the line $8x - 2y = 1$.

**63.** Find all points $(x, y)$ on the graph of $y = x/(x - 2)$ with tangent lines perpendicular to the line $y = 2x + 3$.

**64.** Find all points $(x, y)$ on the graph of $f(x) = x^2$ with tangent lines passing through the point (3, 8).




**65. a.** Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.

**T b.** Graph the curve and tangent line together. The tangent intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.

**T c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (Solver key).

**66. a.** Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.

**T b.** Graph the curve and tangent together. The tangent intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.

**T c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (Solver key).

## Theory and Examples

For Exercises 67 and 68 evaluate each limit by first converting each to a derivative at a particular $x$-value.

**67.** $\displaystyle\lim_{x \to 1} \frac{x^{50} - 1}{x - 1}$ **68.** $\displaystyle\lim_{x \to -1} \frac{x^{2/9} - 1}{x + 1}$

**69.** Find the value of $a$ that makes the following function differentiable for all $x$-values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x, & \text{if } x \ge 0 \end{cases}$$

**70.** Find the values of $a$ and $b$ that make the following function differentiable for all $x$-values.

$$f(x) = \begin{cases} ax + b, & x > -1 \\ bx^2 - 3, & x \le -1 \end{cases}$$

**71.** The general polynomial of degree $n$ has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n \ne 0$. Find $P'(x)$.

**72. The body's reaction to medicine** The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where $C$ is a positive constant and $M$ is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, $R$ is measured in millimeters of mercury. If the reaction is a change in temperature, $R$ is measured in degrees, and so on.

Find $dR/dM$. This derivative, as a function of $M$, is called the sensitivity of the body to the medicine. In Section 4.5, we will see how to find the amount of medicine to which the body is most sensitive.

**73.** Suppose that the function $v$ in the Derivative Product Rule has a constant value $c$. What does the Derivative Product Rule then say? What does this say about the Derivative Constant Multiple Rule?

**74. The Reciprocal Rule**

**a.** The *Reciprocal Rule* says that at any point where the function $v(x)$ is differentiable and different from zero,

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}.$$

Show that the Reciprocal Rule is a special case of the Derivative Quotient Rule.

**b.** Show that the Reciprocal Rule and the Derivative Product Rule together imply the Derivative Quotient Rule.

**75. Generalizing the Product Rule** The Derivative Product Rule gives the formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

for the derivative of the product $uv$ of two differentiable functions of $x$.

**a.** What is the analogous formula for the derivative of the product $uvw$ of *three* differentiable functions of $x$?

**b.** What is the formula for the derivative of the product $u_1u_2u_3u_4$ of *four* differentiable functions of $x$?

**c.** What is the formula for the derivative of a product $u_1u_2u_3\cdots u_n$ of a finite number $n$ of differentiable functions of $x$?

**76. Power Rule for negative integers** Use the Derivative Quotient Rule to prove the Power Rule for negative integers, that is,

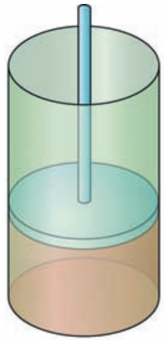
$$\frac{d}{dx}(x^{-m}) = -mx^{-m-1}$$

where $m$ is a positive integer.

**77. Cylinder pressure** If gas in a cylinder is maintained at a constant temperature $T$, the pressure $P$ is related to the volume $V$ by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which $a$, $b$, $n$, and $R$ are constants. Find $dP/dV$. (See accompanying figure.)



**78. The best quantity to order** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where $q$ is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be); $k$ is the cost of placing an order (the same, no matter how often you order); $c$ is the cost of one item (a constant); $m$ is the number of items sold each week (a constant); and $h$ is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find $dA/dq$ and $d^2A/dq^2$.

# 3.4 The Derivative as a Rate of Change

In Section 2.1 we introduced average and instantaneous rates of change. In this section we study further applications in which derivatives model the rates at which things change. It is natural to think of a quantity changing with respect to time, but other variables can be treated in the same way. For example, an economist may want to study how the cost of producing steel varies with the number of tons produced, or an engineer may want to know how the power output of a generator varies with its temperature.

## Instantaneous Rates of Change

If we interpret the difference quotient $(f(x + h) - f(x))/h$ as the average rate of change in $f$ over the interval from $x$ to $x + h$, we can interpret its limit as $h \to 0$ as the rate at which $f$ is changing at the point $x$.

**DEFINITION** The **instantaneous rate of change** of $f$ with respect to $x$ at $x_0$ is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Thus, instantaneous rates are limits of average rates.

It is conventional to use the word *instantaneous* even when $x$ does not represent time. The word is, however, frequently omitted. When we say *rate of change*, we mean *instantaneous rate of change*.

**EXAMPLE 1** The area $A$ of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4} D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

**Solution** The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}.$$

When $D = 10$ m, the area is changing with respect to the diameter at the rate of $(\pi/2)10 = 5\pi \text{ m}^2/\text{m} \approx 15.71 \text{ m}^2/\text{m}$. ■

## Motion Along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object is moving along a coordinate line (an $s$-axis), usually horizontal or vertical, so that we know its position $s$ on that line as a function of time $t$:

$$s = f(t).$$

The **displacement** of the object over the time interval from $t$ to $t + \Delta t$ (Figure 3.14) is

$$\Delta s = f(t + \Delta t) - f(t),$$

and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

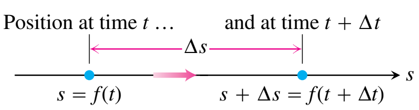



**FIGURE 3.14** The positions of a body moving along a coordinate line at time $t$ and shortly later at time $t + \Delta t$. Here the coordinate line is horizontal.

To find the body's velocity at the exact instant $t$, we take the limit of the average velocity over the interval from $t$ to $t + \Delta t$ as $\Delta t$ shrinks to zero. This limit is the derivative of $f$ with respect to $t$.

**DEFINITION** **Velocity** (**instantaneous velocity**) is the derivative of position with respect to time. If a body's position at time $t$ is $s = f(t)$, then the body's velocity at time $t$ is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

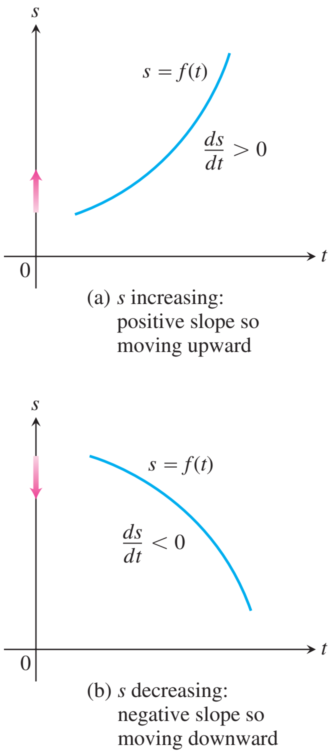



**FIGURE 3.15** For motion $s = f(t)$ along a straight line (the vertical axis), $v = ds/dt$ is (a) positive when $s$ increases and (b) negative when $s$ decreases.

Besides telling how fast an object is moving along the horizontal line in Figure 3.14, its velocity tells the direction of motion. When the object is moving forward ($s$ increasing), the velocity is positive; when the object is moving backward ($s$ decreasing), the velocity is negative. If the coordinate line is vertical, the object moves upward for positive velocity and downward for negative velocity. The blue curves in Figure 3.15 represent position along the line over time; they do not portray the path of motion, which lies along the $s$-axis.

If we drive to a friend's house and back at 30 mph, say, the speedometer will show 30 on the way over but it will not show $-30$ on the way back, even though our distance from home is decreasing. The speedometer always shows *speed*, which is the absolute value of velocity. Speed measures the rate of progress regardless of direction.

**DEFINITION** **Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

**EXAMPLE 2** Figure 3.16 shows the graph of the velocity $v = f'(t)$ of a particle moving along a horizontal line (as opposed to showing a position function $s = f(t)$ such as in Figure 3.15). In the graph of the velocity function, it's not the slope of the curve that tells us if the particle is moving forward or backward along the line (which is not shown in the figure), but rather the sign of the velocity. Looking at Figure 3.16, we see that the particle moves forward for the first 3 sec (when the velocity is positive), moves backward for the next 2 sec (the velocity is negative), stands motionless for a full second, and then moves forward again. The particle is speeding up when its positive velocity increases during the first second, moves at a steady speed during the next second, and then slows down as the velocity decreases to zero during the third second. It stops for an instant at $t = 3$ sec (when the velocity is zero) and reverses direction as the velocity starts to become negative. The particle is now moving backward and gaining in speed until $t = 4$ sec, at which time it achieves its greatest speed during its backward motion. Continuing its backward motion at time $t = 4$, the particle starts to slow down again until it finally stops at time $t = 5$ (when the velocity is once again zero). The particle now remains motionless for one full second, and then moves forward again at $t = 6$ sec, speeding up during the final second of the forward motion indicated in the velocity graph. ■

The rate at which a body's velocity changes is the body's *acceleration*. The acceleration measures how quickly the body picks up or loses speed.

A sudden change in acceleration is called a *jerk*. When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

**HISTORICAL BIOGRAPHY**

Bernard Bolzano
(1781–1848)

**DEFINITIONS** **Acceleration** is the derivative of velocity with respect to time. If a body's position at time $t$ is $s = f(t)$, then the body's acceleration at time $t$ is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

**Jerk** is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Near the surface of the Earth all bodies fall with the same constant acceleration. Galileo's experiments with free fall (see Section 2.1) lead to the equation

$$s = \frac{1}{2}gt^2,$$

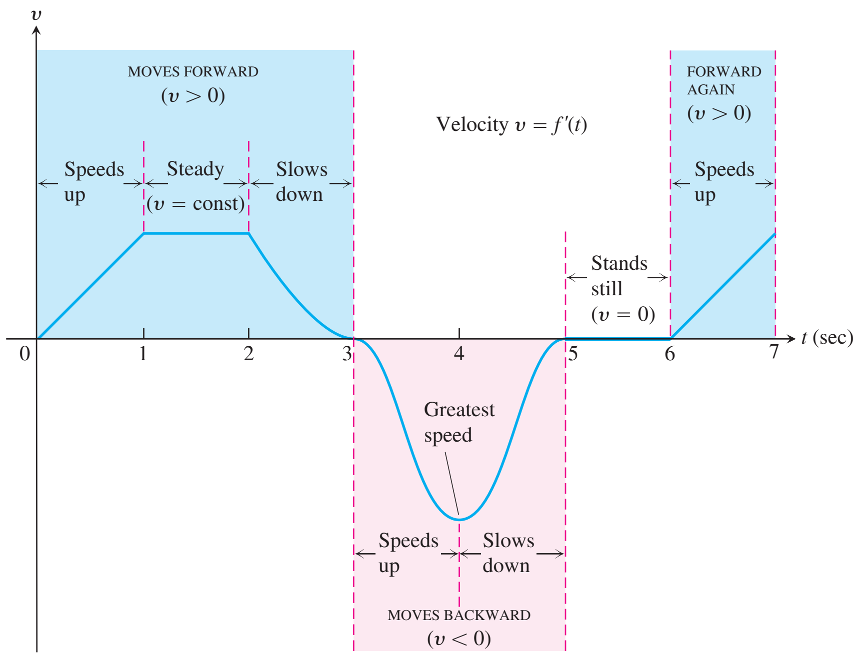



**FIGURE 3.16** The velocity graph of a particle moving along a horizontal line, discussed in Example 2.

where $s$ is the distance fallen and $g$ is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, and closely models the fall of dense, heavy objects, such as rocks or steel tools, for the first few seconds of their fall, before the effects of air resistance are significant.

The value of $g$ in the equation $s = (1/2)gt^2$ depends on the units used to measure $t$ and $s$. With $t$ in seconds (the usual unit), the value of $g$ determined by measurement at sea level is approximately 32 ft/sec$^2$ (feet per second squared) in English units, and $g = 9.8$ m/sec$^2$ (meters per second squared) in metric units. (These gravitational constants depend on the distance from Earth's center of mass, and are slightly lower on top of Mt. Everest, for example.)

The jerk associated with the constant acceleration of gravity ($g = 32$ ft/sec$^2$) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

An object does not exhibit jerkiness during free fall.

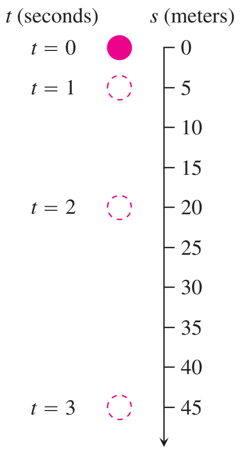



**FIGURE 3.17** A ball bearing falling from rest (Example 3).

**EXAMPLE 3** Figure 3.17 shows the free fall of a heavy ball bearing released from rest at time $t = 0$ sec.

**(a)** How many meters does the ball fall in the first 2 sec?

**(b)** What is its velocity, speed, and acceleration when $t = 2$?

**Solution**

**(a)** The metric free-fall equation is $s = 4.9t^2$. During the first 2 sec, the ball falls

$$s(2) = 4.9(2)^2 = 19.6 \text{ m}.$$

**(b)** At any time $t$, *velocity* is the derivative of position:

$$v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8t.$$

At $t = 2$, the velocity is

$$v(2) = 19.6 \text{ m/sec}$$

in the downward (increasing $s$) direction. The *speed* at $t = 2$ is

$$\text{speed} = |v(2)| = 19.6 \text{ m/sec}.$$

The *acceleration* at any time $t$ is

$$a(t) = v'(t) = s''(t) = 9.8 \text{ m/sec}^2.$$

At $t = 2$, the acceleration is $9.8 \text{ m/sec}^2$. ■

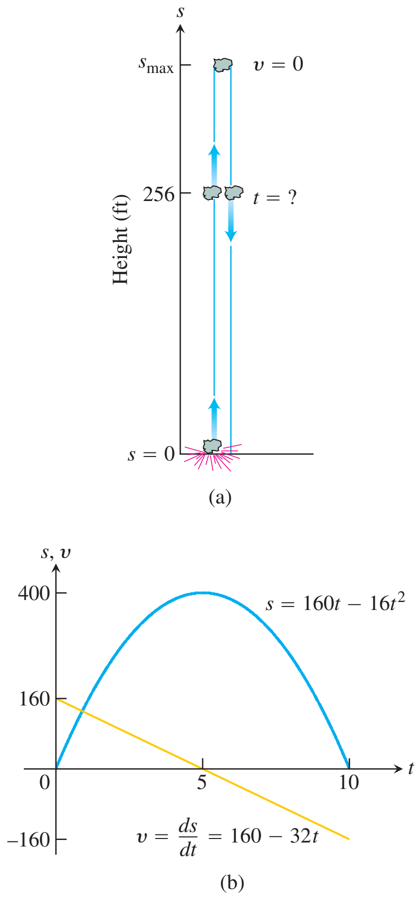



**FIGURE 3.18** (a) The rock in Example 4. (b) The graphs of $s$ and $v$ as functions of time; $s$ is largest when $v = ds/dt = 0$. The graph of $s$ is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

**EXAMPLE 4** A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.18a). It reaches a height of $s = 160t - 16t^2$ ft after $t$ sec.

**(a)** How high does the rock go?

**(b)** What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

**(c)** What is the acceleration of the rock at any time $t$ during its flight (after the blast)?

**(d)** When does the rock hit the ground again?

**Solution**

**(a)** In the coordinate system we have chosen, $s$ measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. To find the maximum height, all we need to do is to find when $v = 0$ and evaluate $s$ at this time.

At any time $t$ during the rock's motion, its velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec}.$$

The velocity is zero when

$$160 - 32t = 0 \quad \text{or} \quad t = 5 \text{ sec}.$$

The rock's height at $t = 5$ sec is

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 800 - 400 = 400 \text{ ft}.$$

See Figure 3.18b.

**(b)** To find the rock's velocity at 256 ft on the way up and again on the way down, we first find the two values of $t$ for which

$$s(t) = 160t - 16t^2 = 256.$$

To solve this equation, we write

$$\begin{aligned} 16t^2 - 160t + 256 &= 0 \\ 16(t^2 - 10t + 16) &= 0 \\ (t - 2)(t - 8) &= 0 \\ t = 2 \text{ sec}, t &= 8 \text{ sec}. \end{aligned}$$

The rock is 256 ft above the ground 2 sec after the explosion and again 8 sec after the explosion. The rock's velocities at these times are

$$v(2) = 160 - 32(2) = 160 - 64 = 96 \text{ ft/sec}.$$
$$v(8) = 160 - 32(8) = 160 - 256 = -96 \text{ ft/sec}.$$

At both instants, the rock's speed is 96 ft/sec. Since $v(2) > 0$, the rock is moving upward ($s$ is increasing) at $t = 2$ sec; it is moving downward ($s$ is decreasing) at $t = 8$ because $v(8) < 0$.

**(c)** At any time during its flight following the explosion, the rock's acceleration is a constant

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. As the rock rises, it slows down; as it falls, it speeds up.

**(d)** The rock hits the ground at the positive time $t$ for which $s = 0$. The equation $160t - 16t^2 = 0$ factors to give $16t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$, the blast occurred and the rock was thrown upward. It returned to the ground 10 sec later. ■

## Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals*.

In a manufacturing operation, the *cost of production* $c(x)$ is a function of $x$, the number of units produced. The **marginal cost of production** is the rate of change of cost with respect to level of production, so it is $dc/dx$.

Suppose that $c(x)$ represents the dollars needed to produce $x$ tons of steel in one week. It costs more to produce $x + h$ tons per week, and the cost difference, divided by $h$, is the average cost of producing each additional ton:

$$\frac{c(x+h) - c(x)}{h} = \text{average cost of each of the additional } h \text{ tons of steel produced.}$$

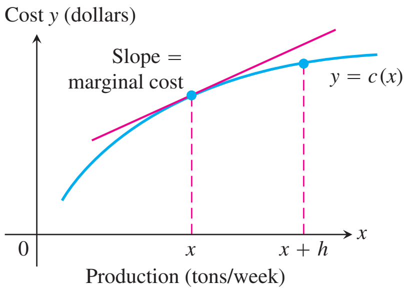



**FIGURE 3.19** Weekly steel production: $c(x)$ is the cost of producing $x$ tons per week. The cost of producing an additional $h$ tons is $c(x + h) - c(x)$.

The limit of this ratio as $h \to 0$ is the *marginal cost* of producing more steel per week when the current weekly production is $x$ tons (Figure 3.19):

$$\frac{dc}{dx} = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h} = \text{marginal cost of production.}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one additional unit:

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1) - c(x)}{1},$$

which is approximated by the value of $dc/dx$ at $x$. This approximation is acceptable if the slope of the graph of $c$ does not change quickly near $x$. Then the difference quotient will be close to its limit $dc/dx$, which is the rise in the tangent line if $\Delta x = 1$ (Figure 3.20). The approximation works best for large values of $x$.

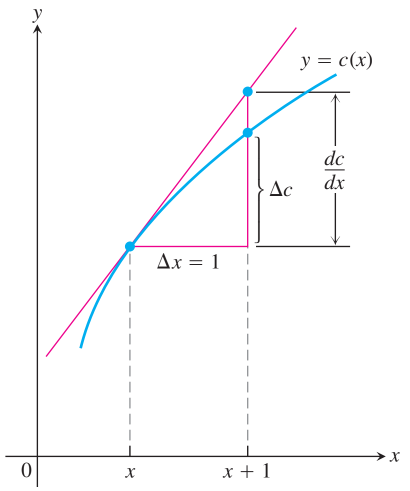



**FIGURE 3.20** The marginal cost $dc/dx$ is approximately the extra cost $\Delta c$ of producing $\Delta x = 1$ more unit.

Economists often represent a total cost function by a cubic polynomial

$$c(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$$

where $\delta$ represents *fixed costs* such as rent, heat, equipment capitalization, and management costs. The other terms represent *variable costs* such as the costs of raw materials, taxes, and labor. Fixed costs are independent of the number of units produced, whereas variable costs depend on the quantity produced. A cubic polynomial is usually adequate to capture the cost behavior on a realistic quantity interval.

**EXAMPLE 5** Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce $x$ radiators when 8 to 30 radiators are produced and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling $x$ radiators. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

**Solution** The cost of producing one more radiator a day when 10 are produced is about $c'(10)$:

$$c'(x) = \frac{d}{dx}\left(x^3 - 6x^2 + 15x\right) = 3x^2 - 12x + 15$$
$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

The additional cost will be about \$195. The marginal revenue is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12.$$

The marginal revenue function estimates the increase in revenue that will result from selling one additional unit. If you currently sell 10 radiators a day, you can expect your revenue to increase by about

$$r'(10) = 3(100) - 6(10) + 12 = \$252$$

if you increase sales to 11 radiators a day. ■

**EXAMPLE 6** To get some feel for the language of marginal rates, consider marginal tax rates. If your marginal income tax rate is 28% and your income increases by \$1000, you can expect to pay an extra \$280 in taxes. This does not mean that you pay 28% of your entire income in taxes. It just means that at your current income level $I$, the rate of increase of taxes $T$ with respect to income is $dT/dI = 0.28$. You will pay \$0.28 in taxes out of every extra dollar you earn. Of course, if you earn a lot more, you may land in a higher tax bracket and your marginal rate will increase. ■

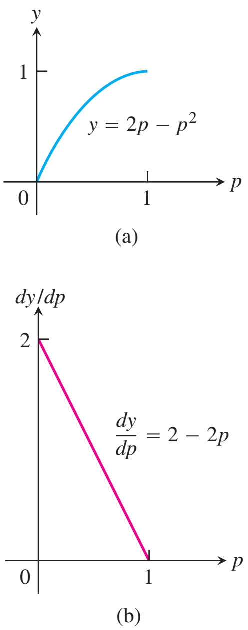



**FIGURE 3.21** (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skinned peas in the next generation. (b) The graph of $dy/dp$ (Example 7).

## Sensitivity to Change

When a small change in $x$ produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in $x$. The derivative $f'(x)$ is a measure of this sensitivity.

**EXAMPLE 7** Genetic Data and Sensitivity to Change

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization.

His careful records showed that if $p$ (a number between 0 and 1) is the frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the frequency of the gene for wrinkled skin in peas, then the proportion of smooth-skinned peas in the next generation will be

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

The graph of $y$ versus $p$ in Figure 3.21a suggests that the value of $y$ is more sensitive to a change in $p$ when $p$ is small than when $p$ is large. Indeed, this fact is borne out by the derivative graph in Figure 3.21b, which shows that $dy/dp$ is close to 2 when $p$ is near 0 and close to 0 when $p$ is near 1.

The implication for genetics is that introducing a few more smooth skin genes into a population where the frequency of wrinkled skin peas is large will have a more dramatic effect on later generations than will a similar increase when the population has a large proportion of smooth skin peas. ■

# Exercises 3.4

## Motion Along a Coordinate Line

Exercises 1–6 give the positions $s = f(t)$ of a body moving on a coordinate line, with $s$ in meters and $t$ in seconds.

**a.** Find the body's displacement and average velocity for the given time interval.

**b.** Find the body's speed and acceleration at the endpoints of the interval.

**c.** When, if ever, during the interval does the body change direction?

**1.** $s = t^2 - 3t + 2, \quad 0 \le t \le 2$

**2.** $s = 6t - t^2, \quad 0 \le t \le 6$

**3.** $s = -t^3 + 3t^2 - 3t, \quad 0 \le t \le 3$

**4.** $s = (t^4/4) - t^3 + t^2, \quad 0 \le t \le 3$

**5.** $s = \dfrac{25}{t^2} - \dfrac{5}{t}, \quad 1 \le t \le 5$

**6.** $s = \dfrac{25}{t + 5}, \quad -4 \le t \le 0$

**7. Particle motion** At time $t$, the position of a body moving along the $s$-axis is $s = t^3 - 6t^2 + 9t$ m.

**a.** Find the body's acceleration each time the velocity is zero.

**b.** Find the body's speed each time the acceleration is zero.

**c.** Find the total distance traveled by the body from $t = 0$ to $t = 2$.

**8. Particle motion** At time $t \ge 0$, the velocity of a body moving along the horizontal $s$-axis is $v = t^2 - 4t + 3$.

**a.** Find the body's acceleration each time the velocity is zero.

**b.** When is the body moving forward? Backward?

**c.** When is the body's velocity increasing? Decreasing?

## Free-Fall Applications

**9. Free fall on Mars and Jupiter** The equations for free fall at the surfaces of Mars and Jupiter ($s$ in meters, $t$ in seconds) are $s = 1.86t^2$ on Mars and $s = 11.44t^2$ on Jupiter. How long does it take a rock falling from rest to reach a velocity of 27.8 m/sec (about 100 km/h) on each planet?

**10. Lunar projectile motion** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ m in $t$ sec.

**a.** Find the rock's velocity and acceleration at time $t$. (The acceleration in this case is the acceleration of gravity on the moon.)

**b.** How long does it take the rock to reach its highest point?

**c.** How high does the rock go?

**d.** How long does it take the rock to reach half its maximum height?

**e.** How long is the rock aloft?

**11. Finding *g* on a small airless planet** Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 15 m/sec. Because the acceleration of gravity at the planet's surface was $g_s$ m/sec$^2$, the explorers expected the ball bearing to reach a height of $s = 15t - (1/2)g_s t^2$ m $t$ sec later. The ball bearing reached its maximum height 20 sec after being launched. What was the value of $g_s$?

**12. Speeding bullet** A 45-caliber bullet shot straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ ft after $t$ sec. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ ft after $t$ sec. How long will the bullet be aloft in each case? How high will the bullet go?

**13. Free fall from the Tower of Pisa** Had Galileo dropped a cannonball from the Tower of Pisa, 179 ft above the ground, the ball's height above the ground $t$ sec into the fall would have been $s = 179 - 16t^2$.

**a.** What would have been the ball's velocity, speed, and acceleration at time $t$?

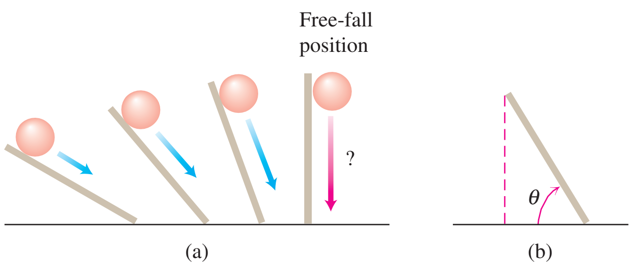
**b.** About how long would it have taken the ball to hit the ground?

**c.** What would have been the ball's velocity at the moment of impact?

**14. Galileo's free-fall formula** Galileo developed a formula for a body's velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict a ball's behavior when the plank was vertical and the ball fell freely; see part (a) of the accompanying figure. He found that, for any given angle of the plank, the ball's velocity $t$ sec into motion was a constant multiple of $t$. That is, the velocity was given by a formula of the form $v = kt$. The value of the constant $k$ depended on the inclination of the plank.

In modern notation—part (b) of the figure—with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle $\theta$, the ball's velocity $t$ sec into the roll was

$$v = 9.8(\sin\theta)t \text{ m/sec}.$$

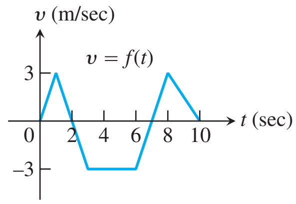



**a.** What is the equation for the ball's velocity during free fall?

**b.** Building on your work in part (a), what constant acceleration does a freely falling body experience near the surface of Earth?

## Understanding Motion from Graphs

**15.** The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/sec) of a body moving along a coordinate line.



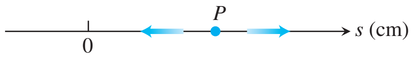

**a.** When does the body reverse direction?

**b.** When (approximately) is the body moving at a constant speed?

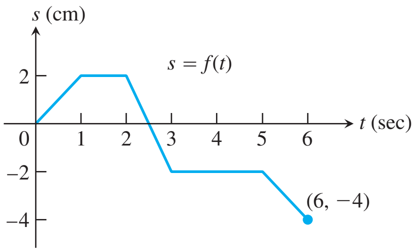
**c.** Graph the body's speed for $0 \le t \le 10$.

**d.** Graph the acceleration, where defined.

**16.** A particle $P$ moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of $P$ as a function of time $t$.




(a)




(b)

**a.** When is $P$ moving to the left? Moving to the right? Standing still?

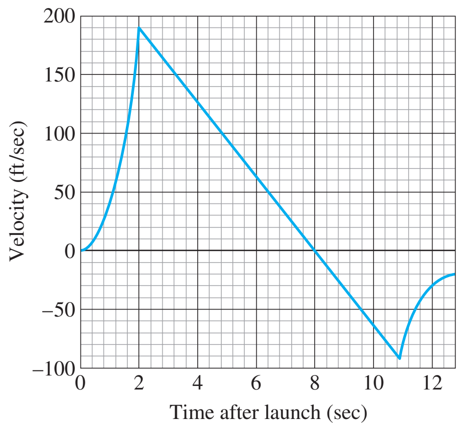
**b.** Graph the particle's velocity and speed (where defined).

**17. Launching a rocket** When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands.

The figure here shows velocity data from the flight of the model rocket. Use the data to answer the following.

**a.** How fast was the rocket climbing when the engine stopped?

**b.** For how many seconds did the engine burn?




**c.** When did the rocket reach its highest point? What was its velocity then?

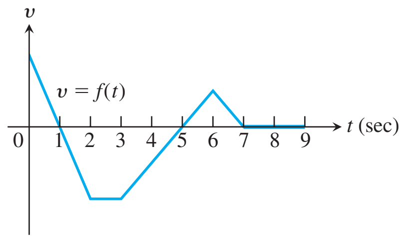
**d.** When did the parachute pop out? How fast was the rocket falling then?

**e.** How long did the rocket fall before the parachute opened?

**f.** When was the rocket's acceleration greatest?

**g.** When was the acceleration constant? What was its value then (to the nearest integer)?

**18.** The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a horizontal coordinate line.



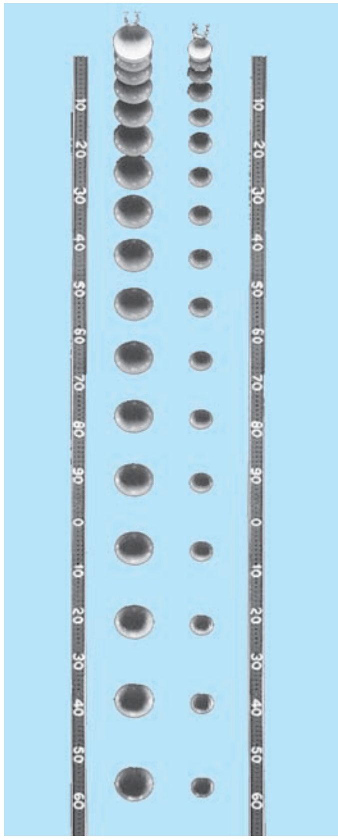

**a.** When does the particle move forward? Move backward? Speed up? Slow down?

**b.** When is the particle's acceleration positive? Negative? Zero?

**c.** When does the particle move at its greatest speed?

**d.** When does the particle stand still for more than an instant?

**19. Two falling balls** The multiflash photograph in the accompanying figure shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for $s$ in centimeters and $t$ in seconds) to answer the following questions.



**a.** How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

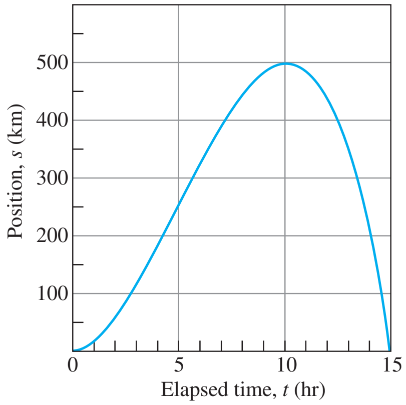
**b.** How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

**c.** About how fast was the light flashing (flashes per second)?

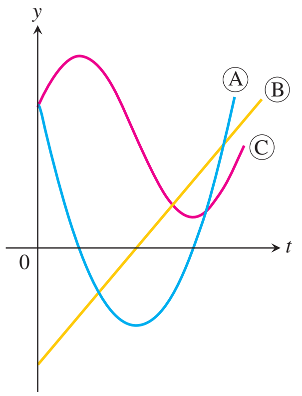
**20. A traveling truck** The accompanying graph shows the position $s$ of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 h later at $t = 15$.

**a.** Use the technique described in Section 3.2, Example 3, to graph the truck's velocity $v = ds/dt$ for $0 \le t \le 15$. Then repeat the process, with the velocity curve, to graph the truck's acceleration $dv/dt$.

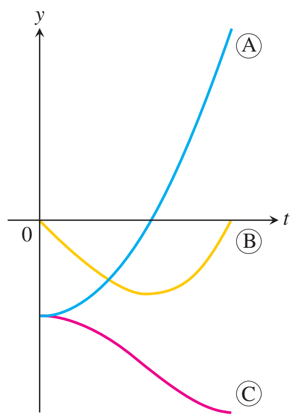
**b.** Suppose that $s = 15t^2 - t^3$. Graph $ds/dt$ and $d^2s/dt^2$ and compare your graphs with those in part (a).




**21.** The graphs in the accompanying figure show the position $s$, velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time $t$. Which graph is which? Give reasons for your answers.




**22.** The graphs in the accompanying figure show the position $s$, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along the coordinate line as functions of time $t$. Which graph is which? Give reasons for your answers.




## Economics

**23. Marginal cost** Suppose that the dollar cost of producing $x$ washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

**a.** Find the average cost per machine of producing the first 100 washing machines.

**b.** Find the marginal cost when 100 washing machines are produced.

**c.** Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

**24. Marginal revenue** Suppose that the revenue from selling $x$ washing machines is

$$r(x) = 20{,}000\left(1 - \frac{1}{x}\right)$$

dollars.

**a.** Find the marginal revenue when 100 machines are produced.

**b.** Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines a week to 101 machines a week.

**c.** Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

## Additional Applications

**25. Bacterium population** When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time $t$ (hours) was $b = 10^6 + 10^4t - 10^3t^2$. Find the growth rates at

**a.** $t = 0$ hours.

**b.** $t = 5$ hours.

**c.** $t = 10$ hours.

**26. Draining a tank** The number of gallons of water in a tank $t$ minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

T **27. Draining a tank** It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth $y$ of fluid in the tank $t$ hours after the valve is opened is given by the formula

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m}.$$

**a.** Find the rate $dy/dt$ (m/h) at which the tank is draining at time $t$.

**b.** When is the fluid level in the tank falling fastest? Slowest? What are the values of $dy/dt$ at these times?

**c.** Graph $y$ and $dy/dt$ together and discuss the behavior of $y$ in relation to the signs and values of $dy/dt$.

**28. Inflating a balloon** The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

**a.** At what rate ($\text{ft}^3/\text{ft}$) does the volume change with respect to the radius when $r = 2$ ft?

**b.** By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

**29. Airplane takeoff** Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where $D$ is measured in meters from the starting point and $t$ is measured in seconds from the time the brakes are released. The aircraft will become airborne when its speed reaches 200 km/h. How long will it take to become airborne, and what distance will it travel in that time?

**30. Volcanic lava fountains** Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a Hawaiian record). What was the lava's exit velocity in feet per second? In miles per hour? (*Hint:* If $v_0$ is the exit velocity of a particle of lava, its height $t$ sec later will be $s = v_0 t - 16t^2$ ft. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.)

### Analyzing Motion Using Graphs

T Exercises 31–34 give the position function $s = f(t)$ of an object moving along the $s$-axis as a function of time $t$. Graph $f$ together with the velocity function $v(t) = ds/dt = f'(t)$ and the acceleration function $a(t) = d^2s/dt^2 = f''(t)$. Comment on the object's behavior in relation to the signs and values of $v$ and $a$. Include in your commentary such topics as the following:

**a.** When is the object momentarily at rest?

**b.** When does it move to the left (down) or to the right (up)?

**c.** When does it change direction?

**d.** When does it speed up and slow down?

**e.** When is it moving fastest (highest speed)? Slowest?

**f.** When is it farthest from the axis origin?

**31.** $s = 200t - 16t^2, \quad 0 \le t \le 12.5$ (a heavy object fired straight up from Earth's surface at 200 ft/sec)

**32.** $s = t^2 - 3t + 2, \quad 0 \le t \le 5$

**33.** $s = t^3 - 6t^2 + 7t, \quad 0 \le t \le 4$

**34.** $s = 4 - 7t + 6t^2 - t^3, \quad 0 \le t \le 4$

# 3.5 Derivatives of Trigonometric Functions

Many phenomena of nature are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

## Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for $x$ measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 2.4 with the angle sum identity for the sine function:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{\text{limit } 0} + \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{\text{limit } 1} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x. \qquad \text{Example 5a and Theorem 7, Section 2.4}$$

**The derivative of the sine function is the cosine function:**

$$\frac{d}{dx}(\sin x) = \cos x.$$

**EXAMPLE 1** We find derivatives of the sine function involving differences, products, and quotients.

**(a)** $y = x^2 - \sin x$: $\quad \dfrac{dy}{dx} = 2x - \dfrac{d}{dx}(\sin x)$ Difference Rule

$$= 2x - \cos x$$

**(b)** $y = e^x \sin x$: $\quad \dfrac{dy}{dx} = e^x \dfrac{d}{dx}(\sin x) + \dfrac{d}{dx}(e^x)\sin x$ Product Rule

$$= e^x \cos x + e^x \sin x$$
$$= e^x(\cos x + \sin x)$$

**(c)** $y = \dfrac{\sin x}{x}$: $\quad \dfrac{dy}{dx} = \dfrac{x \cdot \dfrac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2}$ Quotient Rule

$$= \frac{x \cos x - \sin x}{x^2}$$

■

## Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h}$$ Derivative definition

$$= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$ Cosine angle sum identity

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h \to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1$$ Example 5a and Theorem 7, Section 2.4

$$= -\sin x.$$

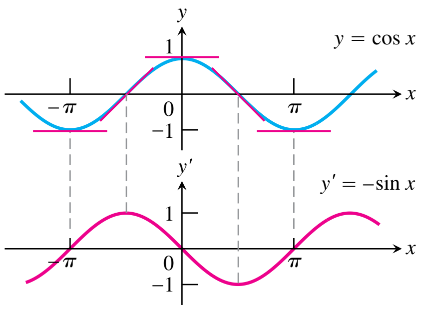



**FIGURE 3.22** The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

**The derivative of the cosine function is the negative of the sine function:**

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Figure 3.22 shows a way to visualize this result in the same way we did for graphing derivatives in Section 3.2, Figure 3.6.

**EXAMPLE 2** We find derivatives of the cosine function in combinations with other functions.

**(a)** $y = 5e^x + \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x) && \text{Sum Rule}\\ &= 5e^x - \sin x\end{aligned}$$

**(b)** $y = \sin x \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) && \text{Product Rule}\\ &= \sin x(-\sin x) + \cos x(\cos x)\\ &= \cos^2 x - \sin^2 x\end{aligned}$$

**(c)** $y = \dfrac{\cos x}{1 - \sin x}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} && \text{Quotient Rule}\\ &= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}\\ &= \frac{1 - \sin x}{(1 - \sin x)^2} && \sin^2 x + \cos^2 x = 1\\ &= \frac{1}{1 - \sin x}\end{aligned}$$

■

## Simple Harmonic Motion

The motion of an object or weight bobbing freely up and down with no resistance on the end of a spring is an example of *simple harmonic motion*. The motion is periodic and repeats indefinitely, so we represent it using trigonometric functions. The next example describes a case in which there are no opposing forces such as friction or buoyancy to slow the motion.

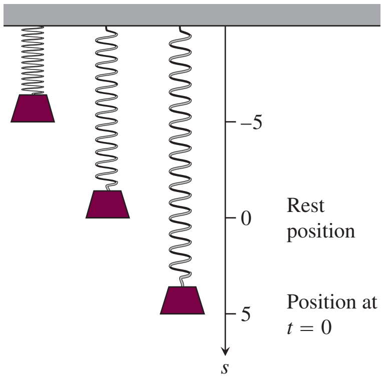



**FIGURE 3.23** A weight hanging from a vertical spring and then displaced oscillates above and below its rest position (Example 3).

**EXAMPLE 3** A weight hanging from a spring (Figure 3.23) is stretched down 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time $t$ is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time $t$?

**Solution** We have

Position: $s = 5 \cos t$

Velocity: $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(5 \cos t) = -5 \sin t$

Acceleration: $a = \dfrac{dv}{dt} = \dfrac{d}{dt}(-5 \sin t) = -5 \cos t.$

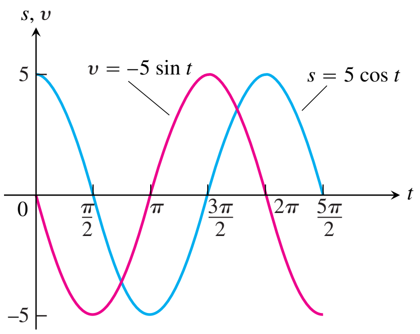



**FIGURE 3.24** The graphs of the position and velocity of the weight in Example 3.

Notice how much we can learn from these equations:

1. As time passes, the weight moves down and up between $s = -5$ and $s = 5$ on the $s$-axis. The amplitude of the motion is 5. The period of the motion is $2\pi$, the period of the cosine function.
2. The velocity $v = -5\sin t$ attains its greatest magnitude, 5, when $\cos t = 0$, as the graphs show in Figure 3.24. Hence, the speed of the weight, $|v| = 5|\sin t|$, is greatest when $\cos t = 0$, that is, when $s = 0$ (the rest position). The speed of the weight is zero when $\sin t = 0$. This occurs when $s = 5\cos t = \pm 5$, at the endpoints of the interval of motion.
3. The acceleration value is always the exact opposite of the position value. When the weight is above the rest position, gravity is pulling it back down; when the weight is below the rest position, the spring is pulling it back up.
4. The acceleration, $a = -5\cos t$, is zero only at the rest position, where $\cos t = 0$ and the force of gravity and the force from the spring balance each other. When the weight is anywhere else, the two forces are unequal and acceleration is nonzero. The acceleration is greatest in magnitude at the points farthest from the rest position, where $\cos t = \pm 1$. ■

**EXAMPLE 4** The jerk associated with the simple harmonic motion in Example 3 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5\cos t) = 5\sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the rest position, where the acceleration changes direction and sign. ■

## Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of $x$, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of $x$ at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

**The derivatives of the other trigonometric functions:**

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

To show a typical calculation, we find the derivative of the tangent function. The other derivations are left to Exercise 60.

**EXAMPLE 5** Find $d(\tan x)/dx$.

**Solution** We use the Derivative Quotient Rule to calculate the derivative:

$$\begin{aligned}
\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} && \text{Quotient Rule} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

**EXAMPLE 6** Find $y''$ if $y = \sec x$.

**Solution** Finding the second derivative involves a combination of trigonometric derivatives.

$$\begin{aligned}
y &= \sec x \\
y' &= \sec x \tan x && \text{Derivative rule for secant function} \\
y'' &= \frac{d}{dx}(\sec x \tan x) \\
&= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) && \text{Derivative Product Rule} \\
&= \sec x(\sec^2 x) + \tan x(\sec x \tan x) && \text{Derivative rules} \\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}$$

The differentiability of the trigonometric functions throughout their domains gives another proof of their continuity at every point in their domains (Theorem 1, Section 3.2). So we can calculate limits of algebraic combinations and composites of trigonometric functions by direct substitution.

**EXAMPLE 7** We can use direct substitution in computing limits provided there is no division by zero, which is algebraically undefined.

$$\lim_{x \to 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2 + 1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

## Exercises 3.5

### Derivatives

In Exercises 1–18, find $dy/dx$.

**1.** $y = -10x + 3\cos x$

**2.** $y = \frac{3}{x} + 5\sin x$

**3.** $y = x^2 \cos x$

**4.** $y = \sqrt{x}\sec x + 3$

**5.** $y = \csc x - 4\sqrt{x} + 7$

**6.** $y = x^2 \cot x - \frac{1}{x^2}$

**7.** $f(x) = \sin x \tan x$

**8.** $g(x) = \csc x \cot x$

**9.** $y = (\sec x + \tan x)(\sec x - \tan x)$

**10.** $y = (\sin x + \cos x)\sec x$

**11.** $y = \dfrac{\cot x}{1 + \cot x}$

**12.** $y = \dfrac{\cos x}{1 + \sin x}$

**13.** $y = \dfrac{4}{\cos x} + \dfrac{1}{\tan x}$

**14.** $y = \dfrac{\cos x}{x} + \dfrac{x}{\cos x}$

**15.** $y = x^2 \sin x + 2x \cos x - 2 \sin x$

**16.** $y = x^2 \cos x - 2x \sin x - 2 \cos x$

**17.** $f(x) = x^3 \sin x \cos x$

**18.** $g(x) = (2 - x) \tan^2 x$

In Exercises 19–22, find $ds/dt$.

**19.** $s = \tan t - e^{-t}$

**20.** $s = t^2 - \sec t + 5e^t$

**21.** $s = \dfrac{1 + \csc t}{1 - \csc t}$

**22.** $s = \dfrac{\sin t}{1 - \cos t}$

In Exercises 23–26, find $dr/d\theta$.

**23.** $r = 4 - \theta^2 \sin \theta$

**24.** $r = \theta \sin \theta + \cos \theta$

**25.** $r = \sec \theta \csc \theta$

**26.** $r = (1 + \sec \theta) \sin \theta$

In Exercises 27–32, find $dp/dq$.

**27.** $p = 5 + \dfrac{1}{\cot q}$

**28.** $p = (1 + \csc q) \cos q$

**29.** $p = \dfrac{\sin q + \cos q}{\cos q}$

**30.** $p = \dfrac{\tan q}{1 + \tan q}$

**31.** $p = \dfrac{q \sin q}{q^2 - 1}$

**32.** $p = \dfrac{3q + \tan q}{q \sec q}$

**33.** Find $y''$ if

**a.** $y = \csc x$.

**b.** $y = \sec x$.

**34.** Find $y^{(4)} = d^4 y/dx^4$ if

**a.** $y = -2 \sin x$.

**b.** $y = 9 \cos x$.

## Tangent Lines

In Exercises 35–38, graph the curves over the given intervals, together with their tangents at the given values of $x$. Label each curve and tangent with its equation.

**35.** $y = \sin x, \quad -3\pi/2 \le x \le 2\pi$

$x = -\pi, 0, 3\pi/2$

**36.** $y = \tan x, \quad -\pi/2 < x < \pi/2$

$x = -\pi/3, 0, \pi/3$

**37.** $y = \sec x, \quad -\pi/2 < x < \pi/2$

$x = -\pi/3, \pi/4$

**38.** $y = 1 + \cos x, \quad -3\pi/2 \le x \le 2\pi$

$x = -\pi/3, 3\pi/2$

**T** Do the graphs of the functions in Exercises 39–42 have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so, where? If not, why not? Visualize your findings by graphing the functions with a grapher.

**39.** $y = x + \sin x$

**40.** $y = 2x + \sin x$

**41.** $y = x - \cot x$

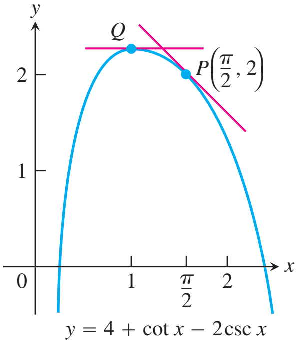
**42.** $y = x + 2 \cos x$

**43.** Find all points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

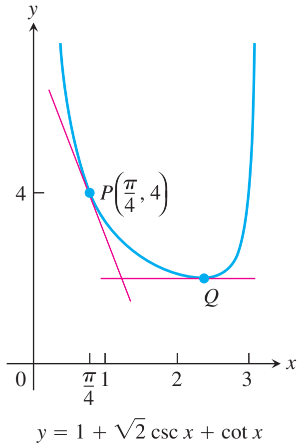
**44.** Find all points on the curve $y = \cot x$, $0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

In Exercises 45 and 46, find an equation for **(a)** the tangent to the curve at $P$ and **(b)** the horizontal tangent to the curve at $Q$.

**45.**




**46.**




## Trigonometric Limits

Find the limits in Exercises 47–54.

**47.** $\lim_{x \to 2} \sin \left( \dfrac{1}{x} - \dfrac{1}{2} \right)$

**48.** $\lim_{x \to -\pi/6} \sqrt{1 + \cos (\pi \csc x)}$

**49.** $\lim_{\theta \to \pi/6} \dfrac{\sin \theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$

**50.** $\lim_{\theta \to \pi/4} \dfrac{\tan \theta - 1}{\theta - \frac{\pi}{4}}$

**51.** $\lim_{x \to 0} \sec \left[ e^x + \pi \tan \left( \dfrac{\pi}{4 \sec x} \right) - 1 \right]$

**52.** $\lim_{x \to 0} \sin \left( \dfrac{\pi + \tan x}{\tan x - 2 \sec x} \right)$

**53.** $\lim_{t \to 0} \tan \left( 1 - \dfrac{\sin t}{t} \right)$

**54.** $\lim_{\theta \to 0} \cos \left( \dfrac{\pi \theta}{\sin \theta} \right)$

## Theory and Examples

The equations in Exercises 55 and 56 give the position $s = f(t)$ of a body moving on a coordinate line ($s$ in meters, $t$ in seconds). Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ sec.

**55.** $s = 2 - 2 \sin t$

**56.** $s = \sin t + \cos t$

**57.** Is there a value of $c$ that will make

$$f(x) = \begin{cases} \dfrac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

**58.** Is there a value of $b$ that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \ge 0 \end{cases}$$

continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answers.

**59.** Find $d^{999}/dx^{999}(\cos x)$.

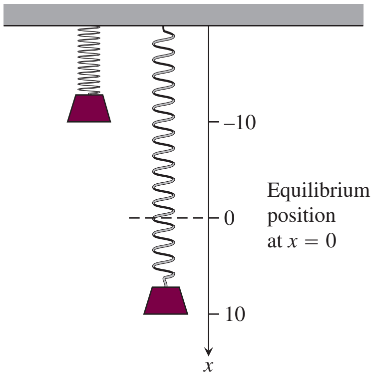
**60.** Derive the formula for the derivative with respect to $x$ of

**a.** $\sec x$. **b.** $\csc x$. **c.** $\cot x$.

**61.** A weight is attached to a spring and reaches its equilibrium position ($x = 0$). It is then set in motion resulting in a displacement of

$$x = 10 \cos t,$$

where $x$ is measured in centimeters and $t$ is measured in seconds. See the accompanying figure.




**a.** Find the spring's displacement when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.

**b.** Find the spring's velocity when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.

**62.** Assume that a particle's position on the $x$-axis is given by

$$x = 3 \cos t + 4 \sin t,$$

where $x$ is measured in feet and $t$ is measured in seconds.

**a.** Find the particle's position when $t = 0$, $t = \pi/2$, and $t = \pi$.

**b.** Find the particle's velocity when $t = 0$, $t = \pi/2$, and $t = \pi$.

T **63.** Graph $y = \cos x$ for $-\pi \le x \le 2\pi$. On the same screen, graph

$$y = \frac{\sin(x + h) - \sin x}{h}$$

for $h = 1, 0.5, 0.3$, and $0.1$. Then, in a new window, try $h = -1, -0.5$, and $-0.3$. What happens as $h \to 0^+$? As $h \to 0^-$? What phenomenon is being illustrated here?

T **64.** Graph $y = -\sin x$ for $-\pi \le x \le 2\pi$. On the same screen, graph

$$y = \frac{\cos(x + h) - \cos x}{h}$$

for $h = 1, 0.5, 0.3$, and $0.1$. Then, in a new window, try $h = -1, -0.5$, and $-0.3$. What happens as $h \to 0^+$? As $h \to 0^-$? What phenomenon is being illustrated here?

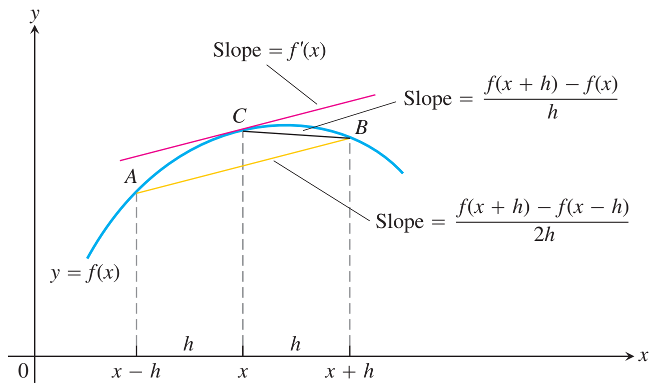
T **65. Centered difference quotients** The *centered difference quotient*

$$\frac{f(x + h) - f(x - h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \to 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of $h$ than the difference quotient

$$\frac{f(x + h) - f(x)}{h}.$$

See the accompanying figure.




**a.** To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x + h) - \sin(x - h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and $0.3$. Compare the results with those obtained in Exercise 63 for the same values of $h$.

**b.** To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x + h) - \cos(x - h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and $0.3$. Compare the results with those obtained in Exercise 64 for the same values of $h$.

**66. A caution about centered difference quotients** (*Continuation of Exercise 65.*) The quotient

$$\frac{f(x + h) - f(x - h)}{2h}$$

may have a limit as $h \to 0$ when $f$ has no derivative at $x$. As a case in point, take $f(x) = |x|$ and calculate

$$\lim_{h \to 0} \frac{|0 + h| - |0 - h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$. *Moral:* Before using a centered difference quotient, be sure the derivative exists.

T **67. Slopes on the graph of the tangent function** Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? A largest slope? Is the slope ever negative? Give reasons for your answers.

T **68. Slopes on the graph of the cotangent function** Graph $y = \cot x$ and its derivative together for $0 < x < \pi$. Does the graph of the cotangent function appear to have a smallest slope? A largest slope? Is the slope ever positive? Give reasons for your answers.

T **69. Exploring $(\sin kx)/x$** Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \le x \le 2$. Where does each graph appear to cross the $y$-axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \to 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of $k$? Give reasons for your answers.

T **70. Radians versus degrees: degree mode derivatives** What happens to the derivatives of $\sin x$ and $\cos x$ if $x$ is measured in degrees instead of radians? To find out, take the following steps.

**a.** With your graphing calculator or computer grapher in *degree mode*, graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h\to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit *should* be $\pi/180$?

**b.** With your grapher still in degree mode, estimate

$$\lim_{h\to 0} \frac{\cos h - 1}{h}.$$

**c.** Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

**d.** Work through the derivation of the formula for the derivative of $\cos x$ using degree-mode limits. What formula do you obtain for the derivative?

**e.** The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. Try it. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?

# 3.6 The Chain Rule

How do we differentiate $F(x) = \sin(x^2 - 4)$? This function is the composite $f \circ g$ of two functions $y = f(u) = \sin u$ and $u = g(x) = x^2 - 4$ that we know how to differentiate. The answer, given by the *Chain Rule*, says that the derivative is the product of the derivatives of $f$ and $g$. We develop the rule in this section.

## Derivative of a Composite Function

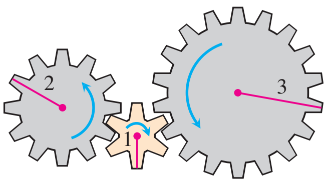
The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composite of the functions $y = \frac{1}{2}u$ and $u = 3x$. We have

$$\frac{dy}{dx} = \frac{3}{2}, \qquad \frac{dy}{du} = \frac{1}{2}, \qquad \text{and} \qquad \frac{du}{dx} = 3.$$

Since $\frac{3}{2} = \frac{1}{2} \cdot 3$, we see in this case that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. If $y = f(u)$ changes half as fast as $u$ and $u = g(x)$ changes three times as fast as $x$, then we expect $y$ to change $3/2$ times as fast as $x$. This effect is much like that of a multiple gear train (Figure 3.25). Let's look at another example.




C: $y$ turns B: $u$ turns A: $x$ turns

**FIGURE 3.25** When gear A makes $x$ turns, gear B makes $u$ turns and gear C makes $y$ turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

**EXAMPLE 1** The function

$$y = (3x^2 + 1)^2$$

is the composite of $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du}\cdot\frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1)\cdot 6x \\ &= 36x^3 + 12x.\end{aligned}$$

Calculating the derivative from the expanded formula $(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$ gives the same result:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x.\end{aligned}$$

The derivative of the composite function $f(g(x))$ at $x$ is the derivative of $f$ at $g(x)$ times the derivative of $g$ at $x$. This is known as the Chain Rule (Figure 3.26).

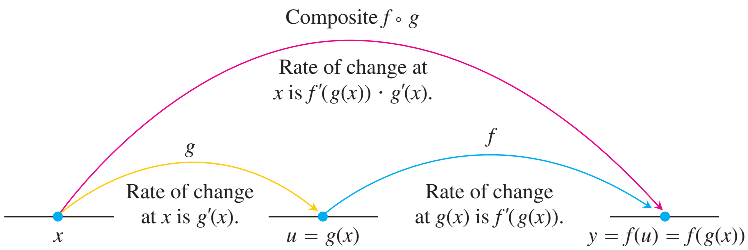



**FIGURE 3.26** Rates of change multiply: The derivative of $f \circ g$ at $x$ is the derivative of $f$ at $g(x)$ times the derivative of $g$ at $x$.

**THEOREM 2—The Chain Rule** If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at $x$, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at $x$, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx},$$

where $dy/du$ is evaluated at $u = g(x)$.

## Intuitive "Proof" of the Chain Rule:

Let $\Delta u$ be the change in $u$ when $x$ changes by $\Delta x$, so that

$$\Delta u = g(x + \Delta x) - g(x).$$

Then the corresponding change in $y$ is

$$\Delta y = f(u + \Delta u) - f(u).$$

If $\Delta u \neq 0$, we can write the fraction $\Delta y/\Delta x$ as the product

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x} \qquad (1)$$

and take the limit as $\Delta x \to 0$:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \qquad \text{(Note that } \Delta u \to 0 \text{ as } \Delta x \to 0 \text{ since } g \text{ is continuous.)} \\
&= \frac{dy}{du} \cdot \frac{du}{dx}.
\end{aligned}$$

The problem with this argument is that it could be true that $\Delta u = 0$ even when $\Delta x \neq 0$, so the cancellation of $\Delta u$ in Equation (1) would be invalid. A proof requires a different approach that avoids this flaw, and we give one such proof in Section 3.11. ■

**EXAMPLE 2** An object moves along the $x$-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of $t$.

**Solution** We know that the velocity is $dx/dt$. In this instance, $x$ is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\begin{aligned}
\frac{dx}{du} &= -\sin(u) \qquad x = \cos(u) \\
\frac{du}{dt} &= 2t. \qquad u = t^2 + 1
\end{aligned}$$

By the Chain Rule,

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\
&= -\sin(u) \cdot 2t \qquad \frac{dx}{du} \text{ evaluated at } u \\
&= -\sin(t^2 + 1) \cdot 2t \\
&= -2t \sin(t^2 + 1).
\end{aligned}$$

■

## "Outside-Inside" Rule

A difficulty with the Leibniz notation is that it doesn't state specifically where the derivatives in the Chain Rule are supposed to be evaluated. So it sometimes helps to think about the Chain Rule using functional notation. If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the "outside" function $f$ and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

**EXAMPLE 3** Differentiate $\sin(x^2 + e^x)$ with respect to $x$.

**Solution** We apply the Chain Rule directly and find

$$\frac{d}{dx} \sin(\underbrace{x^2 + e^x}_{\text{inside}}) = \cos(\underbrace{x^2 + e^x}_{\text{inside left alone}}) \cdot (\underbrace{2x + e^x}_{\text{derivative of the inside}}).$$

■

**EXAMPLE 4** Differentiate $y = e^{\cos x}$.

**Solution** Here the inside function is $u = g(x) = \cos x$ and the outside function is the exponential function $f(x) = e^x$. Applying the Chain Rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x}) = e^{\cos x}\frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -e^{\cos x}\sin x.$$ ■

Generalizing Example 4, we see that the Chain Rule gives the formula

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

Thus, for example,

$$\frac{d}{dx}(e^{kx}) = e^{kx}\cdot\frac{d}{dx}(kx) = ke^{kx}, \qquad \text{for any constant } k$$

and

$$\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2}\cdot\frac{d}{dx}(x^2) = 2xe^{x^2}.$$

## Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

HISTORICAL BIOGRAPHY

Johann Bernoulli
(1667–1748)

**EXAMPLE 5** Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

**Solution** Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of $t$. Therefore, by the Chain Rule,

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t)\cdot\frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\ &= \sec^2(5 - \sin 2t)\cdot\left(0 - \cos 2t\cdot\frac{d}{dt}(2t)\right) \\ &= \sec^2(5 - \sin 2t)\cdot(-\cos 2t)\cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t). \end{aligned}$$ ■

## The Chain Rule with Powers of a Function

If $f$ is a differentiable function of $u$ and if $u$ is a differentiable function of $x$, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}.$$

If $n$ is any real number and $f$ is a power function, $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If $u$ is a differentiable function of $x$, then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}\left(u^n\right) = nu^{n-1}$$

**EXAMPLE 6** The Power Chain Rule simplifies computing the derivative of a power of an expression.

**(a)** $$\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4)$$ Power Chain Rule with $u = 5x^3 - x^4$, $n = 7$

$$= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3)$$

$$= 7(5x^3 - x^4)^6(15x^2 - 4x^3)$$

**(b)** $$\frac{d}{dx}\left(\frac{1}{3x - 2}\right) = \frac{d}{dx}(3x - 2)^{-1}$$

$$= -1(3x - 2)^{-2} \frac{d}{dx}(3x - 2)$$ Power Chain Rule with $u = 3x - 2$, $n = -1$

$$= -1(3x - 2)^{-2}(3)$$

$$= -\frac{3}{(3x - 2)^2}$$

In part (b) we could also find the derivative with the Derivative Quotient Rule.

**(c)** $$\frac{d}{dx}(\sin^5 x) = 5\sin^4 x \cdot \frac{d}{dx}\sin x$$ Power Chain Rule with $u = \sin x$, $n = 5$, because $\sin^n x$ means $(\sin x)^n$, $n \neq -1$.

$$= 5\sin^4 x \cos x$$

**(d)** $$\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}} \cdot \frac{d}{dx}\left(\sqrt{3x + 1}\right)$$

$$= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x + 1)^{-1/2} \cdot 3$$ Power Chain Rule with $u = 3x + 1$, $n = 1/2$

$$= \frac{3}{2\sqrt{3x + 1}} e^{\sqrt{3x+1}}$$ ■

**EXAMPLE 7** In Section 3.2, we saw that the absolute value function $y = |x|$ is not differentiable at $x = 0$. However, the function *is* differentiable at all other real numbers as we now show. Since $|x| = \sqrt{x^2}$, we can derive the following formula:

**Derivative of the Absolute Value Function**

$$\frac{d}{dx}(|x|) = \frac{x}{|x|}, \quad x \neq 0$$

$$\frac{d}{dx}(|x|) = \frac{d}{dx}\sqrt{x^2}$$

$$= \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2)$$ Power Chain Rule with $u = x^2$, $n = 1/2$, $x \neq 0$

$$= \frac{1}{2|x|} \cdot 2x$$ $\sqrt{x^2} = |x|$

$$= \frac{x}{|x|}, \quad x \neq 0.$$ ■

**EXAMPLE 8** Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

**Solution** We find the derivative:

$$\frac{dy}{dx} = \frac{d}{dx}(1 - 2x)^{-3}$$

$$= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x)$$ Power Chain Rule with $u = (1 - 2x)$, $n = -3$

$$= -3(1 - 2x)^{-4} \cdot (-2)$$

$$= \frac{6}{(1 - 2x)^4}.$$

At any point $(x, y)$ on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4},$$

the quotient of two positive numbers. ■

**EXAMPLE 9** The formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that $x$ is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians where $x°$ is the size of the angle measured in degrees.

By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

See Figure 3.27. Similarly, the derivative of $\cos(x°)$ is $-(\pi/180)\sin(x°)$.

The factor $\pi/180$ would compound with repeated differentiation. We see here the advantage for the use of radian measure in computations. ■

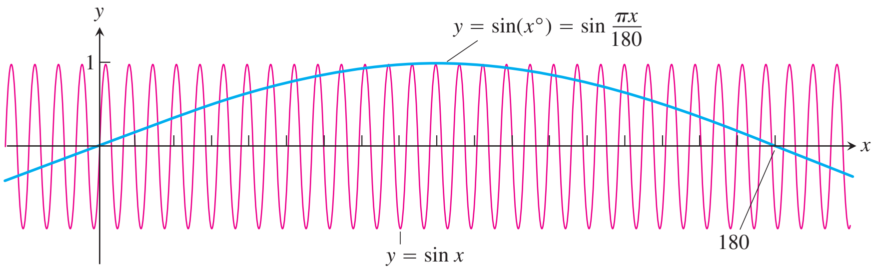



**FIGURE 3.27** $\sin(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 9).

## Exercises 3.6

### Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = 2u^3, \quad u = 8x - 1$
3. $y = \sin u, \quad u = 3x + 1$
4. $y = \cos u, \quad u = -x/3$
5. $y = \cos u, \quad u = \sin x$
6. $y = \sin u, \quad u = x - \cos x$
7. $y = \tan u, \quad u = 10x - 5$
8. $y = -\sec u, \quad u = x^2 + 7x$

In Exercises 9–22, write the function in the form $y = f(u)$ and $u = g(x)$. Then find $dy/dx$ as a function of $x$.

9. $y = (2x + 1)^5$
10. $y = (4 - 3x)^9$
11. $y = \left(1 - \frac{x}{7}\right)^{-7}$
12. $y = \left(\frac{x}{2} - 1\right)^{-10}$
13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
14. $y = \sqrt{3x^2 - 4x + 6}$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \frac{1}{x}\right)$
17. $y = \sin^3 x$
18. $y = 5\cos^{-4} x$
19. $y = e^{-5x}$
20. $y = e^{2x/3}$
21. $y = e^{5-7x}$
22. $y = e^{\left(4\sqrt{x}+x^2\right)}$

Find the derivatives of the functions in Exercises 23–50.

23. $p = \sqrt{3 - t}$
24. $q = \sqrt[3]{2r - r^2}$
25. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$
26. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$
27. $r = (\csc\theta + \cot\theta)^{-1}$
28. $r = 6(\sec\theta - \tan\theta)^{3/2}$
29. $y = x^2\sin^4 x + x\cos^{-2} x$
30. $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x$
31. $y = \frac{1}{21}(3x - 2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1}$
32. $y = (5 - 2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$
33. $y = (4x + 3)^4(x + 1)^{-3}$
34. $y = (2x - 5)^{-1}(x^2 - 5x)^6$
35. $y = xe^{-x} + e^{3x}$
36. $y = (1 + 2x)e^{-2x}$
37. $y = (x^2 - 2x + 2)e^{5x/2}$
38. $y = (9x^2 - 6x + 2)e^{x^3}$
39. $h(x) = x\tan\left(2\sqrt{x}\right) + 7$
40. $k(x) = x^2\sec\left(\frac{1}{x}\right)$

**41.** $f(x) = \sqrt{7 + x \sec x}$

**42.** $g(x) = \dfrac{\tan 3x}{(x + 7)^4}$

**43.** $f(\theta) = \left(\dfrac{\sin\theta}{1 + \cos\theta}\right)^2$

**44.** $g(t) = \left(\dfrac{1 + \sin 3t}{3 - 2t}\right)^{-1}$

**45.** $r = \sin(\theta^2)\cos(2\theta)$

**46.** $r = \sec\sqrt{\theta}\tan\left(\dfrac{1}{\theta}\right)$

**47.** $q = \sin\left(\dfrac{t}{\sqrt{t + 1}}\right)$

**48.** $q = \cot\left(\dfrac{\sin t}{t}\right)$

**49.** $y = \cos\left(e^{-\theta^2}\right)$

**50.** $y = \theta^3 e^{-2\theta}\cos 5\theta$

In Exercises 51–70, find $dy/dt$.

**51.** $y = \sin^2(\pi t - 2)$

**52.** $y = \sec^2 \pi t$

**53.** $y = (1 + \cos 2t)^{-4}$

**54.** $y = (1 + \cot(t/2))^{-2}$

**55.** $y = (t \tan t)^{10}$

**56.** $y = (t^{-3/4}\sin t)^{4/3}$

**57.** $y = e^{\cos^2(\pi t - 1)}$

**58.** $y = \left(e^{\sin(t/2)}\right)^3$

**59.** $y = \left(\dfrac{t^2}{t^3 - 4t}\right)^3$

**60.** $y = \left(\dfrac{3t - 4}{5t + 2}\right)^{-5}$

**61.** $y = \sin(\cos(2t - 5))$

**62.** $y = \cos\left(5\sin\left(\dfrac{t}{3}\right)\right)$

**63.** $y = \left(1 + \tan^4\left(\dfrac{t}{12}\right)\right)^3$

**64.** $y = \dfrac{1}{6}\left(1 + \cos^2(7t)\right)^3$

**65.** $y = \sqrt{1 + \cos(t^2)}$

**66.** $y = 4\sin\left(\sqrt{1 + \sqrt{t}}\right)$

**67.** $y = \tan^2(\sin^3 t)$

**68.** $y = \cos^4(\sec^2 3t)$

**69.** $y = 3t(2t^2 - 5)^4$

**70.** $y = \sqrt{3t + \sqrt{2 + \sqrt{1 - t}}}$

## Second Derivatives

Find $y''$ in Exercises 71–78.

**71.** $y = \left(1 + \dfrac{1}{x}\right)^3$

**72.** $y = \left(1 - \sqrt{x}\right)^{-1}$

**73.** $y = \dfrac{1}{9}\cot(3x - 1)$

**74.** $y = 9\tan\left(\dfrac{x}{3}\right)$

**75.** $y = x(2x + 1)^4$

**76.** $y = x^2(x^3 - 1)^5$

**77.** $y = e^{x^2} + 5x$

**78.** $y = \sin(x^2 e^x)$

## Finding Derivative Values

In Exercises 79–84, find the value of $(f \circ g)'$ at the given value of $x$.

**79.** $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

**80.** $f(u) = 1 - \dfrac{1}{u}, \quad u = g(x) = \dfrac{1}{1 - x}, \quad x = -1$

**81.** $f(u) = \cot\dfrac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

**82.** $f(u) = u + \dfrac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$

**83.** $f(u) = \dfrac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

**84.** $f(u) = \left(\dfrac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \dfrac{1}{x^2} - 1, \quad x = -1$

**85.** Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is $y'$ at $x = 2$?

**86.** If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is $dr/dt$ at $t = 0$?

**87.** Suppose that functions $f$ and $g$ and their derivatives with respect to $x$ have the following values at $x = 2$ and $x = 3$.

| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 2 | 8 | 2 | 1/3 | −3 |
| 3 | 3 | −4 | $2\pi$ | 5 |

Find the derivatives with respect to $x$ of the following combinations at the given value of $x$.

**a.** $2f(x), \quad x = 2$

**b.** $f(x) + g(x), \quad x = 3$

**c.** $f(x) \cdot g(x), \quad x = 3$

**d.** $f(x)/g(x), \quad x = 2$

**e.** $f(g(x)), \quad x = 2$

**f.** $\sqrt{f(x)}, \quad x = 2$

**g.** $1/g^2(x), \quad x = 3$

**h.** $\sqrt{f^2(x) + g^2(x)}, \quad x = 2$

**88.** Suppose that the functions $f$ and $g$ and their derivatives with respect to $x$ have the following values at $x = 0$ and $x = 1$.

| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 5 | 1/3 |
| 1 | 3 | −4 | −1/3 | −8/3 |

Find the derivatives with respect to $x$ of the following combinations at the given value of $x$.

**a.** $5f(x) - g(x), \quad x = 1$

**b.** $f(x)g^3(x), \quad x = 0$

**c.** $\dfrac{f(x)}{g(x) + 1}, \quad x = 1$

**d.** $f(g(x)), \quad x = 0$

**e.** $g(f(x)), \quad x = 0$

**f.** $(x^{11} + f(x))^{-2}, \quad x = 1$

**g.** $f(x + g(x)), \quad x = 0$

**89.** Find $ds/dt$ when $\theta = 3\pi/2$ if $s = \cos\theta$ and $d\theta/dt = 5$.

**90.** Find $dy/dt$ when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

## Theory and Examples

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 91 and 92.

**91.** Find $dy/dx$ if $y = x$ by using the Chain Rule with $y$ as a composite of

**a.** $y = (u/5) + 7$ and $u = 5x - 35$

**b.** $y = 1 + (1/u)$ and $u = 1/(x - 1)$.

**92.** Find $dy/dx$ if $y = x^{3/2}$ by using the Chain Rule with $y$ as a composite of

**a.** $y = u^3$ and $u = \sqrt{x}$

**b.** $y = \sqrt{u}$ and $u = x^3$.

**93.** Find the tangent to $y = ((x - 1)/(x + 1))^2$ at $x = 0$.

**94.** Find the tangent to $y = \sqrt{x^2 - x + 7}$ at $x = 2$.

**95. a.** Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$.

**b. Slopes on a tangent curve** What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

**96. Slopes on sine curves**

**a.** Find equations for the tangents to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangents are related? Give reasons for your answer.

**b.** Can anything be said about the tangents to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin ($m$ a constant $\neq 0$)? Give reasons for your answer.

**c.** For a given $m$, what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.

**d.** The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

**97. Running machinery too fast** Suppose that a piston is moving straight up and down and that its position at time $t$ sec is

$$s = A\cos(2\pi bt),$$

with $A$ and $b$ positive. The value of $A$ is the amplitude of the motion, and $b$ is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why some machinery breaks when you run it too fast.)

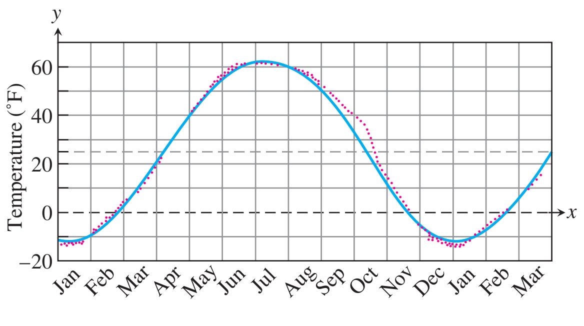
**98. Temperatures in Fairbanks, Alaska** The graph in the accompanying figure shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day $x$ is

$$y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25$$

and is graphed in the accompanying figure.

**a.** On what day is the temperature increasing the fastest?

**b.** About how many degrees per day is the temperature increasing when it is increasing at its fastest?




**99. Particle motion** The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with $s$ in meters and $t$ in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

**100. Constant acceleration** Suppose that the velocity of a falling body is $v = k\sqrt{s}$ m/sec ($k$ a constant) at the instant the body has fallen $s$ m from its starting point. Show that the body's acceleration is constant.

**101. Falling meteorite** The velocity of a heavy meteorite entering Earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is $s$ km from Earth's center. Show that the meteorite's acceleration is inversely proportional to $s^2$.

**102. Particle acceleration** A particle moves along the $x$-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

**103. Temperature and the period of a pendulum** For oscillations of small amplitude (short swings), we may safely model the relationship between the period $T$ and the length $L$ of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where $g$ is the constant acceleration of gravity at the pendulum's location. If we measure $g$ in centimeters per second squared, we measure $L$ in centimeters and $T$ in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to $L$. In symbols, with $u$ being temperature and $k$ the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

**104. Chain Rule** Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though $g$ itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

T **105. The derivative of sin 2x** Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x + h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and $0.2$. Experiment with other values of $h$, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

**106. The derivative of cos (x²)** Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on the same screen, graph

$$y = \frac{\cos((x + h)^2) - \cos(x^2)}{h}$$

for $h = 1.0, 0.7$, and $0.3$. Experiment with other values of $h$. What do you see happening as $h \to 0$? Explain this behavior.

Using the Chain Rule, show that the Power Rule $(d/dx)x^n = nx^{n-1}$ holds for the functions $x^n$ in Exercises 107 and 108.

**107.** $x^{1/4} = \sqrt{\sqrt{x}}$ **108.** $x^{3/4} = \sqrt{x\sqrt{x}}$

## COMPUTER EXPLORATIONS

### Trigonometric Polynomials

**109.** As the accompanying figure shows, the trigonometric "polynomial"

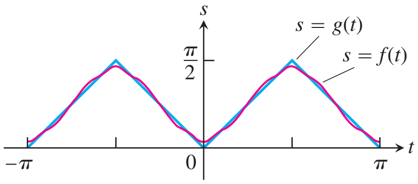
$$s = f(t) = 0.78540 - 0.63662\cos 2t - 0.07074\cos 6t - 0.02546\cos 10t - 0.01299\cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of $f$ approximate the derivative of $g$ at the points where $dg/dt$ is defined? To find out, carry out the following steps.

**a.** Graph $dg/dt$ (where defined) over $[-\pi, \pi]$.

**b.** Find $df/dt$.

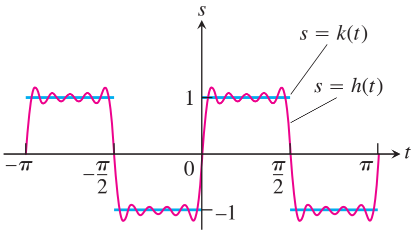
**c.** Graph $df/dt$. Where does the approximation of $dg/dt$ by $df/dt$ seem to be best? Least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.




**110.** (*Continuation of Exercise 109.*) In Exercise 109, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the "polynomial"

$$s = h(t) = 1.2732 \sin 2t + 0.4244 \sin 6t + 0.25465 \sin 10t + 0.18189 \sin 14t + 0.14147 \sin 18t$$

graphed in the accompanying figure approximates the step function $s = k(t)$ shown there. Yet the derivative of $h$ is nothing like the derivative of $k$.




**a.** Graph $dk/dt$ (where defined) over $[-\pi, \pi]$.

**b.** Find $dh/dt$.

**c.** Graph $dh/dt$ to see how badly the graph fits the graph of $dk/dt$. Comment on what you see.

# 3.7 Implicit Differentiation

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses $y$ explicitly in terms of the variable $x$. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0, \qquad y^2 - x = 0, \qquad \text{or} \qquad x^2 + y^2 - 25 = 0.$$

(See Figures 3.28, 3.29, and 3.30.) These equations define an *implicit* relation between the variables $x$ and $y$. In some cases we may be able to solve such an equation for $y$ as an explicit function (or even several functions) of $x$. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find $dy/dx$ by *implicit differentiation*. This section describes the technique.

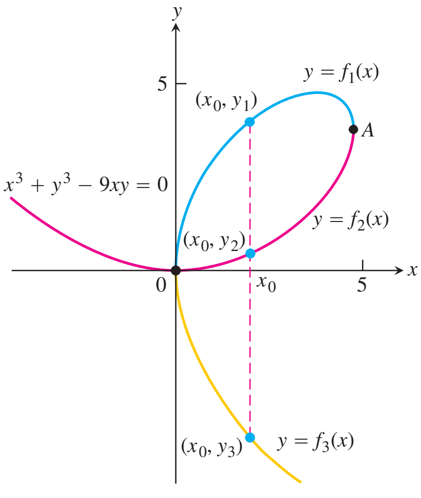



**FIGURE 3.28** The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of $x$. The curve can, however, be divided into separate arcs that *are* the graphs of functions of $x$. This particular curve, called a *folium*, dates to Descartes in 1638.

## Implicitly Defined Functions

We begin with examples involving familiar equations that we can solve for $y$ as a function of $x$ to calculate $dy/dx$ in the usual way. Then we differentiate the equations implicitly, and find the derivative to compare the two methods. Following the examples, we summarize the steps involved in the new method. In the examples and exercises, it is always assumed that the given equation determines $y$ implicitly as a differentiable function of $x$ so that $dy/dx$ exists.

**EXAMPLE 1** Find $dy/dx$ if $y^2 = x$.

**Solution** The equation $y^2 = x$ defines two differentiable functions of $x$ that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Figure 3.29). We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \qquad \text{and} \qquad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

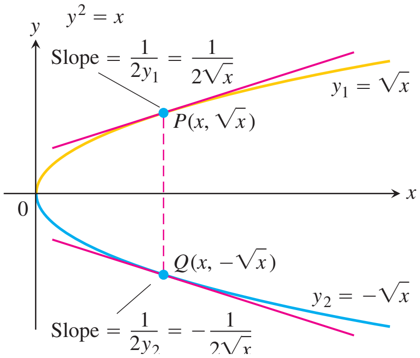



**FIGURE 3.29** The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of $x$ on the interval $x > 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for $y$.

But suppose that we knew only that the equation $y^2 = x$ defined $y$ as one or more differentiable functions of $x$ for $x > 0$ without knowing exactly what these functions were. Could we still find $dy/dx$?

The answer is yes. To find $dy/dx$, we simply differentiate both sides of the equation $y^2 = x$ with respect to $x$, treating $y = f(x)$ as a differentiable function of $x$:

$$y^2 = x$$
$$2y\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{2y}.$$

The Chain Rule gives $\frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}$.

This one formula gives the derivatives we calculated for *both* explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2\left(-\sqrt{x}\right)} = -\frac{1}{2\sqrt{x}}.$$

■

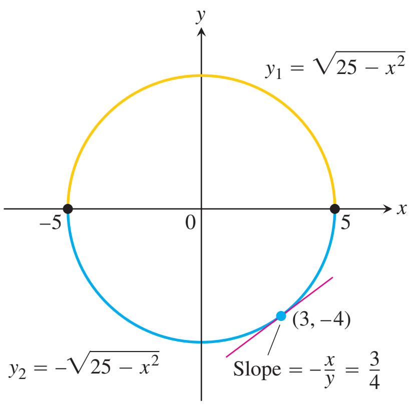



**FIGURE 3.30** The circle combines the graphs of two functions. The graph of $y_2$ is the lower semicircle and passes through $(3, -4)$.

**EXAMPLE 2** Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

**Solution** The circle is not the graph of a single function of $x$. Rather it is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 3.30). The point $(3, -4)$ lies on the graph of $y_2$, so we can find the slope by calculating the derivative directly, using the Power Chain Rule:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

$\frac{d}{dx} - (25 - x^2)^{1/2} = -\frac{1}{2}(25 - x^2)^{-1/2}(-2x)$

We can solve this problem more easily by differentiating the given equation of the circle implicitly with respect to $x$:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $\left.-\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$

Notice that unlike the slope formula for $dy_2/dx$, which applies only to points below the $x$-axis, the formula $dy/dx = -x/y$ applies everywhere the circle has a slope. Notice also that the derivative involves *both* variables $x$ and $y$, not just the independent variable $x$. ■

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2: We treat $y$ as a differentiable implicit function of $x$ and apply the usual rules to differentiate both sides of the defining equation.

> **Implicit Differentiation**
>
> **1.** Differentiate both sides of the equation with respect to $x$, treating $y$ as a differentiable function of $x$.
>
> **2.** Collect the terms with $dy/dx$ on one side of the equation and solve for $dy/dx$.

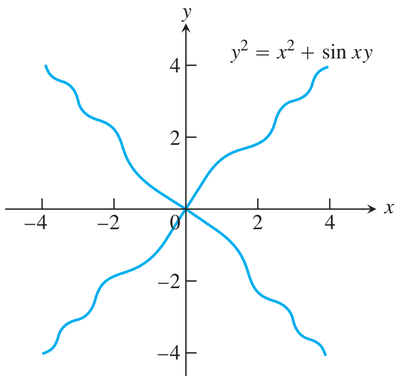



**FIGURE 3.31** The graph of $y^2 = x^2 + \sin xy$ in Example 3.

**EXAMPLE 3** Find $dy/dx$ if $y^2 = x^2 + \sin xy$ (Figure 3.31).

**Solution** We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sin xy\right) \qquad \text{Differentiate both sides with respect to } x \ldots$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy) \qquad \ldots \text{ treating } y \text{ as a function of } x \text{ and using the Chain Rule.}$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right) \qquad \text{Treat } xy \text{ as a product.}$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y \qquad \text{Collect terms with } dy/dx.$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy} \qquad \text{Solve for } dy/dx.$$

Notice that the formula for $dy/dx$ applies everywhere that the implicitly defined curve has a slope. Notice again that the derivative involves *both* variables $x$ and $y$, not just the independent variable $x$. ■

## Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives.

**EXAMPLE 4** Find $d^2y/dx^2$ if $2x^3 - 3y^2 = 8$.

**Solution** To start, we differentiate both sides of the equation with respect to $x$ in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \text{Treat } y \text{ as a function of } x.$$

$$y' = \frac{x^2}{y}, \quad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find $y''$.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\cdot y'$$

Finally, we substitute $y' = x^2/y$ to express $y''$ in terms of $x$ and $y$.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$ ■

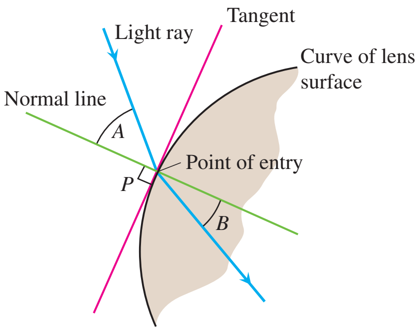



**FIGURE 3.32** The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

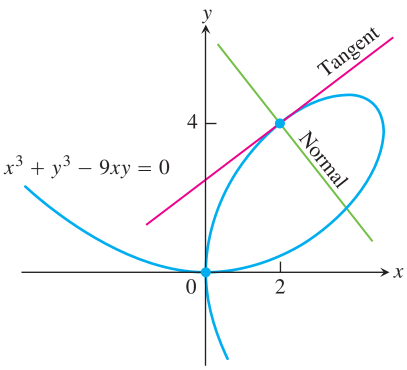



**FIGURE 3.33** Example 5 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

## Lenses, Tangents, and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles $A$ and $B$ in Figure 3.32). This line is called the *normal* to the surface at the point of entry. In a profile view of a lens like the one in Figure 3.32, the **normal** is the line perpendicular to the tangent of the profile curve at the point of entry.

**EXAMPLE 5** Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.33).

**Solution** The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for $dy/dx$:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$

Differentiate both sides with respect to $x$.

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$

Treat $xy$ as a product and $y$ as a function of $x$.

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}.$$

Solve for $dy/dx$.

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at (2, 4) is the line through (2, 4) with slope $4/5$:

$$y = 4 + \frac{4}{5}(x - 2)$$

$$y = \frac{4}{5}x + \frac{12}{5}.$$

The normal to the curve at (2, 4) is the line perpendicular to the tangent there, the line through (2, 4) with slope $-5/4$:

$$y = 4 - \frac{5}{4}(x - 2)$$

$$y = -\frac{5}{4}x + \frac{13}{2}.$$

■

The quadratic formula enables us to solve a second-degree equation like $y^2 - 2xy + 3x^2 = 0$ for $y$ in terms of $x$. There is a formula for the three roots of a cubic equation that is like the quadratic formula but much more complicated. If this formula is used to solve the equation $x^3 + y^3 = 9xy$ in Example 5 for $y$ in terms of $x$, then three functions determined by the equation are

$$y = f(x) = \sqrt[3]{-\frac{x^3}{2} + \sqrt{\frac{x^6}{4} - 27x^3}} + \sqrt[3]{-\frac{x^3}{2} - \sqrt{\frac{x^6}{4} - 27x^3}}$$

and

$$y = \frac{1}{2}\left[-f(x) \pm \sqrt{-3}\left(\sqrt[3]{-\frac{x^3}{2} + \sqrt{\frac{x^6}{4} - 27x^3}} - \sqrt[3]{-\frac{x^3}{2} - \sqrt{\frac{x^6}{4} - 27x^3}}\right)\right].$$

Using implicit differentiation in Example 5 was much simpler than calculating $dy/dx$ directly from any of the above formulas. Finding slopes on curves defined by higher-degree equations usually requires implicit differentiation.

## Exercise 3.7

### Differentiating Implicitly

Use implicit differentiation to find $dy/dx$ in Exercises 1–16.

**1.** $x^2y + xy^2 = 6$

**2.** $x^3 + y^3 = 18xy$

**3.** $2xy + y^2 = x + y$

**4.** $x^3 - xy + y^3 = 1$

**5.** $x^2(x - y)^2 = x^2 - y^2$

**6.** $(3xy + 7)^2 = 6y$

**7.** $y^2 = \dfrac{x - 1}{x + 1}$

**8.** $x^3 = \dfrac{2x - y}{x + 3y}$

**9.** $x = \tan y$

**10.** $xy = \cot(xy)$

**11.** $x + \tan(xy) = 0$

**12.** $x^4 + \sin y = x^3y^2$

**13.** $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$

**14.** $x\cos(2x + 3y) = y\sin x$

**15.** $e^{2x} = \sin(x + 3y)$

**16.** $e^{x^2y} = 2x + 2y$

Find $dr/d\theta$ in Exercises 17–20.

**17.** $\theta^{1/2} + r^{1/2} = 1$

**18.** $r - 2\sqrt{\theta} = \dfrac{3}{2}\theta^{2/3} + \dfrac{4}{3}\theta^{3/4}$

**19.** $\sin(r\theta) = \dfrac{1}{2}$

**20.** $\cos r + \cot\theta = e^{r\theta}$

### Second Derivatives

In Exercises 21–26, use implicit differentiation to find $dy/dx$ and then $d^2y/dx^2$.

**21.** $x^2 + y^2 = 1$

**22.** $x^{2/3} + y^{2/3} = 1$

**23.** $y^2 = e^{x^2} + 2x$

**24.** $y^2 - 2x = 1 - 2y$

**25.** $2\sqrt{y} = x - y$

**26.** $xy + y^2 = 1$

**27.** If $x^3 + y^3 = 16$, find the value of $d^2y/dx^2$ at the point $(2, 2)$.

**28.** If $xy + y^2 = 1$, find the value of $d^2y/dx^2$ at the point $(0, -1)$.

In Exercises 29 and 30, find the slope of the curve at the given points.

**29.** $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$

**30.** $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

### Slopes, Tangents, and Normals

In Exercises 31–40, verify that the given point is on the curve and find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

**31.** $x^2 + xy - y^2 = 1, \quad (2, 3)$

**32.** $x^2 + y^2 = 25, \quad (3, -4)$

**33.** $x^2y^2 = 9, \quad (-1, 3)$

**34.** $y^2 - 2x - 4y - 1 = 0, \quad (-2, 1)$

**35.** $6x^2 + 3xy + 2y^2 + 17y - 6 = 0, \quad (-1, 0)$

**36.** $x^2 - \sqrt{3}xy + 2y^2 = 5, \quad (\sqrt{3}, 2)$

**37.** $2xy + \pi\sin y = 2\pi, \quad (1, \pi/2)$

**38.** $x\sin 2y = y\cos 2x, \quad (\pi/4, \pi/2)$

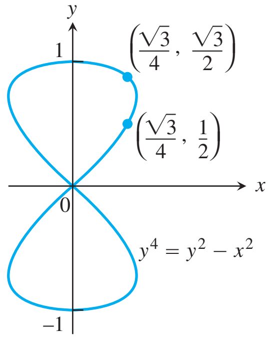
**39.** $y = 2\sin(\pi x - y), \quad (1, 0)$

**40.** $x^2\cos^2 y - \sin y = 0, \quad (0, \pi)$

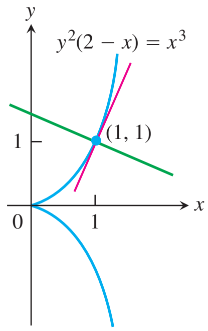
**41. Parallel tangents** Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the $x$-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

**42. Normals parallel to a line** Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

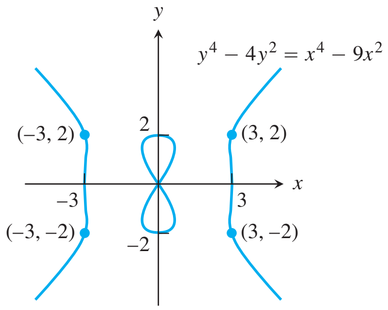
**43. The eight curve** Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.




**44. The cissoid of Diocles (from about 200 B.C.)** Find equations for the tangent and normal to the cissoid of Diocles $y^2(2 - x) = x^3$ at $(1, 1)$.




**45. The devil's curve (Gabriel Cramer, 1750)** Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.




**46. The folium of Descartes** (See Figure 3.28.)

**a.** Find the slope of the folium of Descartes $x^3 + y^3 - 9xy = 0$ at the points (4, 2) and (2, 4).

**b.** At what point other than the origin does the folium have a horizontal tangent?

**c.** Find the coordinates of the point $A$ in Figure 3.28, where the folium has a vertical tangent.

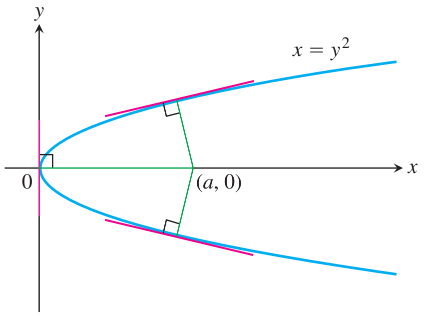
## Theory and Examples

**47. Intersecting normal** The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1, 1) intersects the curve at what other point?

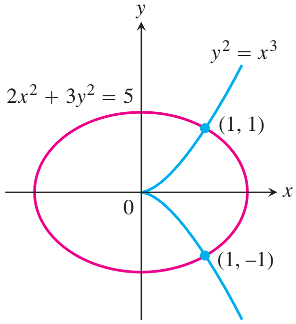
**48. Power rule for rational exponents** Let $p$ and $q$ be integers with $q > 0$. If $y = x^{p/q}$, differentiate the equivalent equation $y^q = x^p$ implicitly and show that, for $y \neq 0$,

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

**49. Normals to a parabola** Show that if it is possible to draw three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown in the accompanying diagram, then $a$ must be greater than 1/2. One of the normals is the $x$-axis. For what value of $a$ are the other two normals perpendicular?




**50.** Is there anything special about the tangents to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for your answer.

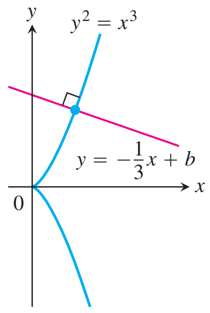



**51.** Verify that the following pairs of curves meet orthogonally.

**a.** $x^2 + y^2 = 4, \quad x^2 = 3y^2$

**b.** $x = 1 - y^2, \quad x = \frac{1}{3}y^2$

**52.** The graph of $y^2 = x^3$ is called a **semicubical parabola** and is shown in the accompanying figure. Determine the constant $b$ so that the line $y = -\frac{1}{3}x + b$ meets this graph orthogonally.




**T** In Exercises 53 and 54, find both $dy/dx$ (treating $y$ as a differentiable function of $x$) and $dx/dy$ (treating $x$ as a differentiable function of $y$). How do $dy/dx$ and $dx/dy$ seem to be related? Explain the relationship geometrically in terms of the graphs.

**53.** $xy^3 + x^2y = 6$

**54.** $x^3 + y^2 = \sin^2 y$

## COMPUTER EXPLORATIONS

Use a CAS to perform the following steps in Exercises 55–62.

**a.** Plot the equation with the implicit plotter of a CAS. Check to see that the given point $P$ satisfies the equation.

**b.** Using implicit differentiation, find a formula for the derivative $dy/dx$ and evaluate it at the given point $P$.

**c.** Use the slope found in part (b) to find an equation for the tangent line to the curve at $P$. Then plot the implicit curve and tangent line together on a single graph.

**55.** $x^3 - xy + y^3 = 7, \quad P(2, 1)$

**56.** $x^5 + y^3x + yx^2 + y^4 = 4, \quad P(1, 1)$

**57.** $y^2 + y = \dfrac{2 + x}{1 - x}, \quad P(0, 1)$

**58.** $y^3 + \cos xy = x^2, \quad P(1, 0)$

**59.** $x + \tan\left(\dfrac{y}{x}\right) = 2, \quad P\left(1, \dfrac{\pi}{4}\right)$

**60.** $xy^3 + \tan(x + y) = 1, \quad P\left(\dfrac{\pi}{4}, 0\right)$

**61.** $2y^2 + (xy)^{1/3} = x^2 + 2, \quad P(1, 1)$

**62.** $x\sqrt{1 + 2y} + y = x^2, \quad P(1, 0)$

# 3.8 Derivatives of Inverse Functions and Logarithms

In Section 1.6 we saw how the inverse of a function undoes, or inverts, the effect of that function. We defined there the natural logarithm function $f^{-1}(x) = \ln x$ as the inverse of the natural exponential function $f(x) = e^x$. This is one of the most important function-inverse pairs in mathematics and science. We learned how to differentiate the exponential function in Section 3.3. Here we learn a rule for differentiating the inverse of a differentiable function and we apply the rule to find the derivative of the natural logarithm function.

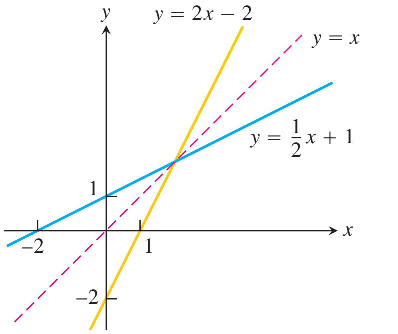



**FIGURE 3.34** Graphing a line and its inverse together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other.

## Derivatives of Inverses of Differentiable Functions

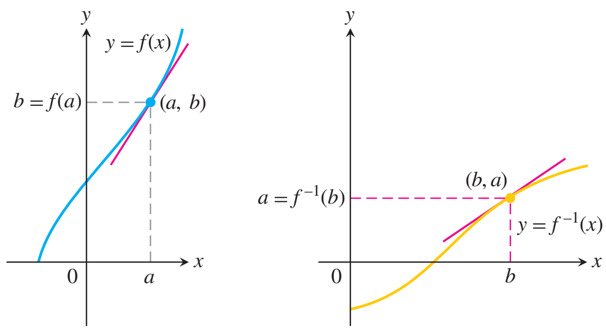
We calculated the inverse of the function $f(x) = (1/2)x + 1$ as $f^{-1}(x) = 2x - 2$ in Example 3 of Section 1.6. Figure 3.34 shows again the graphs of both functions. If we calculate their derivatives, we see that

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$

The derivatives are reciprocals of one another, so the slope of one line is the reciprocal of the slope of its inverse line. (See Figure 3.34.)

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$, the reflected line has slope $1/m$.




The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

**FIGURE 3.35** The graphs of inverse functions have reciprocal slopes at corresponding points.

The reciprocal relationship between the slopes of $f$ and $f^{-1}$ holds for other functions as well, but we must be careful to compare slopes at corresponding points. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$ and $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the point $(f(a), a)$ is the reciprocal $1/f'(a)$ (Figure 3.35). If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

If $y = f(x)$ has a horizontal tangent line at $(a, f(a))$ then the inverse function $f^{-1}$ has a vertical tangent line at $(f(a), a)$, and this infinite slope implies that $f^{-1}$ is not differentiable at $f(a)$. Theorem 3 gives the conditions under which $f^{-1}$ is differentiable in its domain (which is the same as the range of $f$).

**THEOREM 3—The Derivative Rule for Inverses** If $f$ has an interval $I$ as domain and $f'(x)$ exists and is never zero on $I$, then $f^{-1}$ is differentiable at every point in its domain (the range of $f$). The value of $(f^{-1})'$ at a point $b$ in the domain of $f^{-1}$ is the reciprocal of the value of $f'$ at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Theorem 3 makes two assertions. The first of these has to do with the conditions under which $f^{-1}$ is differentiable; the second assertion is a formula for the derivative of $f^{-1}$ when it exists. While we omit the proof of the first assertion, the second one is proved in the following way:

$$f(f^{-1}(x)) = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}f(f^{-1}(x)) = 1 \qquad \text{Differentiating both sides}$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx}f^{-1}(x) = 1 \qquad \text{Chain Rule}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}. \qquad \text{Solving for the derivative}$$

**EXAMPLE 1** The function $f(x) = x^2, x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/\left(2\sqrt{x}\right)$.

Let's verify that Theorem 3 gives the same formula for the derivative of $f^{-1}(x)$:

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}\\ &= \frac{1}{2(f^{-1}(x))} \qquad f'(x) = 2x \text{ with } x \text{ replaced by } f^{-1}(x)\\ &= \frac{1}{2(\sqrt{x})}.\end{aligned}$$

Theorem 3 gives a derivative that agrees with the known derivative of the square root function.

Let's examine Theorem 3 at a specific point. We pick $x = 2$ (the number $a$) and $f(2) = 4$ (the value $b$). Theorem 3 says that the derivative of $f$ at 2, $f'(2) = 4$, and the derivative of $f^{-1}$ at $f(2)$, $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \left.\frac{1}{2x}\right|_{x=2} = \frac{1}{4}.$$

See Figure 3.36. ■

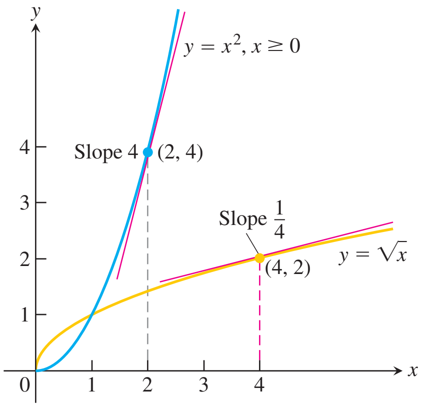



**FIGURE 3.36** The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 1).

We will use the procedure illustrated in Example 1 to calculate formulas for the derivatives of many inverse functions throughout this chapter. Equation (1) sometimes enables us to find specific values of $df^{-1}/dx$ without knowing a formula for $f^{-1}$.

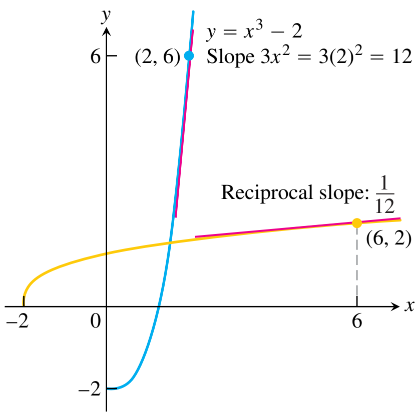



**FIGURE 3.37** The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of $f^{-1}$ at $x = 6$ (Example 2).

**EXAMPLE 2** Let $f(x) = x^3 - 2$. Find the value of $df^{-1}/dx$ at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

**Solution** We apply Theorem 3 to obtain the value of the derivative of $f^{-1}$ at $x = 6$:

$$\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12}. \qquad \text{Eq. (1)}$$

See Figure 3.37. ■

## Derivative of the Natural Logarithm Function

Since we know the exponential function $f(x) = e^x$ is differentiable everywhere, we can apply Theorem 3 to find the derivative of its inverse $f^{-1}(x) = \ln x$:

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
&= \frac{1}{e^{f^{-1}(x)}} && f'(u) = e^u \\
&= \frac{1}{e^{\ln x}} \\
&= \frac{1}{x}. && \text{Inverse function relationship}
\end{aligned}$$

**Alternate Derivation** Instead of applying Theorem 3 directly, we can find the derivative of $y = \ln x$ using implicit differentiation, as follows:

$$\begin{aligned}
y &= \ln x \\
e^y &= x && \text{Inverse function relationship} \\
\frac{d}{dx}(e^y) &= \frac{d}{dx}(x) && \text{Differentiate implicitly} \\
e^y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}. && e^y = x
\end{aligned}$$

No matter which derivation we use, the derivative of $y = \ln x$ with respect to $x$ is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0.$$

The Chain Rule extends this formula for positive functions $u(x)$:

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0. \tag{2}$$

**EXAMPLE 3** We use Equation (2) to find derivatives.

**(a)** $\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$

**(b)** Equation (2) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}. \quad \blacksquare$$

Notice the remarkable occurrence in Example 3a. The function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln bx$ for any constant $b$, provided that $bx > 0$:

$$\frac{d}{dx}\ln bx = \frac{1}{bx}\cdot\frac{d}{dx}(bx) = \frac{1}{bx}(b) = \frac{1}{x}. \qquad (3)$$

If $x < 0$ and $b < 0$, then $bx > 0$ and Equation (3) still applies. In particular, if $x < 0$ and $b = -1$ we get

$$\frac{d}{dx}\ln(-x) = \frac{1}{x} \qquad \text{for } x < 0.$$

Since $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$, we have the following important result.

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad x \neq 0 \qquad (4)$$

**EXAMPLE 4** A line with slope $m$ passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of $m$?

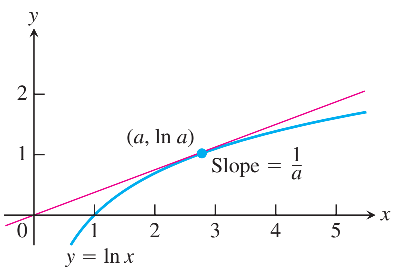



**FIGURE 3.38** The tangent line intersects the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$ (Example 4).

**Solution** Suppose the point of tangency occurs at the unknown point $x = a > 0$. Then we know that the point $(a, \ln a)$ lies on the graph and that the tangent line at that point has slope $m = 1/a$ (Figure 3.38). Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

Setting these two formulas for $m$ equal to each other, we have

$$\begin{aligned} \frac{\ln a}{a} &= \frac{1}{a} \\ \ln a &= 1 \\ e^{\ln a} &= e^1 \\ a &= e \\ m &= \frac{1}{e}. \end{aligned} \quad \blacksquare$$

## The Derivatives of $a^u$ and $\log_a u$

We start with the equation $a^x = e^{\ln(a^x)} = e^{x\ln a}$, which was established in Section 1.6:

$$\begin{aligned} \frac{d}{dx}a^x &= \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot\frac{d}{dx}(x\ln a) \qquad \frac{d}{dx}e^u = e^u\frac{du}{dx} \\ &= a^x\ln a. \end{aligned}$$

If $a > 0$, then

$$\frac{d}{dx} a^x = a^x \ln a.$$

This equation shows why $e^x$ is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and the derivative of $a^x$ simplifies to

$$\frac{d}{dx} e^x = e^x \ln e = e^x.$$

With the Chain Rule, we get a more general form for the derivative of a general exponential function.

> If $a > 0$ and $u$ is a differentiable function of $x$, then $a^u$ is a differentiable function of $x$ and
>
> $$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}. \tag{5}$$

**EXAMPLE 5** We illustrate using Equation (5).

**(a)** $\dfrac{d}{dx} 3^x = 3^x \ln 3$ — Eq. (5) with $a = 3, u = x$

**(b)** $\dfrac{d}{dx} 3^{-x} = 3^{-x}(\ln 3)\dfrac{d}{dx}(-x) = -3^{-x} \ln 3$ — Eq. (5) with $a = 3, u = -x$

**(c)** $\dfrac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3)\dfrac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3) \cos x$ — $\ldots, u = \sin x$ ■

In Section 3.3 we looked at the derivative $f'(0)$ for the exponential functions $f(x) = a^x$ at various values of the base $a$. The number $f'(0)$ is the limit, $\lim_{h\to 0} (a^h - 1)/h$, and gives the slope of the graph of $a^x$ when it crosses the $y$-axis at the point (0, 1). We now see that the value of this slope is

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a. \tag{6}$$

In particular, when $a = e$ we obtain

$$\lim_{h\to 0} \frac{e^h - 1}{h} = \ln e = 1.$$

However, we have not fully justified that these limits actually exist. While all of the arguments given in deriving the derivatives of the exponential and logarithmic functions are correct, they do assume the existence of these limits. In Chapter 7 we will give another development of the theory of logarithmic and exponential functions which fully justifies that both limits do in fact exist and have the values derived above.

To find the derivative of $\log_a u$ for an arbitrary base ($a > 0, a \neq 1$), we start with the change-of-base formula for logarithms (reviewed in Section 1.6) and express $\log_a u$ in terms of natural logarithms,

$$\log_a x = \frac{\ln x}{\ln a}.$$

Taking derivatives, we have

$$\begin{aligned}
\frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\
&= \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x \qquad \text{ln } a \text{ is a constant.} \\
&= \frac{1}{\ln a}\cdot\frac{1}{x} \\
&= \frac{1}{x\ln a}.
\end{aligned}$$

If $u$ is a differentiable function of $x$ and $u > 0$, the Chain Rule gives the following formula.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}. \qquad (7)$$

## Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

**EXAMPLE 6** Find $dy/dx$ if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$$

**Solution** We take the natural logarithm of both sides and simplify the result with the algebraic properties of logarithms from Theorem 1 in Section 1.6:

$$\begin{aligned}
\ln y &= \ln\frac{(x^2+1)(x+3)^{1/2}}{x-1} \\
&= \ln\left((x^2+1)(x+3)^{1/2}\right) - \ln(x-1) \qquad \text{Rule 2} \\
&= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) \qquad \text{Rule 1} \\
&= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). \qquad \text{Rule 4}
\end{aligned}$$

We then take derivatives of both sides with respect to $x$, using Equation (2) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for $dy/dx$:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for $y$:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

## Proof of the Power Rule (General Version)

The definition of the general exponential function enables us to make sense of raising any positive number to a real power $n$, rational or irrational. That is, we can define the power function $y = x^n$ for any exponent $n$.

**DEFINITION** For any $x > 0$ and for any real number $n$,

$$x^n = e^{n \ln x}.$$

Because the logarithm and exponential functions are inverses of each other, the definition gives

$$\ln x^n = n \ln x, \quad \text{for all real numbers } n.$$

That is, the Power Rule for the natural logarithm holds for *all* real exponents $n$, not just for rational exponents.

The definition of the power function also enables us to establish the derivative Power Rule for any real power $n$, as stated in Section 3.3.

### General Power Rule for Derivatives

For $x > 0$ and any real number $n$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $x \le 0$, then the formula holds whenever the derivative, $x^n$, and $x^{n-1}$ all exist.

**Proof** Differentiating $x^n$ with respect to $x$ gives

$$\begin{aligned}
\frac{d}{dx}x^n &= \frac{d}{dx}e^{n \ln x} && \text{Definition of } x^n,\ x > 0 \\
&= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) && \text{Chain Rule for } e^u \\
&= x^n \cdot \frac{n}{x} && \text{Definition and derivative of } \ln x \\
&= nx^{n-1}. && x^n \cdot x^{-1} = x^{n-1}
\end{aligned}$$

In short, whenever $x > 0$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

For $x < 0$, if $y = x^n$, $y'$, and $x^{n-1}$ all exist, then

$$\ln|y| = \ln|x|^n = n \ln|x|.$$

Using implicit differentiation (which *assumes* the existence of the derivative $y'$) and Equation (4), we have

$$\frac{y'}{y} = \frac{n}{x}.$$

Solving for the derivative,

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}.$$

It can be shown directly from the definition of the derivative that the derivative equals 0 when $x = 0$ and $n \geq 1$. This completes the proof of the general version of the Power Rule for all values of $x$. ■

**EXAMPLE 7** Differentiate $f(x) = x^x, x > 0$.

**Solution** We note that $f(x) = x^x = e^{x \ln x}$, so differentiation gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x \ln x}) \\ &= e^{x \ln x}\frac{d}{dx}(x \ln x) && \tfrac{d}{dx}e^u,\ u = x \ln x \\ &= e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right) \\ &= x^x(\ln x + 1). && x > 0 \end{aligned}$$

■

## The Number *e* Expressed as a Limit

In Section 1.5 we defined the number $e$ as the base value for which the exponential function $y = a^x$ has slope 1 when it crosses the $y$-axis at (0, 1). Thus $e$ is the constant that satisfies the equation

$$\lim_{h \to 0} \frac{e^h - 1}{h} = \ln e = 1. \qquad \text{Slope equals } \ln e \text{ from Eq. (6)}$$

We also stated that $e$ could be calculated as $\lim_{y \to \infty}(1 + 1/y)^y$, or by substituting $y = 1/x$, as $\lim_{x \to 0}(1 + x)^{1/x}$. We now prove this result.

**THEOREM 4—The Number *e* as a Limit** The number $e$ can be calculated as the limit

$$e = \lim_{x \to 0}(1 + x)^{1/x}.$$

**Proof** If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1 + x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x}\ln(1 + x) && \ln 1 = 0 \\ &= \lim_{x \to 0} \ln(1 + x)^{1/x} = \ln\left[\lim_{x \to 0}(1 + x)^{1/x}\right]. && \text{ln is continuous, Theorem 10 in Chapter 2} \end{aligned}$$

Because $f'(1) = 1$, we have

$$\ln\left[\lim_{x\to 0}(1+x)^{1/x}\right] = 1.$$

Therefore, exponentiating both sides we get

$$\lim_{x\to 0}(1+x)^{1/x} = e. \qquad \blacksquare$$

Approximating the limit in Theorem 4 by taking $x$ very small gives approximations to $e$. Its value is $e \approx 2.718281828459045$ to 15 decimal places.

## Exercises 3.8

### Derivatives of Inverse Functions

In Exercises 1–4:

**a.** Find $f^{-1}(x)$.

**b.** Graph $f$ and $f^{-1}$ together.

**c.** Evaluate $df/dx$ at $x = a$ and $df^{-1}/dx$ at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

**1.** $f(x) = 2x + 3, \quad a = -1$ **2.** $f(x) = (1/5)x + 7, \quad a = -1$

**3.** $f(x) = 5 - 4x, \quad a = 1/2$ **4.** $f(x) = 2x^2, \quad x \ge 0, \quad a = 5$

**5. a.** Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

**b.** Graph $f$ and $g$ over an $x$-interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry about the line $y = x$.

**c.** Find the slopes of the tangents to the graphs of $f$ and $g$ at $(1, 1)$ and $(-1, -1)$ (four tangents in all).

**d.** What lines are tangent to the curves at the origin?

**6. a.** Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.

**b.** Graph $h$ and $k$ over an $x$-interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.

**c.** Find the slopes of the tangents to the graphs at $h$ and $k$ at $(2, 2)$ and $(-2, -2)$.

**d.** What lines are tangent to the curves at the origin?

**7.** Let $f(x) = x^3 - 3x^2 - 1, x \ge 2$. Find the value of $df^{-1}/dx$ at the point $x = -1 = f(3)$.

**8.** Let $f(x) = x^2 - 4x - 5, x > 2$. Find the value of $df^{-1}/dx$ at the point $x = 0 = f(5)$.

**9.** Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of $f$ passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of $df^{-1}/dx$ at $x = 4$.

**10.** Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of $g$ passes through the origin with slope 2. Find the slope of the graph of $g^{-1}$ at the origin.

### Derivatives of Logarithms

In Exercises 11–40, find the derivative of $y$ with respect to $x$, $t$, or $\theta$, as appropriate.

**11.** $y = \ln 3x$ **12.** $y = \ln kx$, $k$ constant

**13.** $y = \ln(t^2)$ **14.** $y = \ln(t^{3/2})$

**15.** $y = \ln\dfrac{3}{x}$ **16.** $y = \ln\dfrac{10}{x}$

**17.** $y = \ln(\theta + 1)$ **18.** $y = \ln(2\theta + 2)$

**19.** $y = \ln x^3$ **20.** $y = (\ln x)^3$

**21.** $y = t(\ln t)^2$ **22.** $y = t\sqrt{\ln t}$

**23.** $y = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16}$ **24.** $y = (x^2\ln x)^4$

**25.** $y = \dfrac{\ln t}{t}$ **26.** $y = \dfrac{1 + \ln t}{t}$

**27.** $y = \dfrac{\ln x}{1 + \ln x}$ **28.** $y = \dfrac{x\ln x}{1 + \ln x}$

**29.** $y = \ln(\ln x)$ **30.** $y = \ln(\ln(\ln x))$

**31.** $y = \theta(\sin(\ln\theta) + \cos(\ln\theta))$

**32.** $y = \ln(\sec\theta + \tan\theta)$

**33.** $y = \ln\dfrac{1}{x\sqrt{x + 1}}$ **34.** $y = \dfrac{1}{2}\ln\dfrac{1 + x}{1 - x}$

**35.** $y = \dfrac{1 + \ln t}{1 - \ln t}$ **36.** $y = \sqrt{\ln\sqrt{t}}$

**37.** $y = \ln(\sec(\ln\theta))$ **38.** $y = \ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$

**39.** $y = \ln\left(\dfrac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$ **40.** $y = \ln\sqrt{\dfrac{(x + 1)^5}{(x + 2)^{20}}}$

### Logarithmic Differentiation

In Exercises 41–54, use logarithmic differentiation to find the derivative of $y$ with respect to the given independent variable.

**41.** $y = \sqrt{x(x + 1)}$ **42.** $y = \sqrt{(x^2 + 1)(x - 1)^2}$

**43.** $y = \sqrt{\dfrac{t}{t + 1}}$ **44.** $y = \sqrt{\dfrac{1}{t(t + 1)}}$

**45.** $y = \sqrt{\theta + 3}\sin\theta$ **46.** $y = (\tan\theta)\sqrt{2\theta + 1}$

**47.** $y = t(t + 1)(t + 2)$ **48.** $y = \dfrac{1}{t(t + 1)(t + 2)}$

**49.** $y = \dfrac{\theta + 5}{\theta\cos\theta}$ **50.** $y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$

**51.** $y = \dfrac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}}$ **52.** $y = \sqrt{\dfrac{(x + 1)^{10}}{(2x + 1)^5}}$

**53.** $y = \sqrt[3]{\dfrac{x(x-2)}{x^2+1}}$ **54.** $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

### Finding Derivatives

In Exercises 55–62, find the derivative of $y$ with respect to $x$, $t$, or $\theta$, as appropriate.

**55.** $y = \ln(\cos^2\theta)$ **56.** $y = \ln(3\theta e^{-\theta})$

**57.** $y = \ln(3te^{-t})$ **58.** $y = \ln(2e^{-t}\sin t)$

**59.** $y = \ln\left(\dfrac{e^\theta}{1+e^\theta}\right)$ **60.** $y = \ln\left(\dfrac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)$

**61.** $y = e^{(\cos t + \ln t)}$ **62.** $y = e^{\sin t}(\ln t^2 + 1)$

In Exercises 63–66, find $dy/dx$.

**63.** $\ln y = e^y \sin x$ **64.** $\ln xy = e^{x+y}$

**65.** $x^y = y^x$ **66.** $\tan y = e^x + \ln x$

In Exercises 67–88, find the derivative of $y$ with respect to the given independent variable.

**67.** $y = 2^x$ **68.** $y = 3^{-x}$

**69.** $y = 5^{\sqrt{s}}$ **70.** $y = 2^{(s^2)}$

**71.** $y = x^\pi$ **72.** $y = t^{1-e}$

**73.** $y = \log_2 5\theta$ **74.** $y = \log_3(1 + \theta\ln 3)$

**75.** $y = \log_4 x + \log_4 x^2$ **76.** $y = \log_{25} e^x - \log_5\sqrt{x}$

**77.** $y = \log_2 r \cdot \log_4 r$ **78.** $y = \log_3 r \cdot \log_9 r$

**79.** $y = \log_3\left(\left(\dfrac{x+1}{x-1}\right)^{\ln 3}\right)$ **80.** $y = \log_5\sqrt{\left(\dfrac{7x}{3x+2}\right)^{\ln 5}}$

**81.** $y = \theta\sin(\log_7\theta)$ **82.** $y = \log_7\left(\dfrac{\sin\theta\cos\theta}{e^\theta 2^\theta}\right)$

**83.** $y = \log_5 e^x$ **84.** $y = \log_2\left(\dfrac{x^2e^2}{2\sqrt{x+1}}\right)$

**85.** $y = 3^{\log_2 t}$ **86.** $y = 3\log_8(\log_2 t)$

**87.** $y = \log_2(8t^{\ln 2})$ **88.** $y = t\log_3\left(e^{(\sin t)(\ln 3)}\right)$

### Logarithmic Differentiation with Exponentials

In Exercises 89–96, use logarithmic differentiation to find the derivative of $y$ with respect to the given independent variable.

**89.** $y = (x+1)^x$ **90.** $y = x^{(x+1)}$

**91.** $y = (\sqrt{t})^t$ **92.** $y = t^{\sqrt{t}}$

**93.** $y = (\sin x)^x$ **94.** $y = x^{\sin x}$

**95.** $y = x^{\ln x}$ **96.** $y = (\ln x)^{\ln x}$

### Theory and Applications

**97.** If we write $g(x)$ for $f^{-1}(x)$, Equation (1) can be written as

$$g'(f(a)) = \frac{1}{f'(a)}, \quad \text{or} \quad g'(f(a)) \cdot f'(a) = 1.$$

If we then write $x$ for $a$, we get

$$g'(f(x)) \cdot f'(x) = 1.$$

The latter equation may remind you of the Chain Rule, and indeed there is a connection.

Assume that $f$ and $g$ are differentiable functions that are inverses of one another, so that $(g \circ f)(x) = x$. Differentiate both sides of this equation with respect to $x$, using the Chain Rule to express $(g \circ f)'(x)$ as a product of derivatives of $g$ and $f$. What do you find? (This is not a proof of Theorem 3 because we assume here the theorem's conclusion that $g = f^{-1}$ is differentiable.)

**98.** Show that $\lim_{n\to\infty}\left(1 + \dfrac{x}{n}\right)^n = e^x$ for any $x > 0$.

**99.** If $y = A\sin(\ln x) + B\cos(\ln x)$, where $A$ and $B$ are constants, show that

$$x^2y'' + xy' + y = 0.$$

**100.** Using mathematical induction, show that

$$\frac{d^n}{dx^n}\ln x = (-1)^{n-1}\frac{(n-1)!}{x^n}.$$

### COMPUTER EXPLORATIONS

In Exercises 101–108, you will explore some functions and their inverses together with their derivatives and tangent line approximations at specified points. Perform the following steps using your CAS:

**a.** Plot the function $y = f(x)$ together with its derivative over the given interval. Explain why you know that $f$ is one-to-one over the interval.

**b.** Solve the equation $y = f(x)$ for $x$ as a function of $y$, and name the resulting inverse function $g$.

**c.** Find the equation for the tangent line to $f$ at the specified point $(x_0, f(x_0))$.

**d.** Find the equation for the tangent line to $g$ at the point $(f(x_0), x_0)$ located symmetrically across the 45° line $y = x$ (which is the graph of the identity function). Use Theorem 3 to find the slope of this tangent line.

**e.** Plot the functions $f$ and $g$, the identity, the two tangent lines, and the line segment joining the points $(x_0, f(x_0))$ and $(f(x_0), x_0)$. Discuss the symmetries you see across the main diagonal.

**101.** $y = \sqrt{3x-2}, \quad \dfrac{2}{3} \le x \le 4, \quad x_0 = 3$

**102.** $y = \dfrac{3x+2}{2x-11}, \quad -2 \le x \le 2, \quad x_0 = 1/2$

**103.** $y = \dfrac{4x}{x^2+1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

**104.** $y = \dfrac{x^3}{x^2+1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

**105.** $y = x^3 - 3x^2 - 1, \quad 2 \le x \le 5, \quad x_0 = \dfrac{27}{10}$

**106.** $y = 2 - x - x^3, \quad -2 \le x \le 2, \quad x_0 = \dfrac{3}{2}$

**107.** $y = e^x, \quad -3 \le x \le 5, \quad x_0 = 1$

**108.** $y = \sin x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}, \quad x_0 = 1$

In Exercises 109 and 110, repeat the steps above to solve for the functions $y = f(x)$ and $x = f^{-1}(y)$ defined implicitly by the given equations over the interval.

**109.** $y^{1/3} - 1 = (x+2)^3, \quad -5 \le x \le 5, \quad x_0 = -3/2$

**110.** $\cos y = x^{1/5}, \quad 0 \le x \le 1, \quad x_0 = 1/2$

# 3.9 Inverse Trigonometric Functions

We introduced the six basic inverse trigonometric functions in Section 1.6, but focused there on the arcsine and arccosine functions. Here we complete the study of how all six inverse trigonometric functions are defined, graphed, and evaluated, and how their derivatives are computed.

## Inverses of tan *x*, cot *x*, sec *x*, and csc *x*

The graphs of all six basic inverse trigonometric functions are shown in Figure 3.39. We obtain these graphs by reflecting the graphs of the restricted trigonometric functions (as discussed in Section 1.6) through the line $y = x$. Let's take a closer look at the arctangent, arccotangent, arcsecant, and arccosecant functions.

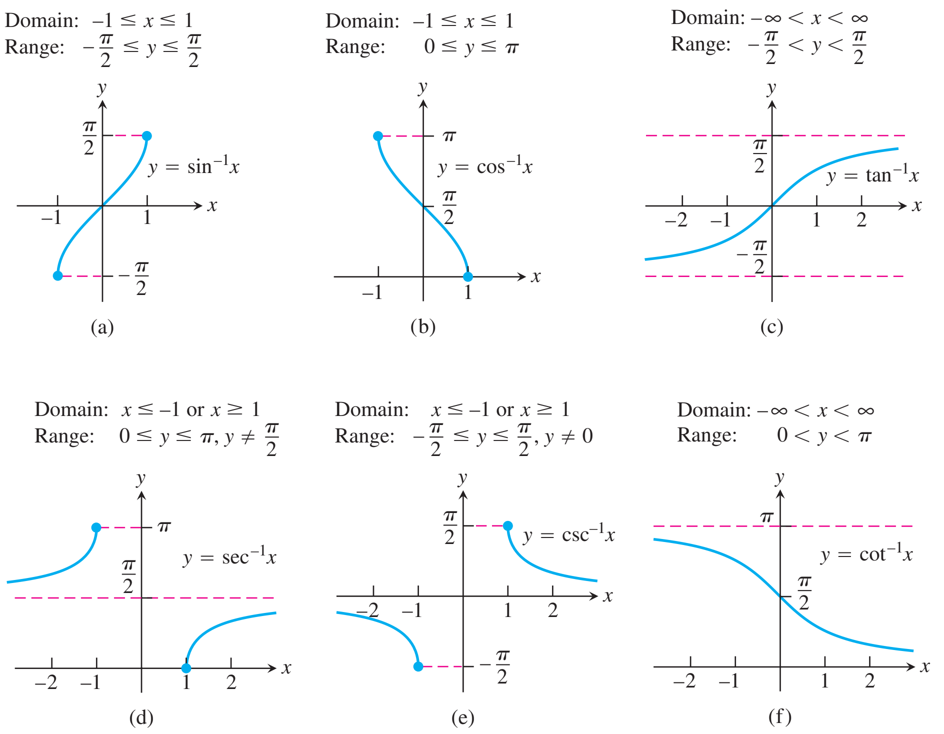



**FIGURE 3.39** Graphs of the six basic inverse trigonometric functions.

The arctangent of $x$ is a radian angle whose tangent is $x$. The arccotangent of $x$ is an angle whose cotangent is $x$. The angles belong to the restricted domains of the tangent and cotangent functions.

**DEFINITION**

$\boldsymbol{y = \tan^{-1} x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$\boldsymbol{y = \cot^{-1} x}$ is the number in $(0, \pi)$ for which $\cot y = x$.

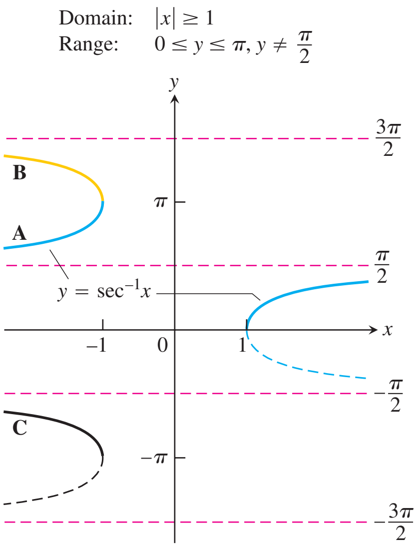



**FIGURE 3.40** There are several logical choices for the left-hand branch of $y = \sec^{-1}x$. With choice **A**, $\sec^{-1}x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

We use open intervals to avoid values where the tangent and cotangent are undefined.

The graph of $y = \tan^{-1}x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Figure 3.39c). Algebraically this means that

$$\tan^{-1}(-x) = -\tan^{-1}x;$$

the arctangent is an odd function. The graph of $y = \cot^{-1}x$ has no such symmetry (Figure 3.39f). Notice from Figure 3.39c that the graph of the arctangent function has two horizontal asymptotes; one at $y = \pi/2$ and the other at $y = -\pi/2$.

The inverses of the restricted forms of $\sec x$ and $\csc x$ are chosen to be the functions graphed in Figures 3.39d and 3.39e.

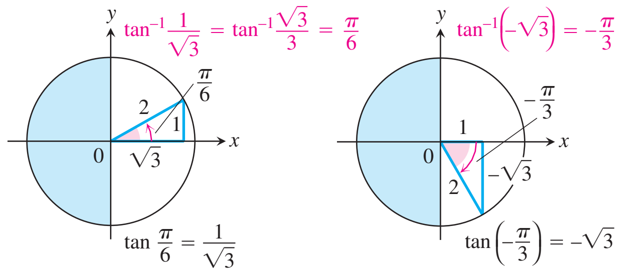
**Caution** There is no general agreement about how to define $\sec^{-1}x$ for negative values of $x$. We chose angles in the second quadrant between $\pi/2$ and $\pi$. This choice makes $\sec^{-1}x = \cos^{-1}(1/x)$. It also makes $\sec^{-1}x$ an increasing function on each interval of its domain. Some tables choose $\sec^{-1}x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Figure 3.40). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1}x = \cos^{-1}(1/x)$. From this, we can derive the identity

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right) \tag{1}$$

by applying Equation (5) in Section 1.6.

**EXAMPLE 1** The accompanying figures show two values of $\tan^{-1}x$.

| $x$ | $\tan^{-1}x$ |
|---|---|
| $\sqrt{3}$ | $\pi/3$ |
| $1$ | $\pi/4$ |
| $\sqrt{3}/3$ | $\pi/6$ |
| $-\sqrt{3}/3$ | $-\pi/6$ |
| $-1$ | $-\pi/4$ |
| $-\sqrt{3}$ | $-\pi/3$ |




The angles come from the first and fourth quadrants because the range of $\tan^{-1}x$ is $(-\pi/2, \pi/2)$. ■

## The Derivative of $y = \sin^{-1}u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 3 in Section 3.8 therefore assures us that the inverse function $y = \sin^{-1}x$ is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points (see Figure 3.41).

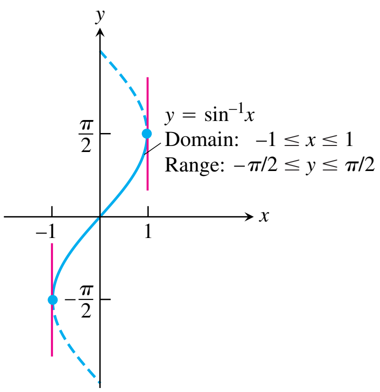



**FIGURE 3.41** The graph of $y = \sin^{-1} x$ has vertical tangents at $x = -1$ and $x = 1$.

We find the derivative of $y = \sin^{-1} x$ by applying Theorem 3 with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$:

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
&= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u} \\
&= \frac{1}{\sqrt{1 - x^2}}. && \sin(\sin^{-1} x) = x
\end{aligned}
$$

If $u$ is a differentiable function of $x$ with $|u| < 1$, we apply the Chain Rule to get

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \qquad |u| < 1.$$

**EXAMPLE 2** Using the Chain Rule, we calculate the derivative

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1 - x^4}}.$$ ■

## The Derivative of $y = \tan^{-1} u$

We find the derivative of $y = \tan^{-1} x$ by applying Theorem 3 with $f(x) = \tan x$ and $f^{-1}(x) = \tan^{-1} x$. Theorem 3 can be applied because the derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$:

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
&= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u \\
&= \frac{1}{1 + \tan^2(\tan^{-1} x)} && \sec^2 u = 1 + \tan^2 u \\
&= \frac{1}{1 + x^2}. && \tan(\tan^{-1} x) = x
\end{aligned}
$$

The derivative is defined for all real numbers. If $u$ is a differentiable function of $x$, we get the Chain Rule form:

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2} \frac{du}{dx}.$$

## The Derivative of $y = \sec^{-1} u$

Since the derivative of $\sec x$ is positive for $0 < x < \pi/2$ and $\pi/2 < x < \pi$, Theorem 3 says that the inverse function $y = \sec^{-1} x$ is differentiable. Instead of applying the formula in Theorem 3 directly, we find the derivative of $y = \sec^{-1} x$, $|x| > 1$, using implicit differentiation and the Chain Rule as follows:

$$\begin{aligned} y &= \sec^{-1} x \\ \sec y &= x && \text{Inverse function relationship} \\ \frac{d}{dx}(\sec y) &= \frac{d}{dx} x && \text{Differentiate both sides.} \\ \sec y \tan y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y}. && \text{Since } |x| > 1,\ y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \neq 0. \end{aligned}$$

To express the result in terms of $x$, we use the relationships

$$\sec y = x \qquad \text{and} \qquad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

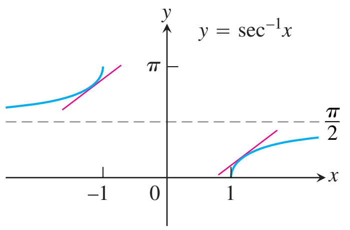



**FIGURE 3.42** The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

Can we do anything about the ± sign? A glance at Figure 3.42 shows that the slope of the graph $y = \sec^{-1} x$ is always positive. Thus,

$$\frac{d}{dx} \sec^{-1} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol, we can write a single expression that eliminates the "±" ambiguity:

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

If $u$ is a differentiable function of $x$ with $|u| > 1$, we have the formula

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \qquad |u| > 1.$$

**EXAMPLE 3** Using the Chain Rule and derivative of the arcsecant function, we find

$$\begin{aligned} \frac{d}{dx} \sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}} \frac{d}{dx}(5x^4) \\ &= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) && 5x^4 > 1 > 0 \\ &= \frac{4}{x\sqrt{25x^8 - 1}}. \end{aligned}$$

■

## Derivatives of the Other Three Inverse Trigonometric Functions

We could use the same techniques to find the derivatives of the other three inverse trigonometric functions—arccosine, arccotangent, and arccosecant—but there is an easier way, thanks to the following identities.

**Inverse Function–Inverse Cofunction Identities**

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

We saw the first of these identities in Equation (5) of Section 1.6. The others are derived in a similar way. It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions. For example, the derivative of $\cos^{-1} x$ is calculated as follows:

$$\frac{d}{dx}(\cos^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right) \qquad \text{Identity}$$
$$= -\frac{d}{dx}(\sin^{-1} x)$$
$$= -\frac{1}{\sqrt{1 - x^2}}. \qquad \text{Derivative of arcsine}$$

The derivatives of the inverse trigonometric functions are summarized in Table 3.1.

**TABLE 3.1** Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1 - u^2}}\dfrac{du}{dx}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1 - u^2}}\dfrac{du}{dx}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1 + u^2}\dfrac{du}{dx}$

4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1 + u^2}\dfrac{du}{dx}$

5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{|u|\sqrt{u^2 - 1}}\dfrac{du}{dx}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{|u|\sqrt{u^2 - 1}}\dfrac{du}{dx}, \quad |u| > 1$

## Exercises 3.9

### Common Values

Use reference triangles like those in Example 1 to find the angles in Exercises 1–8.

**1. a.** $\tan^{-1} 1$ **b.** $\tan^{-1}(-\sqrt{3})$ **c.** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

**2. a.** $\tan^{-1}(-1)$ **b.** $\tan^{-1}\sqrt{3}$ **c.** $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

**3. a.** $\sin^{-1}\left(\frac{-1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

**4. a.** $\sin^{-1}\left(\frac{1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

**5. a.** $\cos^{-1}\left(\frac{1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

**6. a.** $\csc^{-1}\sqrt{2}$ **b.** $\csc^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ **c.** $\csc^{-1} 2$

**7. a.** $\sec^{-1}(-\sqrt{2})$ **b.** $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **c.** $\sec^{-1}(-2)$

**8. a.** $\cot^{-1}(-1)$ **b.** $\cot^{-1}(\sqrt{3})$ **c.** $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

### Evaluations

Find the values in Exercises 9–12.

**9.** $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

**10.** $\sec\left(\cos^{-1}\frac{1}{2}\right)$

**11.** $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

**12.** $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

### Limits

Find the limits in Exercises 13–20. (If in doubt, look at the function's graph.)

**13.** $\lim_{x\to 1^-} \sin^{-1} x$

**14.** $\lim_{x\to -1^+} \cos^{-1} x$

**15.** $\lim_{x\to\infty} \tan^{-1} x$

**16.** $\lim_{x\to -\infty} \tan^{-1} x$

**17.** $\lim_{x\to\infty} \sec^{-1} x$

**18.** $\lim_{x\to -\infty} \sec^{-1} x$

**19.** $\lim_{x\to\infty} \csc^{-1} x$

**20.** $\lim_{x\to -\infty} \csc^{-1} x$

### Finding Derivatives

In Exercises 21–42, find the derivative of $y$ with respect to the appropriate variable.

**21.** $y = \cos^{-1}(x^2)$

**22.** $y = \cos^{-1}(1/x)$

**23.** $y = \sin^{-1}\sqrt{2}\,t$

**24.** $y = \sin^{-1}(1 - t)$

**25.** $y = \sec^{-1}(2s + 1)$

**26.** $y = \sec^{-1} 5s$

**27.** $y = \csc^{-1}(x^2 + 1), \quad x > 0$

**28.** $y = \csc^{-1}\frac{x}{2}$

**29.** $y = \sec^{-1}\frac{1}{t}, \quad 0 < t < 1$

**30.** $y = \sin^{-1}\frac{3}{t^2}$

**31.** $y = \cot^{-1}\sqrt{t}$

**32.** $y = \cot^{-1}\sqrt{t - 1}$

**33.** $y = \ln(\tan^{-1} x)$

**34.** $y = \tan^{-1}(\ln x)$

**35.** $y = \csc^{-1}(e^t)$

**36.** $y = \cos^{-1}(e^{-t})$

**37.** $y = s\sqrt{1 - s^2} + \cos^{-1} s$

**38.** $y = \sqrt{s^2 - 1} - \sec^{-1} s$

**39.** $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x, \quad x > 1$

**40.** $y = \cot^{-1}\frac{1}{x} - \tan^{-1} x$

**41.** $y = x\sin^{-1} x + \sqrt{1 - x^2}$

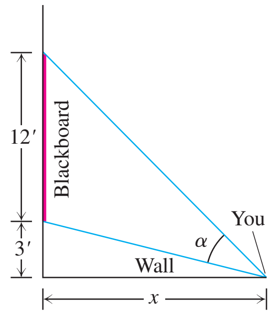
**42.** $y = \ln(x^2 + 4) - x\tan^{-1}\left(\frac{x}{2}\right)$

### Theory and Examples

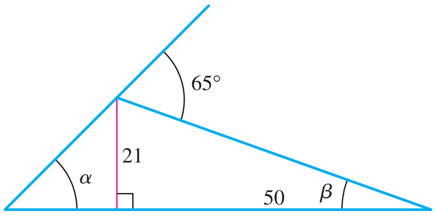
**43.** You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$$

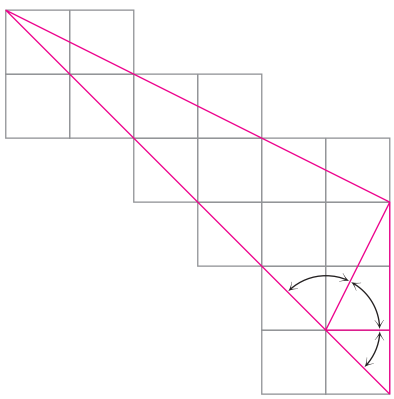
if you are $x$ ft from the front wall.




**44.** Find the angle $\alpha$.

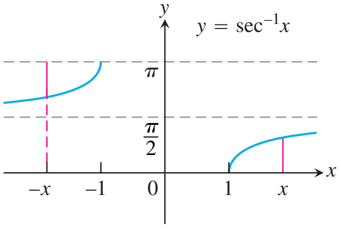



**45.** Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



**46. Two derivations of the identity $\sec^{-1}(-x) = \pi - \sec^{-1}x$**

**a.** (*Geometric*) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1}x$. See if you can tell what is going on.




**b.** (*Algebraic*) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1}x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1}x \qquad \text{Eq. (4), Section 1.6}$$

$$\sec^{-1}x = \cos^{-1}(1/x) \qquad \text{Eq. (1)}$$

Which of the expressions in Exercises 47–50 are defined, and which are not? Give reasons for your answers.

**47. a.** $\tan^{-1}2$ **b.** $\cos^{-1}2$

**48. a.** $\csc^{-1}(1/2)$ **b.** $\csc^{-1}2$

**49. a.** $\sec^{-1}0$ **b.** $\sin^{-1}\sqrt{2}$

**50. a.** $\cot^{-1}(-1/2)$ **b.** $\cos^{-1}(-5)$

**51.** Use the identity

$$\csc^{-1}u = \frac{\pi}{2} - \sec^{-1}u$$

to derive the formula for the derivative of $\csc^{-1}u$ in Table 3.1 from the formula for the derivative of $\sec^{-1}u$.

**52.** Derive the formula

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

for the derivative of $y = \tan^{-1}x$ by differentiating both sides of the equivalent equation $\tan y = x$.

**53.** Use the Derivative Rule in Section 3.8, Theorem 3, to derive

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2 - 1}}, \qquad |x| > 1.$$

**54.** Use the identity

$$\cot^{-1}u = \frac{\pi}{2} - \tan^{-1}u$$

to derive the formula for the derivative of $\cot^{-1}u$ in Table 3.1 from the formula for the derivative of $\tan^{-1}u$.

**55.** What is special about the functions

$$f(x) = \sin^{-1}\frac{x - 1}{x + 1}, \quad x \geq 0, \quad \text{and} \quad g(x) = 2\tan^{-1}\sqrt{x}?$$

Explain.

**56.** What is special about the functions

$$f(x) = \sin^{-1}\frac{1}{\sqrt{x^2 + 1}} \quad \text{and} \quad g(x) = \tan^{-1}\frac{1}{x}?$$

Explain.

T **57.** Find the values of

**a.** $\sec^{-1}1.5$ **b.** $\csc^{-1}(-1.5)$ **c.** $\cot^{-1}2$

T **58.** Find the values of

**a.** $\sec^{-1}(-3)$ **b.** $\csc^{-1}1.7$ **c.** $\cot^{-1}(-2)$

T In Exercises 59–61, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

**59. a.** $y = \tan^{-1}(\tan x)$ **b.** $y = \tan(\tan^{-1}x)$

**60. a.** $y = \sin^{-1}(\sin x)$ **b.** $y = \sin(\sin^{-1}x)$

**61. a.** $y = \cos^{-1}(\cos x)$ **b.** $y = \cos(\cos^{-1}x)$

T Use your graphing utility for Exercises 62–66.

**62.** Graph $y = \sec(\sec^{-1}x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

**63. Newton's serpentine** Graph Newton's serpentine, $y = 4x/(x^2 + 1)$. Then graph $y = 2\sin(2\tan^{-1}x)$ in the same graphing window. What do you see? Explain.

**64.** Graph the rational function $y = (2 - x^2)/x^2$. Then graph $y = \cos(2\sec^{-1}x)$ in the same graphing window. What do you see? Explain.

**65.** Graph $f(x) = \sin^{-1}x$ together with its first two derivatives. Comment on the behavior of $f$ and the shape of its graph in relation to the signs and values of $f'$ and $f''$.

**66.** Graph $f(x) = \tan^{-1}x$ together with its first two derivatives. Comment on the behavior of $f$ and the shape of its graph in relation to the signs and values of $f'$ and $f''$.

# 3.10 Related Rates

In this section we look at problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable (or perhaps several variables) changes. The problem of finding a rate of change from other known rates of change is called a *related rates problem*.

## Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If $V$ is the volume and $r$ is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

Using the Chain Rule, we differentiate both sides with respect to $t$ to find an equation relating the rates of change of $V$ and $r$,

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius $r$ of the balloon and the rate $dV/dt$ at which the volume is increasing at a given instant of time, then we can solve this last equation for $dr/dt$ to find how fast the radius is increasing at that instant. Note that it is easier to directly measure the rate of increase of the volume (the rate at which air is being pumped into the balloon) than it is to measure the increase in the radius. The related rates equation allows us to calculate $dr/dt$ from $dV/dt$.

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

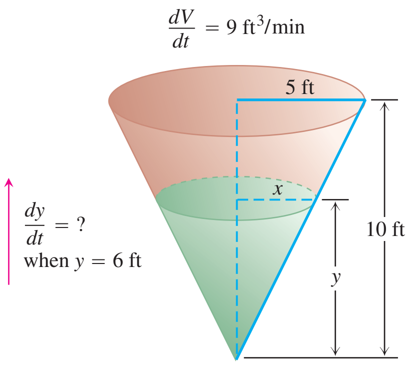



**FIGURE 3.43** The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

**EXAMPLE 1** Water runs into a conical tank at the rate of $9\ \text{ft}^3/\text{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

**Solution** Figure 3.43 shows a partially filled conical tank. The variables in the problem are

$V$ = volume ($\text{ft}^3$) of the water in the tank at time $t$ (min)

$x$ = radius (ft) of the surface of the water at time $t$

$y$ = depth (ft) of the water in the tank at time $t$.

We assume that $V$, $x$, and $y$ are differentiable functions of $t$. The constants are the dimensions of the tank. We are asked for $dy/dt$ when

$$y = 6 \text{ ft} \qquad \text{and} \qquad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves $x$ as well as $V$ and $y$. Because no information is given about $x$ and $dx/dt$ at the time in question, we need to eliminate $x$. The similar triangles in Figure 3.43 give us a way to express $x$ in terms of $y$:

$$\frac{x}{y} = \frac{5}{10} \qquad \text{or} \qquad x = \frac{y}{2}.$$

Therefore, find

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}.$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for $dy/dt$.

$$9 = \frac{\pi}{4}(6)^2 \frac{dy}{dt}$$
$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft/min. ■

## Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use $t$ for time. Assume that all variables are differentiable functions of $t$.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t*. Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate*. Use known values to find the unknown rate.

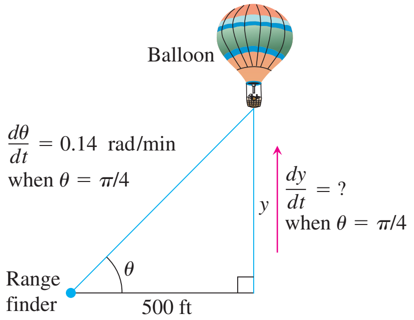



**FIGURE 3.44** The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

**EXAMPLE 2** A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

**Solution** We answer the question in six steps.

1. *Draw a picture and name the variables and constants* (Figure 3.44). The variables in the picture are

   $\theta$ = the angle in radians the range finder makes with the ground.

   $y$ = the height in feet of the balloon.

We let $t$ represent time in minutes and assume that $\theta$ and $y$ are differentiable functions of $t$.

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

2. *Write down the additional numerical information.*

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

3. *Write down what we are to find.* We want $dy/dt$ when $\theta = \pi/4$.
4. *Write an equation that relates the variables y and $\theta$.*

$$\frac{y}{500} = \tan\theta \qquad \text{or} \qquad y = 500\tan\theta$$

5. *Differentiate with respect to t using the Chain Rule.* The result tells how $dy/dt$ (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500(\sec^2\theta)\frac{d\theta}{dt}$$

6. *Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find $dy/dt$.*

$$\frac{dy}{dt} = 500\left(\sqrt{2}\right)^2(0.14) = 140 \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 140 ft/min. ■

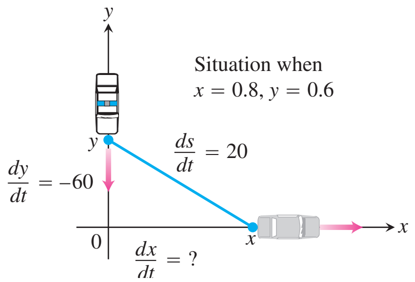



**FIGURE 3.45** The speed of the car is related to the speed of the police cruiser and the rate of change of the distance between them (Example 3).

**EXAMPLE 3** A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

**Solution** We picture the car and cruiser in the coordinate plane, using the positive $x$-axis as the eastbound highway and the positive $y$-axis as the southbound highway (Figure 3.45). We let $t$ represent time and set

$$x = \text{position of car at time } t$$
$$y = \text{position of cruiser at time } t$$
$$s = \text{distance between car and cruiser at time } t.$$

We assume that $x$, $y$, and $s$ are differentiable functions of $t$.

We want to find $dx/dt$ when

$$x = 0.8 \text{ mi}, \qquad y = 0.6 \text{ mi}, \qquad \frac{dy}{dt} = -60 \text{ mph}, \qquad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that $dy/dt$ is negative because $y$ is decreasing.

We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$
$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

Finally, we use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for $dx/dt$.

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$
$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph. ■

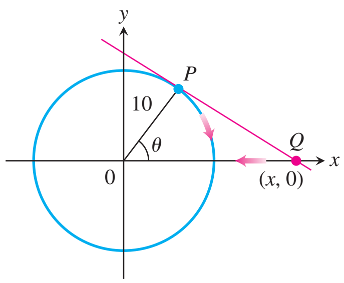



**FIGURE 3.46** The particle $P$ travels clockwise along the circle (Example 4).

**EXAMPLE 4** A particle $P$ moves clockwise at a constant rate along a circle of radius 10 ft centered at the origin. The particle's initial position is (0, 10) on the $y$-axis and its final destination is the point (10, 0) on the $x$-axis. Once the particle is in motion, the tangent line at $P$ intersects the $x$-axis at a point $Q$ (which moves over time). If it takes the particle 30 sec to travel from start to finish, how fast is the point $Q$ moving along the $x$-axis when it is 20 ft from the center of the circle?

**Solution** We picture the situation in the coordinate plane with the circle centered at the origin (see Figure 3.46). We let $t$ represent time and let $\theta$ denote the angle from the $x$-axis to the radial line joining the origin to $P$. Since the particle travels from start to finish in 30 sec, it is traveling along the circle at a constant rate of $\pi/2$ radians in $1/2$ min, or $\pi$ rad/min. In other words, $d\theta/dt = -\pi$, with $t$ being measured in minutes. The negative sign appears because $\theta$ is decreasing over time.

Setting $x(t)$ to be the distance at time $t$ from the point $Q$ to the origin, we want to find $dx/dt$ when

$$x = 20 \text{ ft} \qquad \text{and} \qquad \frac{d\theta}{dt} = -\pi \text{ rad/min}.$$

To relate the variables $x$ and $\theta$, we see from Figure 3.46 that $x\cos\theta = 10$, or $x = 10\sec\theta$. Differentiation of this last equation gives

$$\frac{dx}{dt} = 10\sec\theta\tan\theta\frac{d\theta}{dt} = -10\pi\sec\theta\tan\theta.$$

Note that $dx/dt$ is negative because $x$ is decreasing ($Q$ is moving towards the origin).

When $x = 20$, $\cos\theta = 1/2$ and $\sec\theta = 2$. Also, $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{3}$. It follows that

$$\frac{dx}{dt} = (-10\pi)(2)\left(\sqrt{3}\right) = -20\sqrt{3}\pi.$$

At the moment in question, the point $Q$ is moving towards the origin at the speed of $20\sqrt{3}\pi \approx 108.8$ ft/min. ■

**EXAMPLE 5** A jet airliner is flying at a constant altitude of 12,000 ft above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30°. How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of 2/3 deg/sec in order to keep the aircraft within its direct line of sight?

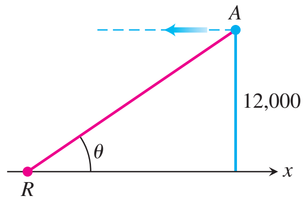



**FIGURE 3.47** Jet airliner $A$ traveling at constant altitude toward radar station $R$ (Example 5).

**Solution** The aircraft $A$ and radar station $R$ are pictured in the coordinate plane, using the positive $x$-axis as the horizontal distance at sea level from $R$ to $A$, and the positive $y$-axis as the vertical altitude above sea level. We let $t$ represent time and observe that $y = 12{,}000$ is a constant. The general situation and line-of-sight angle $\theta$ are depicted in Figure 3.47. We want to find $dx/dt$ when $\theta = \pi/6$ rad and $d\theta/dt = 2/3$ deg/sec.

From Figure 3.47, we see that

$$\frac{12{,}000}{x} = \tan\theta \qquad \text{or} \qquad x = 12{,}000\cot\theta.$$

Using miles instead of feet for our distance units, the last equation translates to

$$x = \frac{12{,}000}{5280}\cot\theta.$$

Differentiation with respect to $t$ gives

$$\frac{dx}{dt} = -\frac{1200}{528}\csc^2\theta\frac{d\theta}{dt}.$$

When $\theta = \pi/6$, $\sin^2\theta = 1/4$, so $\csc^2\theta = 4$. Converting $d\theta/dt = 2/3$ deg/sec to radians per hour, we find

$$\frac{d\theta}{dt} = \frac{2}{3}\left(\frac{\pi}{180}\right)(3600) \text{ rad/hr}. \qquad 1 \text{ hr} = 3600 \text{ sec}, 1 \text{ deg} = \pi/180 \text{ rad}$$

Substitution into the equation for $dx/dt$ then gives

$$\frac{dx}{dt} = \left(-\frac{1200}{528}\right)(4)\left(\frac{2}{3}\right)\left(\frac{\pi}{180}\right)(3600) \approx -380.$$

The negative sign appears because the distance $x$ is decreasing, so the aircraft is approaching the island at a speed of approximately 380 mi/hr when first detected by the radar. ■

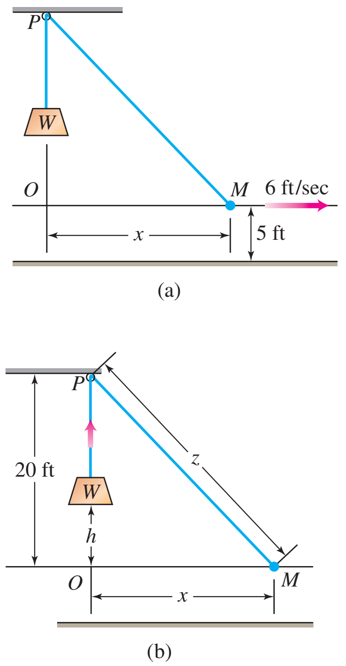



**FIGURE 3.48** A worker at $M$ walks to the right pulling the weight $W$ upwards as the rope moves through the pulley $P$ (Example 6).

**EXAMPLE 6** Figure 3.48(a) shows a rope running through a pulley at $P$ and bearing a weight $W$ at one end. The other end is held 5 ft above the ground in the hand $M$ of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the worker is walking rapidly away from the vertical line $PW$ at the rate of 6 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from $PW$?

**Solution** We let $OM$ be the horizontal line of length $x$ ft from a point $O$ directly below the pulley to the worker's hand $M$ at any instant of time (Figure 3.48). Let $h$ be the height of the weight $W$ above $O$, and let $z$ denote the length of rope from the pulley $P$ to the worker's hand. We want to know $dh/dt$ when $x = 21$ given that $dx/dt = 6$. Note that the height of $P$ above $O$ is 20 ft because $O$ is 5 ft above the ground. We assume the angle at $O$ is a right angle.

At any instant of time $t$ we have the following relationships (see Figure 3.48b):

$$20 - h + z = 45 \qquad \text{Total length of rope is 45 ft.}$$
$$20^2 + x^2 = z^2. \qquad \text{Angle at } O \text{ is a right angle.}$$

If we solve for $z = 25 + h$ in the first equation, and substitute into the second equation, we have

$$20^2 + x^2 = (25 + h)^2. \tag{1}$$

Differentiating both sides with respect to $t$ gives

$$2x\frac{dx}{dt} = 2(25 + h)\frac{dh}{dt},$$

and solving this last equation for $dh/dt$ we find

$$\frac{dh}{dt} = \frac{x}{25 + h}\frac{dx}{dt}. \tag{2}$$

Since we know $dx/dt$, it remains only to find $25 + h$ at the instant when $x = 21$. From Equation (1),

$$20^2 + 21^2 = (25 + h)^2$$

so that

$$(25 + h)^2 = 841, \qquad \text{or} \qquad 25 + h = 29.$$

Equation (2) now gives

$$\frac{dh}{dt} = \frac{21}{29} \cdot 6 = \frac{126}{29} \approx 4.3 \text{ ft/sec}$$

as the rate at which the weight is being raised when $x = 21$ ft. ■

## Exercises 3.10

1. **Area** Suppose that the radius $r$ and area $A = \pi r^2$ of a circle are differentiable functions of $t$. Write an equation that relates $dA/dt$ to $dr/dt$.
2. **Surface area** Suppose that the radius $r$ and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of $t$. Write an equation that relates $dS/dt$ to $dr/dt$.
3. Assume that $y = 5x$ and $dx/dt = 2$. Find $dy/dt$.
4. Assume that $2x + 3y = 12$ and $dy/dt = -2$. Find $dx/dt$.
5. If $y = x^2$ and $dx/dt = 3$, then what is $dy/dt$ when $x = -1$?
6. If $x = y^3 - y$ and $dy/dt = 5$, then what is $dx/dt$ when $y = 2$?
7. If $x^2 + y^2 = 25$ and $dx/dt = -2$, then what is $dy/dt$ when $x = 3$ and $y = -4$?
8. If $x^2y^3 = 4/27$ and $dy/dt = 1/2$, then what is $dx/dt$ when $x = 2$?
9. If $L = \sqrt{x^2 + y^2}$, $dx/dt = -1$, and $dy/dt = 3$, find $dL/dt$ when $x = 5$ and $y = 12$.
10. If $r + s^2 + v^3 = 12$, $dr/dt = 4$, and $ds/dt = -3$, find $dv/dt$ when $r = 3$ and $s = 1$.

**11.** If the original 24 m edge length $x$ of a cube decreases at the rate of 5 m/min, when $x = 3$ m at what rate does the cube's

**a.** surface area change?

**b.** volume change?

**12.** A cube's surface area increases at the rate of 72 $\text{in}^2/\text{sec}$. At what rate is the cube's volume changing when the edge length is $x = 3$ in?

**13. Volume** The radius $r$ and height $h$ of a right circular cylinder are related to the cylinder's volume $V$ by the formula $V = \pi r^2 h$.

**a.** How is $dV/dt$ related to $dh/dt$ if $r$ is constant?

**b.** How is $dV/dt$ related to $dr/dt$ if $h$ is constant?

**c.** How is $dV/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

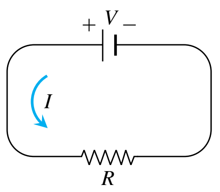
**14. Volume** The radius $r$ and height $h$ of a right circular cone are related to the cone's volume $V$ by the equation $V = (1/3)\pi r^2 h$.

**a.** How is $dV/dt$ related to $dh/dt$ if $r$ is constant?

**b.** How is $dV/dt$ related to $dr/dt$ if $h$ is constant?

**c.** How is $dV/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

**15. Changing voltage** The voltage $V$ (volts), current $I$ (amperes), and resistance $R$ (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that $V$ is increasing at the rate of 1 volt/sec while $I$ is decreasing at the rate of 1/3 amp/sec. Let $t$ denote time in seconds.




**a.** What is the value of $dV/dt$?

**b.** What is the value of $dI/dt$?

**c.** What equation relates $dR/dt$ to $dV/dt$ and $dI/dt$?

**d.** Find the rate at which $R$ is changing when $V = 12$ volts and $I = 2$ amp. Is $R$ increasing, or decreasing?

**16. Electrical power** The power $P$ (watts) of an electric circuit is related to the circuit's resistance $R$ (ohms) and current $I$ (amperes) by the equation $P = RI^2$.

**a.** How are $dP/dt$, $dR/dt$, and $dI/dt$ related if none of $P$, $R$, and $I$ are constant?

**b.** How is $dR/dt$ related to $dI/dt$ if $P$ is constant?

**17. Distance** Let $x$ and $y$ be differentiable functions of $t$ and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the $xy$-plane.

**a.** How is $ds/dt$ related to $dx/dt$ if $y$ is constant?

**b.** How is $ds/dt$ related to $dx/dt$ and $dy/dt$ if neither $x$ nor $y$ is constant?

**c.** How is $dx/dt$ related to $dy/dt$ if $s$ is constant?

**18. Diagonals** If $x$, $y$, and $z$ are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.

**a.** Assuming that $x$, $y$, and $z$ are differentiable functions of $t$, how is $ds/dt$ related to $dx/dt$, $dy/dt$, and $dz/dt$?

**b.** How is $ds/dt$ related to $dy/dt$ and $dz/dt$ if $x$ is constant?

**c.** How are $dx/dt$, $dy/dt$, and $dz/dt$ related if $s$ is constant?

**19. Area** The area $A$ of a triangle with sides of lengths $a$ and $b$ enclosing an angle of measure $\theta$ is

$$A = \frac{1}{2} ab \sin \theta .$$

**a.** How is $dA/dt$ related to $d\theta/dt$ if $a$ and $b$ are constant?

**b.** How is $dA/dt$ related to $d\theta/dt$ and $da/dt$ if only $b$ is constant?

**c.** How is $dA/dt$ related to $d\theta/dt$, $da/dt$, and $db/dt$ if none of $a$, $b$, and $\theta$ are constant?

**20. Heating a plate** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

**21. Changing dimensions in a rectangle** The length $l$ of a rectangle is decreasing at the rate of 2 cm/sec while the width $w$ is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change of **(a)** the area, **(b)** the perimeter, and **(c)** the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

**22. Changing dimensions in a rectangular box** Suppose that the edge lengths $x$, $y$, and $z$ of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

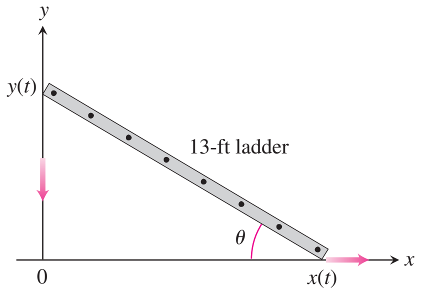
Find the rates at which the box's **(a)** volume, **(b)** surface area, and **(c)** diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

**23. A sliding ladder** A 13-ft ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

**a.** How fast is the top of the ladder sliding down the wall then?

**b.** At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

**c.** At what rate is the angle $\theta$ between the ladder and the ground changing then?




**24. Commercial air traffic** Two commercial airplanes are flying at an altitude of 40,000 ft along straight-line courses that intersect at right angles. Plane $A$ is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane $B$ is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when $A$ is 5 nautical miles from the intersection point and $B$ is 12 nautical miles from the intersection point?

**25. Flying a kite** A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

**26. Boring a cylinder** The mechanics at Lincoln Automotive are reboring a 6-in.-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

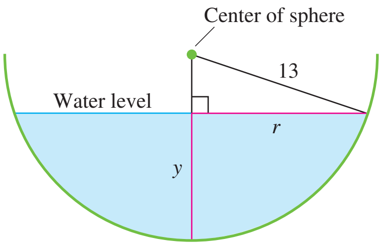
**27. A growing sand pile** Sand falls from a conveyor belt at the rate of 10 $m^3$/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Answer in centimeters per minute.

**28. A draining conical reservoir** Water is flowing at the rate of 50 $m^3$/min from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.

**a.** How fast (centimeters per minute) is the water level falling when the water is 5 m deep?

**b.** How fast is the radius of the water's surface changing then? Answer in centimeters per minute.

**29. A draining hemispherical reservoir** Water is flowing at the rate of 6 $m^3$/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius $R$ is $V = (\pi/3)y^2(3R - y)$ when the water is $y$ meters deep.




**a.** At what rate is the water level changing when the water is 8 m deep?

**b.** What is the radius $r$ of the water's surface when the water is $y$ m deep?

**c.** At what rate is the radius $r$ changing when the water is 8 m deep?

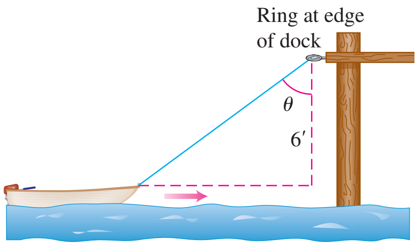
**30. A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.

**31. The radius of an inflating balloon** A spherical balloon is inflated with helium at the rate of $100\pi$ $ft^3$/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

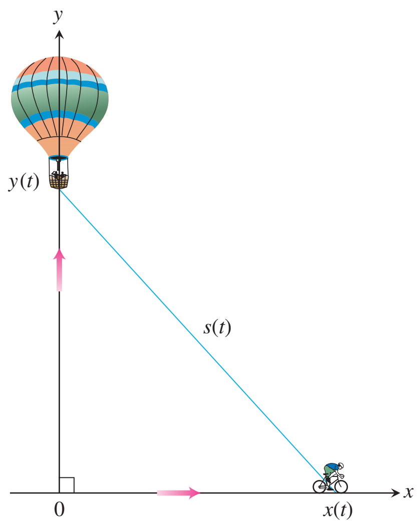
**32. Hauling in a dinghy** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec.

**a.** How fast is the boat approaching the dock when 10 ft of rope are out?

**b.** At what rate is the angle $\theta$ changing at this instant (see the figure)?



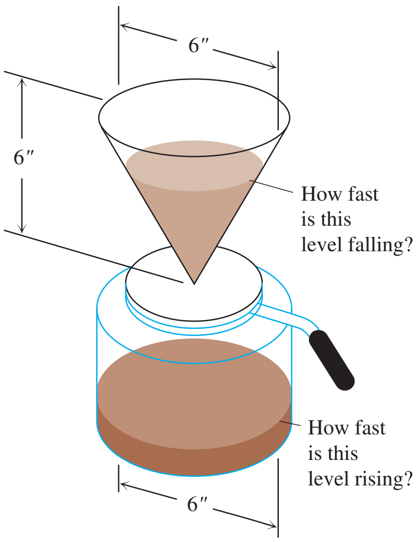

**33. A balloon and a bicycle** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 sec later?




**34. Making coffee** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 $in^3$/min.

**a.** How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

**b.** How fast is the level in the cone falling then?



**35. Cardiac output** In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where $Q$ is the number of milliliters of $CO_2$ you exhale in a minute and $D$ is the difference between the $CO_2$ concentration (ml/L) in the blood pumped to the lungs and the $CO_2$ concentration in the blood returning from the lungs. With $Q = 233$ ml/min and $D = 97 - 56 = 41$ ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

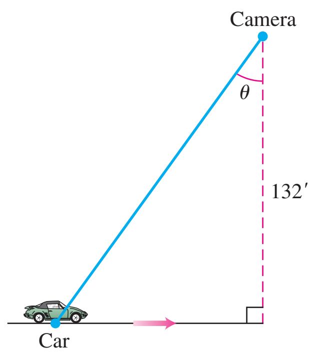
fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that $D$ is decreasing at the rate of 2 units a minute but that $Q$ remains unchanged. What is happening to the cardiac output?

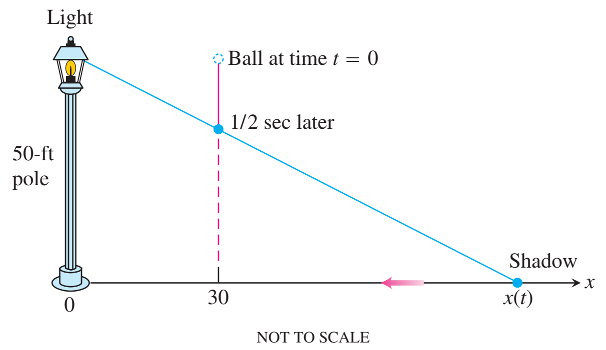
**36. Moving along a parabola** A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its $x$-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination $\theta$ of the line joining the particle to the origin changing when $x = 3$ m?

**37. Motion in the plane** The coordinates of a particle in the metric $xy$-plane are differentiable functions of time $t$ with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

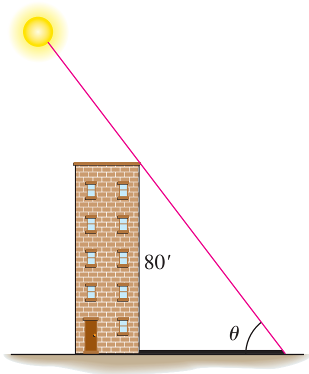
**38. Videotaping a moving car** You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mi/h (264 ft/sec), as shown in the accompanying figure. How fast will your camera angle $\theta$ be changing when the car is right in front of you? A half second later?




**39. A moving shadow** A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in $t$ sec.)




**40. A building's shadow** On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long. At the moment in question, the angle $\theta$ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in inches per minute, to the nearest tenth.)




**41. A melting ice layer** A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of $10 \text{ in}^3/\text{min}$, how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?

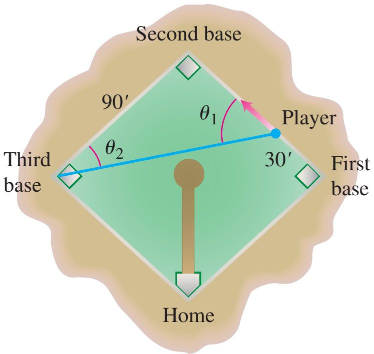
**42. Highway patrol** A highway patrol plane flies 3 mi above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi, the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.

**43. Baseball players** A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec.

**a.** At what rate is the player's distance from third base changing when the player is 30 ft from first base?

**b.** At what rates are angles $\theta_1$ and $\theta_2$ (see the figure) changing at that time?

**c.** The player slides into second base at the rate of 15 ft/sec. At what rates are angles $\theta_1$ and $\theta_2$ changing as the player touches base?




**44. Ships** Two ships are steaming straight away from a point $O$ along routes that make a 120° angle. Ship $A$ moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yd). Ship $B$ moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

# 3.11 Linearization and Differentials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed in this section are called *linearizations*, and they are based on tangent lines. Other approximating functions, such as polynomials, are discussed in Chapter 10.

We introduce new variables $dx$ and $dy$, called *differentials*, and define them in a way that makes Leibniz's notation for the derivative $dy/dx$ a true ratio. We use $dy$ to estimate error in measurement, which then provides for a precise proof of the Chain Rule (Section 3.6).

## Linearization

As you can see in Figure 3.49, the tangent to the curve $y = x^2$ lies close to the curve near the point of tangency. For a brief interval to either side, the $y$-values along the tangent line

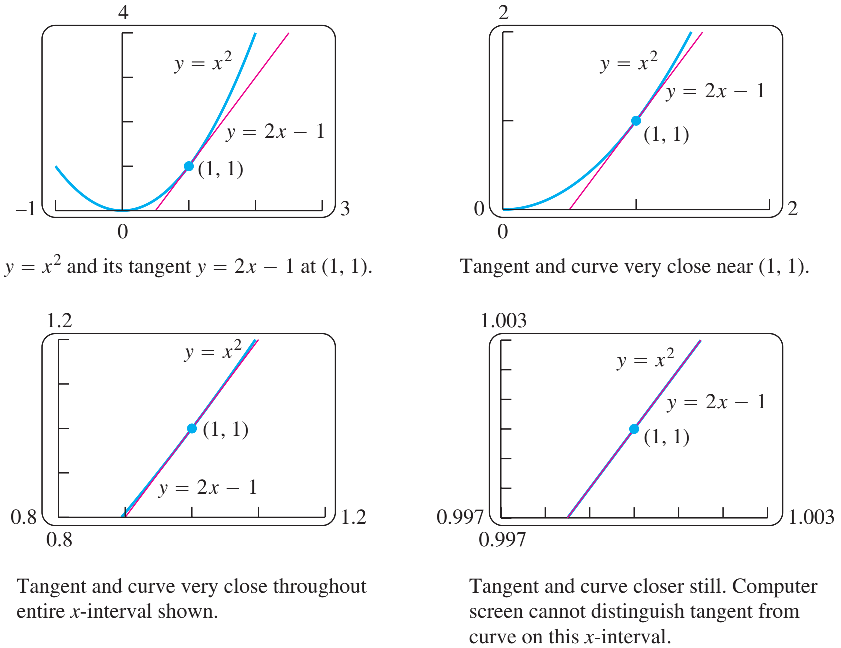



**FIGURE 3.49** The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

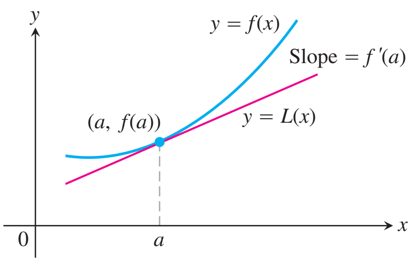



**FIGURE 3.50** The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

give good approximations to the $y$-values on the curve. We observe this phenomenon by zooming in on the two graphs at the point of tangency or by looking at tables of values for the difference between $f(x)$ and its tangent line near the $x$-coordinate of the point of tangency. The phenomenon is true not just for parabolas; every differentiable curve behaves locally like its tangent line.

In general, the tangent to $y = f(x)$ at a point $x = a$, where $f$ is differentiable (Figure 3.50), passes through the point $(a, f(a))$, so its point-slope equation is

$$y = f(a) + f'(a)(x - a).$$

Thus, this tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

For as long as this line remains close to the graph of $f$, $L(x)$ gives a good approximation to $f(x)$.

**DEFINITIONS** If $f$ is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of $f$ at $a$. The approximation

$$f(x) \approx L(x)$$

of $f$ by $L$ is the **standard linear approximation** of $f$ at $a$. The point $x = a$ is the **center** of the approximation.

**EXAMPLE 1** Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$ (Figure 3.51).

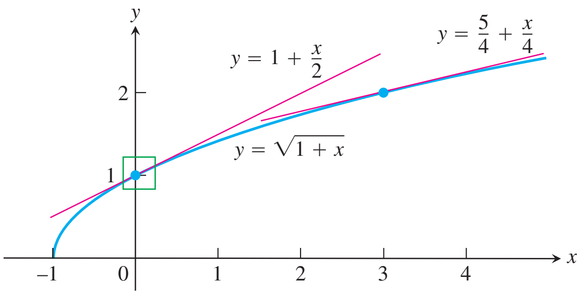



**FIGURE 3.51** The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.52 shows a magnified view of the small window about 1 on the $y$-axis.

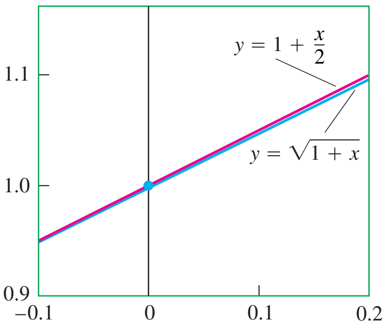



**FIGURE 3.52** Magnified view of the window in Figure 3.51.

**Solution** Since

$$f'(x) = \frac{1}{2}(1 + x)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

See Figure 3.52. ■

The following table shows how accurate the approximation $\sqrt{1 + x} \approx 1 + (x/2)$ from Example 1 is for some values of $x$ near 0. As we move away from zero, we lose accuracy. For example, for $x = 2$, the linearization gives 2 as the approximation for $\sqrt{3}$, which is not even accurate to one decimal place.

| Approximation | True value | $\lvert$True value − approximation$\rvert$ |
|---|---|---|
| $\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$ | 1.095445 | $<10^{-2}$ |
| $\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$ | 1.024695 | $<10^{-3}$ |
| $\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$ | 1.002497 | $<10^{-5}$ |

Do not be misled by the preceding calculations into thinking that whatever we do with a linearization is better done with a calculator. In practice, we would never use a linearization to find a particular square root. The utility of a linearization is its ability to replace a complicated formula by a simpler one over an entire interval of values. If we have to work with $\sqrt{1 + x}$ for $x$ close to 0 and can tolerate the small amount of error involved, we can work with $1 + (x/2)$ instead. Of course, we then need to know how much error there is. We further examine the estimation of error in Chapter 10.

A linear approximation normally loses accuracy away from its center. As Figure 3.51 suggests, the approximation $\sqrt{1 + x} \approx 1 + (x/2)$ will probably be too crude to be useful near $x = 3$. There, we need the linearization at $x = 3$.

**EXAMPLE 2** Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 3$.

**Solution** We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \qquad f'(3) = \frac{1}{2}(1 + x)^{-1/2}\bigg|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x - 3) = \frac{5}{4} + \frac{x}{4}. \qquad \blacksquare$$

At $x = 3.2$, the linearization in Example 2 gives

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%.

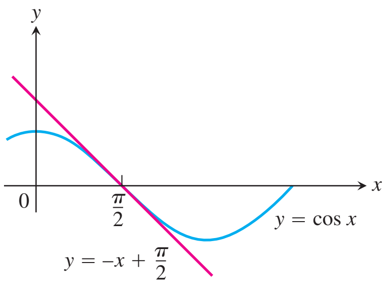



**FIGURE 3.53** The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$ (Example 3).

**EXAMPLE 3** Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 3.53).

**Solution** Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned} \qquad \blacksquare$$

An important linear approximation for roots and powers is

$$(1 + x)^k \approx 1 + kx \qquad (x \text{ near } 0; \text{ any number } k)$$

(Exercise 15). This approximation, good for values of $x$ sufficiently close to zero, has broad application. For example, when $x$ is small,

$$\sqrt{1 + x} \approx 1 + \frac{1}{2}x \qquad k = 1/2$$

$$\frac{1}{1 - x} = (1 - x)^{-1} \approx 1 + (-1)(-x) = 1 + x \qquad k = -1;\ \text{replace } x \text{ by } -x.$$

$$\sqrt[3]{1 + 5x^4} = (1 + 5x^4)^{1/3} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4 \qquad k = 1/3;\ \text{replace } x \text{ by } 5x^4.$$

$$\frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2 \qquad k = -1/2;\ \text{replace } x \text{ by } -x^2.$$

## Differentials

We sometimes use the Leibniz notation $dy/dx$ to represent the derivative of $y$ with respect to $x$. Contrary to its appearance, it is not a ratio. We now introduce two new variables $dx$ and $dy$ with the property that when their ratio exists, it is equal to the derivative.

**DEFINITION** Let $y = f(x)$ be a differentiable function. The **differential $dx$** is an independent variable. The **differential $dy$** is

$$dy = f'(x)\,dx.$$

Unlike the independent variable $dx$, the variable $dy$ is always a dependent variable. It depends on both $x$ and $dx$. If $dx$ is given a specific value and $x$ is a particular number in the domain of the function $f$, then these values determine the numerical value of $dy$.

### EXAMPLE 4

**(a)** Find $dy$ if $y = x^5 + 37x$.

**(b)** Find the value of $dy$ when $x = 1$ and $dx = 0.2$.

#### Solution

**(a)** $dy = (5x^4 + 37)\,dx$

**(b)** Substituting $x = 1$ and $dx = 0.2$ in the expression for $dy$, we have

$$dy = (5 \cdot 1^4 + 37)0.2 = 8.4.$$

■

The geometric meaning of differentials is shown in Figure 3.54. Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a).$$

The corresponding change in the tangent line $L$ is

$$\begin{aligned}\Delta L &= L(a + dx) - L(a)\\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a + dx)} - \underbrace{f(a)}_{L(a)}\\ &= f'(a)\,dx.\end{aligned}$$

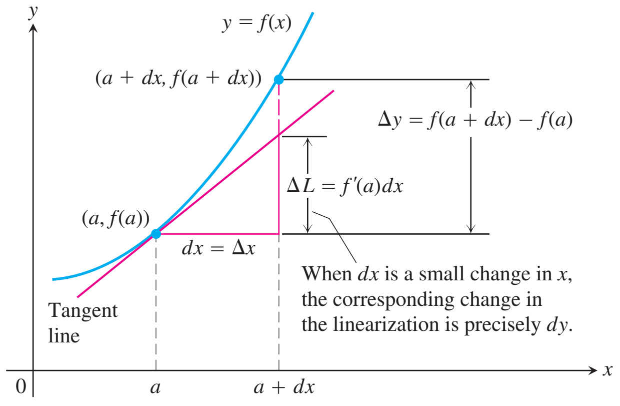



**FIGURE 3.54** Geometrically, the differential $dy$ is the change $\Delta L$ in the linearization of $f$ when $x = a$ changes by an amount $dx = \Delta x$.

That is, the change in the linearization of $f$ is precisely the value of the differential $dy$ when $x = a$ and $dx = \Delta x$. Therefore, $dy$ represents the amount the tangent line rises or falls when $x$ changes by an amount $dx = \Delta x$.

If $dx \neq 0$, then the quotient of the differential $dy$ by the differential $dx$ is equal to the derivative $f'(x)$ because

$$dy \div dx = \frac{f'(x)\,dx}{dx} = f'(x) = \frac{dy}{dx}.$$

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, calling $df$ the **differential of $f$**. For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Every differentiation formula like

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{or} \quad \frac{d(\sin u)}{dx} = \cos u\,\frac{du}{dx}$$

has a corresponding differential form like

$$d(u + v) = du + dv \quad \text{or} \quad d(\sin u) = \cos u\,du.$$

**EXAMPLE 5** We can use the Chain Rule and other differentiation rules to find differentials of functions.

**(a)** $d(\tan 2x) = \sec^2(2x)\,d(2x) = 2\sec^2 2x\,dx$

**(b)** $d\left(\dfrac{x}{x + 1}\right) = \dfrac{(x + 1)\,dx - x\,d(x + 1)}{(x + 1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x + 1)^2} = \dfrac{dx}{(x + 1)^2}$ ■

## Estimating with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point $a$ and want to estimate how much this value will change if we move to a nearby point $a + dx$. If $dx = \Delta x$ is small, then we can see from Figure 3.54 that $\Delta y$ is approximately equal to the differential $dy$. Since

$$f(a + dx) = f(a) + \Delta y, \qquad \Delta x = dx$$

the differential approximation gives

$$f(a + dx) \approx f(a) + dy$$

when $dx = \Delta x$. Thus the approximation $\Delta y \approx dy$ can be used to estimate $f(a + dx)$ when $f(a)$ is known and $dx$ is small.

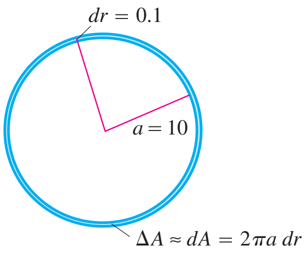



**FIGURE 3.55** When $dr$ is small compared with $a$, the differential $dA$ gives the estimate $A(a + dr) = \pi a^2 + dA$ (Example 6).

**EXAMPLE 6** The radius $r$ of a circle increases from $a = 10$ m to 10.1 m (Figure 3.55). Use $dA$ to estimate the increase in the circle's area $A$. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

**Solution** Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a)\,dr = 2\pi a\,dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

Thus, since $A(r + \Delta r) \approx A(r) + dA$, we have

$$\begin{aligned} A(10 + 0.1) &\approx A(10) + 2\pi \\ &= \pi(10)^2 + 2\pi = 102\pi. \end{aligned}$$

The area of a circle of radius 10.1 m is approximately $102\pi$ m$^2$.

The true area is

$$\begin{aligned} A(10.1) &= \pi(10.1)^2 \\ &= 102.01\pi \text{ m}^2. \end{aligned}$$

The error in our estimate is $0.01\pi$ m$^2$, which is the difference $\Delta A - dA$. ■

## Error in Differential Approximation

Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = \Delta x$ is an increment of $x$. We have two ways to describe the change in $f$ as $x$ changes from $a$ to $a + \Delta x$:

| | |
|---|---|
| The true change: | $\Delta f = f(a + \Delta x) - f(a)$ |
| The differential estimate: | $df = f'(a)\,\Delta x.$ |

How well does $df$ approximate $\Delta f$?

We measure the approximation error by subtracting $df$ from $\Delta f$:

$$\begin{aligned} \text{Approximation error} &= \Delta f - df \\ &= \Delta f - f'(a)\Delta x \\ &= \underbrace{f(a + \Delta x) - f(a)}_{\Delta f} - f'(a)\Delta x \\ &= \underbrace{\left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a)\right)}_{\text{Call this part } \epsilon.} \cdot \Delta x \\ &= \epsilon \cdot \Delta x. \end{aligned}$$

As $\Delta x \to 0$, the difference quotient

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

approaches $f'(a)$ (remember the definition of $f'(a)$), so the quantity in parentheses becomes a very small number (which is why we called it $\epsilon$). In fact, $\epsilon \to 0$ as $\Delta x \to 0$. When $\Delta x$ is small, the approximation error $\epsilon\,\Delta x$ is smaller still.

$$\underbrace{\Delta f}_{\substack{\text{true}\\ \text{change}}} = \underbrace{f'(a)\Delta x}_{\substack{\text{estimated}\\ \text{change}}} + \underbrace{\epsilon\,\Delta x}_{\text{error}}$$

Although we do not know the exact size of the error, it is the product $\epsilon \cdot \Delta x$ of two small quantities that both approach zero as $\Delta x \to 0$. For many common functions, whenever $\Delta x$ is small, the error is still smaller.

**Change in $y = f(x)$ near $x = a$**

If $y = f(x)$ is differentiable at $x = a$ and $x$ changes from $a$ to $a + \Delta x$, the change $\Delta y$ in $f$ is given by

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \qquad (1)$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

In Example 6 we found that

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = (\underbrace{2\pi}_{dA} + \underbrace{0.01\pi}_{\text{error}})\ \text{m}^2$$

so the approximation error is $\Delta A - dA = \epsilon\Delta r = 0.01\pi$ and $\epsilon = 0.01\pi/\Delta r = 0.01\pi/0.1 = 0.1\pi$ m.

## Proof of the Chain Rule

Equation (1) enables us to prove the Chain Rule correctly. Our goal is to show that if $f(u)$ is a differentiable function of $u$ and $u = g(x)$ is a differentiable function of $x$, then the composite $y = f(g(x))$ is a differentiable function of $x$. Since a function is differentiable if and only if it has a derivative at each point in its domain, we must show that whenever $g$ is differentiable at $x_0$ and $f$ is differentiable at $g(x_0)$, then the composite is differentiable at $x_0$ and the derivative of the composite satisfies the equation

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0).$$

Let $\Delta x$ be an increment in $x$ and let $\Delta u$ and $\Delta y$ be the corresponding increments in $u$ and $y$. Applying Equation (1) we have

$$\Delta u = g'(x_0)\Delta x + \epsilon_1\,\Delta x = (g'(x_0) + \epsilon_1)\Delta x,$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(u_0)\Delta u + \epsilon_2\,\Delta u = (f'(u_0) + \epsilon_2)\Delta u,$$

where $\epsilon_2 \to 0$ as $\Delta u \to 0$. Notice also that $\Delta u \to 0$ as $\Delta x \to 0$. Combining the equations for $\Delta u$ and $\Delta y$ gives

$$\Delta y = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x,$$

so

$$\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \epsilon_2\,g'(x_0) + f'(u_0)\epsilon_1 + \epsilon_2\epsilon_1.$$

Since $\epsilon_1$ and $\epsilon_2$ go to zero as $\Delta x$ goes to zero, three of the four terms on the right vanish in the limit, leaving

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0)) \cdot g'(x_0). \quad \blacksquare$$

## Sensitivity to Change

The equation $df = f'(x)\,dx$ tells how *sensitive* the output of $f$ is to a change in input at different values of $x$. The larger the value of $f'$ at $x$, the greater the effect of a given change $dx$. As we move from $a$ to a nearby point $a + dx$, we can describe the change in $f$ in three ways:

| | **True** | **Estimated** |
|---|---|---|
| Absolute change | $\Delta f = f(a + dx) - f(a)$ | $df = f'(a)\,dx$ |
| Relative change | $\dfrac{\Delta f}{f(a)}$ | $\dfrac{df}{f(a)}$ |
| Percentage change | $\dfrac{\Delta f}{f(a)} \times 100$ | $\dfrac{df}{f(a)} \times 100$ |

**EXAMPLE 7** You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1-sec error in measuring the time?

**Solution** The size of $ds$ in the equation

$$ds = 32t\,dt$$

depends on how big $t$ is. If $t = 2$ sec, the change caused by $dt = 0.1$ is about

$$ds = 32(2)(0.1) = 6.4 \text{ ft}.$$

Three seconds later at $t = 5$ sec, the change caused by the same $dt$ is

$$ds = 32(5)(0.1) = 16 \text{ ft}.$$

For a fixed error in the time measurement, the error in using $ds$ to estimate the depth is larger when the time it takes until the stone splashes into the water is longer. ■

**EXAMPLE 8** In the late 1830s, French physiologist Jean Poiseuille ("pwa-ZOY") discovered the formula we use today to predict how much the radius of a partially clogged artery decreases the normal volume of flow. His formula,

$$V = kr^4,$$

says that the volume $V$ of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius $r$. How does a 10% decrease in $r$ affect $V$? (See Figure 3.56.)

**Solution** The differentials of $r$ and $V$ are related by the equation

$$dV = \frac{dV}{dr}\,dr = 4kr^3\,dr.$$

The relative change in $V$ is

$$\frac{dV}{V} = \frac{4kr^3\,dr}{kr^4} = 4\frac{dr}{r}.$$

The relative change in $V$ is 4 times the relative change in $r$, so a 10% decrease in $r$ will result in a 40% decrease in the flow. ■

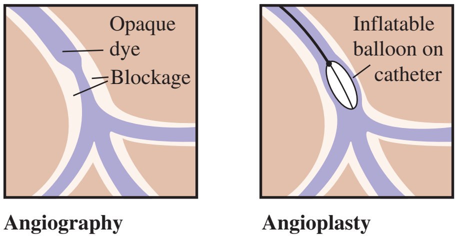



**FIGURE 3.56** To unblock a clogged artery, an opaque dye is injected into it to make the inside visible under X-rays. Then a balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.

**EXAMPLE 9** Newton's second law,

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma,$$

is stated with the assumption that mass is constant, but we know this is not strictly true because the mass of a body increases with velocity. In Einstein's corrected formula, mass has the value

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where the "rest mass" $m_0$ represents the mass of a body that is not moving and $c$ is the speed of light, which is about 300,000 km/sec. Use the approximation

$$\frac{1}{\sqrt{1 - x^2}} \approx 1 + \frac{1}{2}x^2 \tag{2}$$

to estimate the increase $\Delta m$ in mass resulting from the added velocity $v$.

**Solution** When $v$ is very small compared with $c$, $v^2/c^2$ is close to zero and it is safe to use the approximation

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) \qquad \text{Eq. (2) with } x = \frac{v}{c}$$

to obtain

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0\left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)\right] = m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right),$$

or

$$m \approx m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right). \tag{3}$$

Equation (3) expresses the increase in mass that results from the added velocity $v$. ■

## Converting Mass to Energy

Equation (3) derived in Example 9 has an important interpretation. In Newtonian physics, $(1/2)m_0 v^2$ is the kinetic energy (KE) of the body, and if we rewrite Equation (3) in the form

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2,$$

we see that

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2 = \frac{1}{2}m_0 v^2 - \frac{1}{2}m_0(0)^2 = \Delta(\text{KE}),$$

or

$$(\Delta m)c^2 \approx \Delta(\text{KE}).$$

So the change in kinetic energy $\Delta(\text{KE})$ in going from velocity 0 to velocity $v$ is approximately equal to $(\Delta m)c^2$, the change in mass times the square of the speed of light. Using $c \approx 3 \times 10^8$ m/sec, we see that a small change in mass can create a large change in energy.

# Exercises 3.11

## Finding Linearizations

In Exercises 1–5, find the linearization $L(x)$ of $f(x)$ at $x = a$.

**1.** $f(x) = x^3 - 2x + 3, \quad a = 2$

**2.** $f(x) = \sqrt{x^2 + 9}, \quad a = -4$

**3.** $f(x) = x + \frac{1}{x}, \quad a = 1$

**4.** $f(x) = \sqrt[3]{x}, \quad a = -8$

**5.** $f(x) = \tan x, \quad a = \pi$

**6. Common linear approximations at $x = 0$** Find the linearizations of the following functions at $x = 0$.

**(a)** $\sin x$ **(b)** $\cos x$ **(c)** $\tan x$ **(d)** $e^x$ **(e)** $\ln(1 + x)$

## Linearization for Approximation

In Exercises 7–14, find a linearization at a suitably chosen integer near $x_0$ at which the given function and its derivative are easy to evaluate.

**7.** $f(x) = x^2 + 2x, \quad x_0 = 0.1$

**8.** $f(x) = x^{-1}, \quad x_0 = 0.9$

**9.** $f(x) = 2x^2 + 3x - 3, \quad x_0 = -0.9$

**10.** $f(x) = 1 + x, \quad x_0 = 8.1$

**11.** $f(x) = \sqrt[3]{x}, \quad x_0 = 8.5$

**12.** $f(x) = \frac{x}{x + 1}, \quad x_0 = 1.3$

**13.** $f(x) = e^{-x}, \quad x_0 = -0.1$

**14.** $f(x) = \sin^{-1} x, \quad x_0 = \pi/12$

**15.** Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

**16.** Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of $x$ near zero.

**a.** $f(x) = (1 - x)^6$ **b.** $f(x) = \frac{2}{1 - x}$

**c.** $f(x) = \frac{1}{\sqrt{1 + x}}$ **d.** $f(x) = \sqrt{2 + x^2}$

**e.** $f(x) = (4 + 3x)^{1/3}$ **f.** $f(x) = \sqrt[3]{\left(1 - \frac{1}{2 + x}\right)^2}$

**17. Faster than a calculator** Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate the following.

**a.** $(1.0002)^{50}$ **b.** $\sqrt[3]{1.009}$

**18.** Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations of $\sqrt{x + 1}$ and $\sin x$ at $x = 0$?

## Derivatives in Differential Form

In Exercises 19–38, find $dy$.

**19.** $y = x^3 - 3\sqrt{x}$

**20.** $y = x\sqrt{1 - x^2}$

**21.** $y = \frac{2x}{1 + x^2}$

**22.** $y = \frac{2\sqrt{x}}{3(1 + \sqrt{x})}$

**23.** $2y^{3/2} + xy - x = 0$

**24.** $xy^2 - 4x^{3/2} - y = 0$

**25.** $y = \sin(5\sqrt{x})$

**26.** $y = \cos(x^2)$

**27.** $y = 4\tan(x^3/3)$

**28.** $y = \sec(x^2 - 1)$

**29.** $y = 3\csc(1 - 2\sqrt{x})$

**30.** $y = 2\cot\left(\frac{1}{\sqrt{x}}\right)$

**31.** $y = e^{\sqrt{x}}$

**32.** $y = xe^{-x}$

**33.** $y = \ln(1 + x^2)$

**34.** $y = \ln\left(\frac{x + 1}{\sqrt{x - 1}}\right)$

**35.** $y = \tan^{-1}(e^{x^2})$

**36.** $y = \cot^{-1}\left(\frac{1}{x^2}\right) + \cos^{-1} 2x$

**37.** $y = \sec^{-1}(e^{-x})$

**38.** $y = e^{\tan^{-1}\sqrt{x^2+1}}$

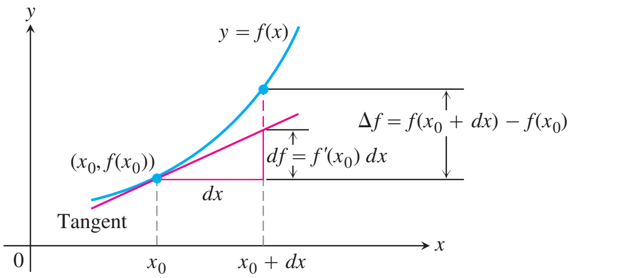
## Approximation Error

In Exercises 39–44, each function $f(x)$ changes value when $x$ changes from $x_0$ to $x_0 + dx$. Find

**a.** the change $\Delta f = f(x_0 + dx) - f(x_0)$;

**b.** the value of the estimate $df = f'(x_0)\,dx$; and

**c.** the approximation error $|\Delta f - df|$.



**39.** $f(x) = x^2 + 2x, \quad x_0 = 1, \quad dx = 0.1$

**40.** $f(x) = 2x^2 + 4x - 3, \quad x_0 = -1, \quad dx = 0.1$

**41.** $f(x) = x^3 - x, \quad x_0 = 1, \quad dx = 0.1$

**42.** $f(x) = x^4, \quad x_0 = 1, \quad dx = 0.1$

**43.** $f(x) = x^{-1}, \quad x_0 = 0.5, \quad dx = 0.1$

**44.** $f(x) = x^3 - 2x + 3, \quad x_0 = 2, \quad dx = 0.1$

## Differential Estimates of Change

In Exercises 45–50, write a differential formula that estimates the given change in volume or surface area.

**45.** The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from $r_0$ to $r_0 + dr$

**46.** The change in the volume $V = x^3$ of a cube when the edge lengths change from $x_0$ to $x_0 + dx$

**47.** The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from $x_0$ to $x_0 + dx$

**48.** The change in the lateral surface area $S = \pi r\sqrt{r^2 + h^2}$ of a right circular cone when the radius changes from $r_0$ to $r_0 + dr$ and the height does not change

**49.** The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from $r_0$ to $r_0 + dr$ and the height does not change

**50.** The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from $h_0$ to $h_0 + dh$ and the radius does not change

## Applications

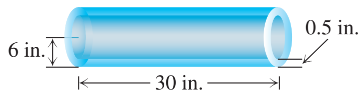
**51.** The radius of a circle is increased from 2.00 to 2.02 m.

**a.** Estimate the resulting change in area.

**b.** Express the estimate as a percentage of the circle's original area.

**52.** The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? The tree's cross-section area?

**53. Estimating volume** Estimate the volume of material in a cylindrical shell with length 30 in., radius 6 in., and shell thickness 0.5 in.




**54. Estimating height of a building** A surveyor, standing 30 ft from the base of a building, measures the angle of elevation to the top of the building to be 75°. How accurately must the angle be measured for the percentage error in estimating the height of the building to be less than 4%?

**55. Tolerance** The radius $r$ of a circle is measured with an error of at most 2%. What is the maximum corresponding percentage error in computing the circle's

**a.** circumference? **b.** area?

**56. Tolerance** The edge $x$ of a cube is measured with an error of at most 0.5%. What is the maximum corresponding percentage error in computing the cube's

**a.** surface area? **b.** volume?

**57. Tolerance** The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of $h$, expressed as a percentage of $h$.

**58. Tolerance**

**a.** About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

**b.** About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

**59.** The diameter of a sphere is measured as $100 \pm 1$ cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

**60.** Estimate the allowable percentage error in measuring the diameter $D$ of a sphere if the volume is to be calculated correctly to within 3%.

**61. The effect of flight maneuvers on the heart** The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where $W$ is the work per unit time, $P$ is the average blood pressure, $V$ is the volume of blood pumped out during the unit of time, $\delta$ ("delta") is the weight density of the blood, $v$ is the average velocity of the exiting blood, and $g$ is the acceleration of gravity.

When $P$, $V$, $\delta$, and $v$ remain constant, $W$ becomes a function of $g$, and the equation takes the simplified form

$$W = a + \frac{b}{g} \quad (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive $W$ is to apparent changes in $g$ caused by flight maneuvers, and this depends on the initial value of $g$. As part of your investigation, you decide to compare the effect on $W$ of a given change $dg$ on the moon, where $g = 5.2$ ft/sec$^2$, with the effect the same change $dg$ would have on Earth, where $g = 32$ ft/sec$^2$. Use the simplified equation above to find the ratio of $dW_{\text{moon}}$ to $dW_{\text{Earth}}$.

**62. Measuring acceleration of gravity** When the length $L$ of a clock pendulum is held constant by controlling its temperature, the pendulum's period $T$ depends on the acceleration of gravity $g$. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in $g$. By keeping track of $\Delta T$, we can estimate the variation in $g$ from the equation $T = 2\pi(L/g)^{1/2}$ that relates $T$, $g$, and $L$.

**a.** With $L$ held constant and $g$ as the independent variable, calculate $dT$ and use it to answer parts (b) and (c).

**b.** If $g$ increases, will $T$ increase or decrease? Will a pendulum clock speed up or slow down? Explain.

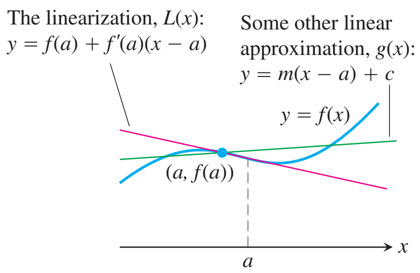
**c.** A clock with a 100-cm pendulum is moved from a location where $g = 980$ cm/sec$^2$ to a new location. This increases the period by $dT = 0.001$ sec. Find $dg$ and estimate the value of $g$ at the new location.

**63. The linearization is the best linear approximation** Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ is a linear function in which $m$ and $c$ are constants.

If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using $g$ as a linear approximation of $f$ instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on $g$ the conditions

1. $E(a) = 0$ — The approximation error is zero at $x = a$.
2. $\lim_{x \to a} \dfrac{E(x)}{x - a} = 0$ — The error is negligible when compared with $x - a$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $x - a$.




**64. Quadratic approximations**

**a.** Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

**i)** $Q(a) = f(a)$

**ii)** $Q'(a) = f'(a)$

**iii)** $Q''(a) = f''(a)$.

Determine the coefficients $b_0$, $b_1$, and $b_2$.

**b.** Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

T **c.** Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point (0, 1). Comment on what you see.

T **d.** Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph $g$ and its quadratic approximation together. Comment on what you see.

T **e.** Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph $h$ and its quadratic approximation together. Comment on what you see.

**f.** What are the linearizations of $f$, $g$, and $h$ at the respective points in parts (b), (d), and (e)?

**65. The linearization of $2^x$**

**a.** Find the linearization of $f(x) = 2^x$ at $x = 0$. Then round its coefficients to two decimal places.

T **b.** Graph the linearization and function together for $-3 \le x \le 3$ and $-1 \le x \le 1$.

**66. The linearization of $\log_3 x$**

**a.** Find the linearization of $f(x) = \log_3 x$ at $x = 3$. Then round its coefficients to two decimal places.

T **b.** Graph the linearization and function together in the window $0 \le x \le 8$ and $2 \le x \le 4$.

## COMPUTER EXPLORATIONS

In Exercises 67–72, use a CAS to estimate the magnitude of the error in using the linearization in place of the function over a specified interval $I$. Perform the following steps:

**a.** Plot the function $f$ over $I$.

**b.** Find the linearization $L$ of the function at the point $a$.

**c.** Plot $f$ and $L$ together on a single graph.

**d.** Plot the absolute error $|f(x) - L(x)|$ over $I$ and find its maximum value.

**e.** From your graph in part (d), estimate as large a $\delta > 0$ as you can, satisfying

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - L(x)| < \epsilon$$

for $\epsilon = 0.5, 0.1$, and $0.01$. Then check graphically to see if your $\delta$-estimate holds true.

**67.** $f(x) = x^3 + x^2 - 2x, \quad [-1, 2], \quad a = 1$

**68.** $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad \left[-\dfrac{3}{4}, 1\right], \quad a = \dfrac{1}{2}$

**69.** $f(x) = x^{2/3}(x - 2), \quad [-2, 3], \quad a = 2$

**70.** $f(x) = \sqrt{x} - \sin x, \quad [0, 2\pi], \quad a = 2$

**71.** $f(x) = x2^x, \quad [0, 2], \quad a = 1$

**72.** $f(x) = \sqrt{x} \sin^{-1} x, \quad [0, 1], \quad a = \dfrac{1}{2}$

# Chapter 3 Questions to Guide Your Review

1. What is the derivative of a function $f$? How is its domain related to the domain of $f$? Give examples.
2. What role does the derivative play in defining slopes, tangents, and rates of change?
3. How can you sometimes graph the derivative of a function when all you have is a table of the function's values?
4. What does it mean for a function to be differentiable on an open interval? On a closed interval?
5. How are derivatives and one-sided derivatives related?
6. Describe geometrically when a function typically does *not* have a derivative at a point.
7. How is a function's differentiability at a point related to its continuity there, if at all?
8. What rules do you know for calculating derivatives? Give some examples.

**9.** Explain how the three formulas

**a.** $\frac{d}{dx}(x^n) = nx^{n-1}$

**b.** $\frac{d}{dx}(cu) = c\frac{du}{dx}$

**c.** $\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}$

enable us to differentiate any polynomial.

**10.** What formula do we need, in addition to the three listed in Question 9, to differentiate rational functions?

**11.** What is a second derivative? A third derivative? How many derivatives do the functions you know have? Give examples.

**12.** What is the derivative of the exponential function $e^x$? How does the domain of the derivative compare with the domain of the function?

**13.** What is the relationship between a function's average and instantaneous rates of change? Give an example.

**14.** How do derivatives arise in the study of motion? What can you learn about a body's motion along a line by examining the derivatives of the body's position function? Give examples.

**15.** How can derivatives arise in economics?

**16.** Give examples of still other applications of derivatives.

**17.** What do the limits $\lim_{h\to 0}((\sin h)/h)$ and $\lim_{h\to 0}((\cos h - 1)/h)$ have to do with the derivatives of the sine and cosine functions? What *are* the derivatives of these functions?

**18.** Once you know the derivatives of $\sin x$ and $\cos x$, how can you find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$? What *are* the derivatives of these functions?

**19.** At what points are the six basic trigonometric functions continuous? How do you know?

**20.** What is the rule for calculating the derivative of a composite of two differentiable functions? How is such a derivative evaluated? Give examples.

**21.** If $u$ is a differentiable function of $x$, how do you find $(d/dx)(u^n)$ if $n$ is an integer? If $n$ is a real number? Give examples.

**22.** What is implicit differentiation? When do you need it? Give examples.

**23.** What is the derivative of the natural logarithm function $\ln x$? How does the domain of the derivative compare with the domain of the function?

**24.** What is the derivative of the exponential function $a^x$, $a > 0$ and $a \neq 1$? What is the geometric significance of the limit of $(a^h - 1)/h$ as $h \to 0$? What is the limit when $a$ is the number $e$?

**25.** What is the derivative of $\log_a x$? Are there any restrictions on $a$?

**26.** What is logarithmic differentiation? Give an example.

**27.** How can you write any real power of $x$ as a power of $e$? Are there any restrictions on $x$? How does this lead to the Power Rule for differentiating arbitrary real powers?

**28.** What is one way of expressing the special number $e$ as a limit? What is an approximate numerical value of $e$ correct to 7 decimal places?

**29.** What are the derivatives of the inverse trigonometric functions? How do the domains of the derivatives compare with the domains of the functions?

**30.** How do related rates problems arise? Give examples.

**31.** Outline a strategy for solving related rates problems. Illustrate with an example.

**32.** What is the linearization $L(x)$ of a function $f(x)$ at a point $x = a$? What is required of $f$ at $a$ for the linearization to exist? How are linearizations used? Give examples.

**33.** If $x$ moves from $a$ to a nearby value $a + dx$, how do you estimate the corresponding change in the value of a differentiable function $f(x)$? How do you estimate the relative change? The percentage change? Give an example.

# Chapter 3 Practice Exercises

## Derivatives of Functions

Find the derivatives of the functions in Exercises 1–64.

**1.** $y = x^5 - 0.125x^2 + 0.25x$

**2.** $y = 3 - 0.7x^3 + 0.3x^7$

**3.** $y = x^3 - 3(x^2 + \pi^2)$

**4.** $y = x^7 + \sqrt{7}x - \frac{1}{\pi + 1}$

**5.** $y = (x + 1)^2(x^2 + 2x)$

**6.** $y = (2x - 5)(4 - x)^{-1}$

**7.** $y = (\theta^2 + \sec\theta + 1)^3$

**8.** $y = \left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)^2$

**9.** $s = \frac{\sqrt{t}}{1 + \sqrt{t}}$

**10.** $s = \frac{1}{\sqrt{t} - 1}$

**11.** $y = 2\tan^2 x - \sec^2 x$

**12.** $y = \frac{1}{\sin^2 x} - \frac{2}{\sin x}$

**13.** $s = \cos^4(1 - 2t)$

**14.** $s = \cot^3\left(\frac{2}{t}\right)$

**15.** $s = (\sec t + \tan t)^5$

**16.** $s = \csc^5(1 - t + 3t^2)$

**17.** $r = \sqrt{2\theta\sin\theta}$

**18.** $r = 2\theta\sqrt{\cos\theta}$

**19.** $r = \sin\sqrt{2\theta}$

**20.** $r = \sin\left(\theta + \sqrt{\theta + 1}\right)$

**21.** $y = \frac{1}{2}x^2\csc\frac{2}{x}$

**22.** $y = 2\sqrt{x}\sin\sqrt{x}$

**23.** $y = x^{-1/2}\sec(2x)^2$

**24.** $y = \sqrt{x}\csc(x + 1)^3$

**25.** $y = 5\cot x^2$

**26.** $y = x^2\cot 5x$

**27.** $y = x^2\sin^2(2x^2)$

**28.** $y = x^{-2}\sin^2(x^3)$

**29.** $s = \left(\frac{4t}{t + 1}\right)^{-2}$

**30.** $s = \frac{-1}{15(15t - 1)^3}$

**31.** $y = \left(\frac{\sqrt{x}}{1 + x}\right)^2$

**32.** $y = \left(\frac{2\sqrt{x}}{2\sqrt{x} + 1}\right)^2$

**33.** $y = \sqrt{\frac{x^2 + x}{x^2}}$

**34.** $y = 4x\sqrt{x + \sqrt{x}}$

35. $r = \left(\dfrac{\sin\theta}{\cos\theta - 1}\right)^2$

36. $r = \left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)^2$

37. $y = (2x+1)\sqrt{2x+1}$

38. $y = 20(3x-4)^{1/4}(3x-4)^{-1/5}$

39. $y = \dfrac{3}{(5x^2+\sin 2x)^{3/2}}$

40. $y = (3+\cos^3 3x)^{-1/3}$

41. $y = 10e^{-x/5}$

42. $y = \sqrt{2}e^{\sqrt{2}x}$

43. $y = \dfrac{1}{4}xe^{4x} - \dfrac{1}{16}e^{4x}$

44. $y = x^2e^{-2/x}$

45. $y = \ln(\sin^2\theta)$

46. $y = \ln(\sec^2\theta)$

47. $y = \log_2(x^2/2)$

48. $y = \log_5(3x-7)$

49. $y = 8^{-t}$

50. $y = 9^{2t}$

51. $y = 5x^{3.6}$

52. $y = \sqrt{2}x^{-\sqrt{2}}$

53. $y = (x+2)^{x+2}$

54. $y = 2(\ln x)^{x/2}$

55. $y = \sin^{-1}\sqrt{1-u^2},\quad 0 < u < 1$

56. $y = \sin^{-1}\left(\dfrac{1}{\sqrt{v}}\right),\quad v > 1$

57. $y = \ln\cos^{-1}x$

58. $y = z\cos^{-1}z - \sqrt{1-z^2}$

59. $y = t\tan^{-1}t - \dfrac{1}{2}\ln t$

60. $y = (1+t^2)\cot^{-1}2t$

61. $y = z\sec^{-1}z - \sqrt{z^2-1},\quad z > 1$

62. $y = 2\sqrt{x-1}\sec^{-1}\sqrt{x}$

63. $y = \csc^{-1}(\sec\theta),\quad 0 < \theta < \pi/2$

64. $y = (1+x^2)e^{\tan^{-1}x}$

## Implicit Differentiation

In Exercises 65–78, find $dy/dx$ by implicit differentiation.

65. $xy + 2x + 3y = 1$

66. $x^2 + xy + y^2 - 5x = 2$

67. $x^3 + 4xy - 3y^{4/3} = 2x$

68. $5x^{4/5} + 10y^{6/5} = 15$

69. $\sqrt{xy} = 1$

70. $x^2y^2 = 1$

71. $y^2 = \dfrac{x}{x+1}$

72. $y^2 = \sqrt{\dfrac{1+x}{1-x}}$

73. $e^{x+2y} = 1$

74. $y^2 = 2e^{-1/x}$

75. $\ln(x/y) = 1$

76. $x\sin^{-1}y = 1 + x^2$

77. $ye^{\tan^{-1}x} = 2$

78. $x^y = \sqrt{2}$

In Exercises 79 and 80, find $dp/dq$.

79. $p^3 + 4pq - 3q^2 = 2$

80. $q = (5p^2 + 2p)^{-3/2}$

In Exercises 81 and 82, find $dr/ds$.

81. $r\cos 2s + \sin^2 s = \pi$

82. $2rs - r - s + s^2 = -3$

83. Find $d^2y/dx^2$ by implicit differentiation:

**a.** $x^3 + y^3 = 1$ **b.** $y^2 = 1 - \dfrac{2}{x}$

84. **a.** By differentiating $x^2 - y^2 = 1$ implicitly, show that $dy/dx = x/y$.

**b.** Then show that $d^2y/dx^2 = -1/y^3$.

## Numerical Values of Derivatives

85. Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 0 | 1 | 1 | $-3$ | $1/2$ |
| 1 | 3 | 5 | $1/2$ | $-4$ |

Find the first derivatives of the following combinations at the given value of $x$.

**a.** $6f(x) - g(x),\quad x = 1$ **b.** $f(x)g^2(x),\quad x = 0$

**c.** $\dfrac{f(x)}{g(x)+1},\quad x = 1$ **d.** $f(g(x)),\quad x = 0$

**e.** $g(f(x)),\quad x = 0$ **f.** $(x + f(x))^{3/2},\quad x = 1$

**g.** $f(x + g(x)),\quad x = 0$

86. Suppose that the function $f(x)$ and its first derivative have the following values at $x = 0$ and $x = 1$.

| $x$ | $f(x)$ | $f'(x)$ |
|---|---|---|
| 0 | 9 | $-2$ |
| 1 | $-3$ | $1/5$ |

Find the first derivatives of the following combinations at the given value of $x$.

**a.** $\sqrt{x}\,f(x),\quad x = 1$ **b.** $\sqrt{f(x)},\quad x = 0$

**c.** $f(\sqrt{x}),\quad x = 1$ **d.** $f(1 - 5\tan x),\quad x = 0$

**e.** $\dfrac{f(x)}{2+\cos x},\quad x = 0$ **f.** $10\sin\left(\dfrac{\pi x}{2}\right)f^2(x),\quad x = 1$

87. Find the value of $dy/dt$ at $t = 0$ if $y = 3\sin 2x$ and $x = t^2 + \pi$.

88. Find the value of $ds/du$ at $u = 2$ if $s = t^2 + 5t$ and $t = (u^2 + 2u)^{1/3}$.

89. Find the value of $dw/ds$ at $s = 0$ if $w = \sin\left(e^{\sqrt{r}}\right)$ and $r = 3\sin(s + \pi/6)$.

90. Find the value of $dr/dt$ at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

91. If $y^3 + y = 2\cos x$, find the value of $d^2y/dx^2$ at the point (0, 1).

92. If $x^{1/3} + y^{1/3} = 4$, find $d^2y/dx^2$ at the point (8, 8).

## Applying the Derivative Definition

In Exercises 93 and 94, find the derivative using the definition.

93. $f(t) = \dfrac{1}{2t+1}$

94. $g(x) = 2x^2 + 1$

95. **a.** Graph the function

$$f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1. \end{cases}$$

**b.** Is $f$ continuous at $x = 0$?

**c.** Is $f$ differentiable at $x = 0$?

Give reasons for your answers.

**96. a.** Graph the function

$$f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$$

**b.** Is $f$ continuous at $x = 0$?

**c.** Is $f$ differentiable at $x = 0$?

Give reasons for your answers.

**97. a.** Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

**b.** Is $f$ continuous at $x = 1$?

**c.** Is $f$ differentiable at $x = 1$?

Give reasons for your answers.

**98.** For what value or values of the constant $m$, if any, is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

**a.** continuous at $x = 0$?

**b.** differentiable at $x = 0$?

Give reasons for your answers.

## Slopes, Tangents, and Normals

**99. Tangents with specified slope** Are there any points on the curve $y = (x/2) + 1/(2x - 4)$ where the slope is $-3/2$? If so, find them.

**100. Tangents with specified slope** Are there any points on the curve $y = x - e^{-x}$ where the slope is 2? If so, find them.

**101. Horizontal tangents** Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the $x$-axis.

**102. Tangent intercepts** Find the $x$- and $y$-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

**103. Tangents perpendicular or parallel to lines** Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is

**a.** perpendicular to the line $y = 1 - (x/24)$.

**b.** parallel to the line $y = \sqrt{2} - 12x$.

**104. Intersecting tangents** Show that the tangents to the curve $y = (\pi \sin x)/x$ at $x = \pi$ and $x = -\pi$ intersect at right angles.

**105. Normals parallel to a line** Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the normal is parallel to the line $y = -x/2$. Sketch the curve and normals together, labeling each with its equation.

**106. Tangent and normal lines** Find equations for the tangent and normal to the curve $y = 1 + \cos x$ at the point $(\pi/2, 1)$. Sketch the curve, tangent, and normal together, labeling each with its equation.

**107. Tangent parabola** The parabola $y = x^2 + C$ is to be tangent to the line $y = x$. Find $C$.

**108. Slope of tangent** Show that the tangent to the curve $y = x^3$ at any point $(a, a^3)$ meets the curve again at a point where the slope is four times the slope at $(a, a^3)$.

**109. Tangent curve** For what value of $c$ is the curve $y = c/(x + 1)$ tangent to the line through the points $(0, 3)$ and $(5, -2)$?

**110. Normal to a circle** Show that the normal line at any point of the circle $x^2 + y^2 = a^2$ passes through the origin.

In Exercises 111–116, find equations for the lines that are tangent and normal to the curve at the given point.

**111.** $x^2 + 2y^2 = 9$, $(1, 2)$

**112.** $e^x + y^2 = 2$, $(0, 1)$

**113.** $xy + 2x - 5y = 2$, $(3, 2)$

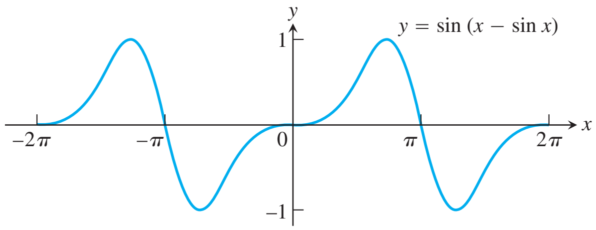
**114.** $(y - x)^2 = 2x + 4$, $(6, 2)$

**115.** $x + \sqrt{xy} = 6$, $(4, 1)$

**116.** $x^{3/2} + 2y^{3/2} = 17$, $(1, 4)$

**117.** Find the slope of the curve $x^3y^3 + y^2 = x + y$ at the points $(1, 1)$ and $(1, -1)$.

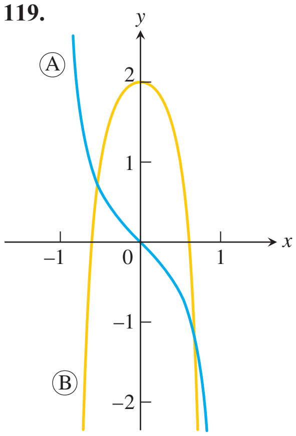
**118.** The graph shown suggests that the curve $y = \sin(x - \sin x)$ might have horizontal tangents at the $x$-axis. Does it? Give reasons for your answer.



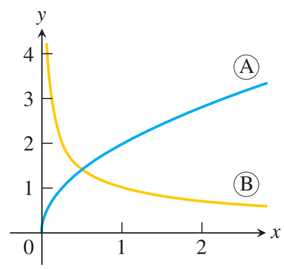

## Analyzing Graphs

Each of the figures in Exercises 119 and 120 shows two graphs, the graph of a function $y = f(x)$ together with the graph of its derivative $f'(x)$. Which graph is which? How do you know?

**119.**




**120.**

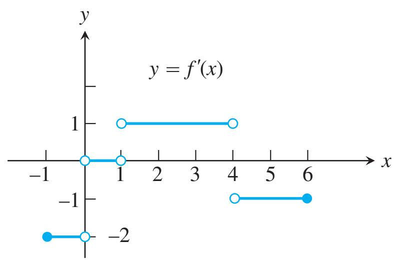



**121.** Use the following information to graph the function $y = f(x)$ for $-1 \le x \le 6$.

**i)** The graph of $f$ is made of line segments joined end to end.

**ii)** The graph starts at the point $(-1, 2)$.

**iii)** The derivative of $f$, where defined, agrees with the step function shown here.



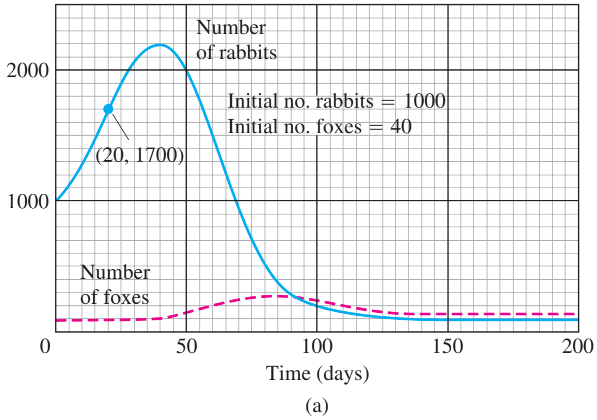
**122.** Repeat Exercise 121, supposing that the graph starts at $(-1, 0)$ instead of $(-1, 2)$.

Exercises 123 and 124 are about the accompanying graphs. The graphs in part (a) show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Part (b) shows the graph of the derivative of the rabbit population, made by plotting slopes.

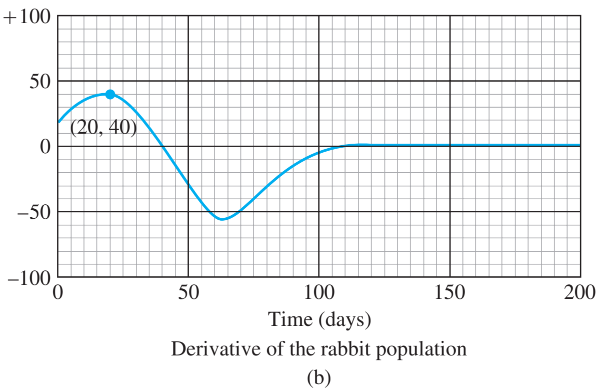
**123. a.** What is the value of the derivative of the rabbit population when the number of rabbits is largest? Smallest?

**b.** What is the size of the rabbit population when its derivative is largest? Smallest (negative value)?

**124.** In what units should the slopes of the rabbit and fox population curves be measured?



(a)



Derivative of the rabbit population

(b)

## Trigonometric Limits

Find the limits in Exercises 125–132.

**125.** $\lim_{x\to 0} \dfrac{\sin x}{2x^2 - x}$ **126.** $\lim_{x\to 0} \dfrac{3x - \tan 7x}{2x}$

**127.** $\lim_{r\to 0} \dfrac{\sin r}{\tan 2r}$ **128.** $\lim_{\theta\to 0} \dfrac{\sin(\sin\theta)}{\theta}$

**129.** $\lim_{\theta\to(\pi/2)^-} \dfrac{4\tan^2\theta + \tan\theta + 1}{\tan^2\theta + 5}$

**130.** $\lim_{\theta\to 0^+} \dfrac{1 - 2\cot^2\theta}{5\cot^2\theta - 7\cot\theta - 8}$

**131.** $\lim_{x\to 0} \dfrac{x\sin x}{2 - 2\cos x}$ **132.** $\lim_{\theta\to 0} \dfrac{1 - \cos\theta}{\theta^2}$

Show how to extend the functions in Exercises 133 and 134 to be continuous at the origin.

**133.** $g(x) = \dfrac{\tan(\tan x)}{\tan x}$ **134.** $f(x) = \dfrac{\tan(\tan x)}{\sin(\sin x)}$

## Logarithmic Differentiation

In Exercises 135–140, use logarithmic differentiation to find the derivative of $y$ with respect to the appropriate variable.

**135.** $y = \dfrac{2(x^2 + 1)}{\sqrt{\cos 2x}}$ **136.** $y = \sqrt[10]{\dfrac{3x + 4}{2x - 4}}$

**137.** $y = \left(\dfrac{(t + 1)(t - 1)}{(t - 2)(t + 3)}\right)^5, \quad t > 2$

**138.** $y = \dfrac{2u2^u}{\sqrt{u^2 + 1}}$

**139.** $y = (\sin\theta)^{\sqrt{\theta}}$ **140.** $y = (\ln x)^{1/(\ln x)}$

## Related Rates

**141. Right circular cylinder** The total surface area $S$ of a right circular cylinder is related to the base radius $r$ and height $h$ by the equation $S = 2\pi r^2 + 2\pi rh$.

**a.** How is $dS/dt$ related to $dr/dt$ if $h$ is constant?

**b.** How is $dS/dt$ related to $dh/dt$ if $r$ is constant?

**c.** How is $dS/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

**d.** How is $dr/dt$ related to $dh/dt$ if $S$ is constant?

**142. Right circular cone** The lateral surface area $S$ of a right circular cone is related to the base radius $r$ and height $h$ by the equation $S = \pi r\sqrt{r^2 + h^2}$.

**a.** How is $dS/dt$ related to $dr/dt$ if $h$ is constant?

**b.** How is $dS/dt$ related to $dh/dt$ if $r$ is constant?

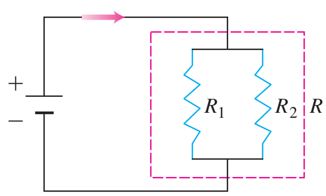
**c.** How is $dS/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

**143. Circle's changing area** The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?

**144. Cube's changing edges** The volume of a cube is increasing at the rate of 1200 cm$^3$/min at the instant its edges are 20 cm long. At what rate are the lengths of the edges changing at that instant?

**145. Resistors connected in parallel** If two resistors of $R_1$ and $R_2$ ohms are connected in parallel in an electric circuit to make an $R$-ohm resistor, the value of $R$ can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$




If $R_1$ is decreasing at the rate of 1 ohm/sec and $R_2$ is increasing at the rate of 0.5 ohm/sec, at what rate is $R$ changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?

**146. Impedance in a series circuit** The impedance $Z$ (ohms) in a series circuit is related to the resistance $R$ (ohms) and reactance $X$ (ohms) by the equation $Z = \sqrt{R^2 + X^2}$. If $R$ is increasing at 3 ohms/sec and $X$ is decreasing at 2 ohms/sec, at what rate is $Z$ changing when $R = 10$ ohms and $X = 20$ ohms?

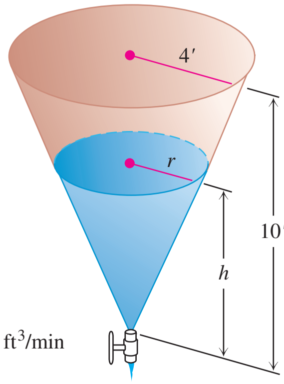
**147. Speed of moving particle** The coordinates of a particle moving in the metric $xy$-plane are differentiable functions of time $t$ with $dx/dt = 10$ m/sec and $dy/dt = 5$ m/sec. How fast is the particle moving away from the origin as it passes through the point $(3, -4)$?

**148. Motion of a particle** A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find $dx/dt$ when $x = 3$.

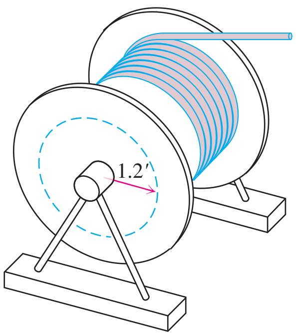
**149. Draining a tank** Water drains from the conical tank shown in the accompanying figure at the rate of 5 ft$^3$/min.

**a.** What is the relation between the variables $h$ and $r$ in the figure?

**b.** How fast is the water level dropping when $h = 6$ ft?



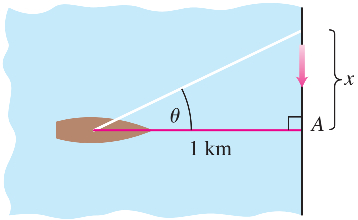

**150. Rotating spool** As television cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius (see accompanying figure). If the truck pulling the cable moves at a steady 6 ft/sec (a touch over 4 mph), use the equation $s = r\theta$ to find how fast (radians per second) the spool is turning when the layer of radius 1.2 ft is being unwound.




**151. Moving searchlight beam** The figure shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

**a.** How fast is the light moving along the shore when it reaches point $A$?

**b.** How many revolutions per minute is 0.6 rad/sec?




**152. Points moving on coordinate axes** Points $A$ and $B$ move along the $x$- and $y$-axes, respectively, in such a way that the distance $r$ (meters) along the perpendicular from the origin to the line $AB$ remains constant. How fast is $OA$ changing, and is it increasing, or decreasing, when $OB = 2r$ and $B$ is moving toward $O$ at the rate of $0.3r$ m/sec?

## Linearization

**153.** Find the linearizations of

**a.** $\tan x$ at $x = -\pi/4$  **b.** $\sec x$ at $x = -\pi/4$.

Graph the curves and linearizations together.

**154.** We can obtain a useful linear approximation of the function $f(x) = 1/(1 + \tan x)$ at $x = 0$ by combining the approximations

$$\frac{1}{1+x} \approx 1 - x \qquad \text{and} \qquad \tan x \approx x$$

to get

$$\frac{1}{1+\tan x} \approx 1 - x.$$

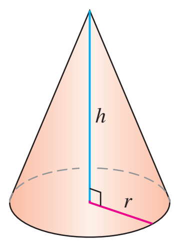
Show that this result is the standard linear approximation of $1/(1 + \tan x)$ at $x = 0$.

**155.** Find the linearization of $f(x) = \sqrt{1+x} + \sin x - 0.5$ at $x = 0$.

**156.** Find the linearization of $f(x) = 2/(1 - x) + \sqrt{1+x} - 3.1$ at $x = 0$.

## Differential Estimates of Change

**157. Surface area of a cone** Write a formula that estimates the change that occurs in the lateral surface area of a right circular cone when the height changes from $h_0$ to $h_0 + dh$ and the radius does not change.




$V = \frac{1}{3}\pi r^2 h$

$S = \pi r\sqrt{r^2 + h^2}$

(Lateral surface area)

**158. Controlling error**

**a.** How accurately should you measure the edge of a cube to be reasonably sure of calculating the cube's surface area with an error of no more than 2%?

**b.** Suppose that the edge is measured with the accuracy required in part (a). About how accurately can the cube's volume be calculated from the edge measurement? To find out, estimate the percentage error in the volume calculation that might result from using the edge measurement.

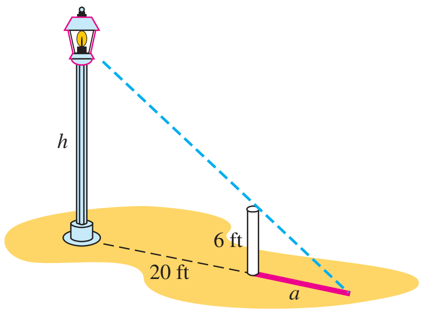
**159. Compounding error** The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is then used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage errors in the calculated values of

**a.** the radius.

**b.** the surface area.

**c.** the volume.

**160. Finding height** To find the height of a lamppost (see accompanying figure), you stand a 6 ft pole 20 ft from the lamp and measure the length $a$ of its shadow, finding it to be 15 ft, give or take an inch. Calculate the height of the lamppost using the value $a = 15$ and estimate the possible error in the result.




## Chapter 3 Additional and Advanced Exercises

**1.** An equation like $\sin^2\theta + \cos^2\theta = 1$ is called an **identity** because it holds for all values of $\theta$. An equation like $\sin\theta = 0.5$ is not an identity because it holds only for selected values of $\theta$, not all. If you differentiate both sides of a trigonometric identity in $\theta$ with respect to $\theta$, the resulting new equation will also be an identity.

Differentiate the following to show that the resulting equations hold for all $\theta$.

**a.** $\sin 2\theta = 2\sin\theta\cos\theta$

**b.** $\cos 2\theta = \cos^2\theta - \sin^2\theta$

**2.** If the identity $\sin(x + a) = \sin x \cos a + \cos x \sin a$ is differentiated with respect to $x$, is the resulting equation also an identity? Does this principle apply to the equation $x^2 - 2x - 8 = 0$? Explain.

**3. a.** Find values for the constants $a$, $b$, and $c$ that will make

$$f(x) = \cos x \quad \text{and} \quad g(x) = a + bx + cx^2$$

satisfy the conditions

$$f(0) = g(0), \quad f'(0) = g'(0), \quad \text{and} \quad f''(0) = g''(0).$$

**b.** Find values for $b$ and $c$ that will make

$$f(x) = \sin(x + a) \quad \text{and} \quad g(x) = b\sin x + c\cos x$$

satisfy the conditions

$$f(0) = g(0) \quad \text{and} \quad f'(0) = g'(0).$$

**c.** For the determined values of $a$, $b$, and $c$, what happens for the third and fourth derivatives of $f$ and $g$ in each of parts (a) and (b)?

**4. Solutions to differential equations**

**a.** Show that $y = \sin x$, $y = \cos x$, and $y = a\cos x + b\sin x$ ($a$ and $b$ constants) all satisfy the equation

$$y'' + y = 0.$$

**b.** How would you modify the functions in part (a) to satisfy the equation

$$y'' + 4y = 0?$$

Generalize this result.

**5. An osculating circle** Find the values of $h$, $k$, and $a$ that make the circle $(x - h)^2 + (y - k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point $(1, 2)$ and that also make the second derivatives $d^2y/dx^2$ have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called *osculating circles* (from the Latin *osculari*, meaning "to kiss"). We encounter them again in Chapter 13.

**6. Marginal revenue** A bus will hold 60 people. The number $x$ of people per trip who use the bus is related to the fare charged ($p$ dollars) by the law $p = [3 - (x/40)]^2$. Write an expression for the total revenue $r(x)$ per trip received by the bus company. What number of people per trip will make the marginal revenue $dr/dx$ equal to zero? What is the corresponding fare? (This fare is the one that maximizes the revenue, so the bus company should probably rethink its fare policy.)

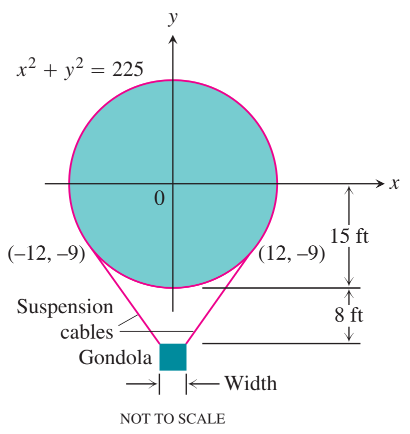
**7. Industrial production**

**a.** Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time $t$ in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time $t$. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, $y$.

b. Suppose that the labor force in part (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

8. **Designing a gondola** The designer of a 30-ft-diameter spherical hot air balloon wants to suspend the gondola 8 ft below the bottom of the balloon with cables tangent to the surface of the balloon, as shown. Two of the cables are shown running from the top edges of the gondola to their points of tangency, $(-12, -9)$ and $(12, -9)$. How wide should the gondola be?




9. **Pisa by parachute** On August 5, 1988, Mike McCarthy of London jumped from the top of the Tower of Pisa. He then opened his parachute in what he said was a world record low-level parachute jump of 179 ft. Make a rough sketch to show the shape of the graph of his speed during the jump. (*Source: Boston Globe*, Aug. 6, 1988.)

10. **Motion of a particle** The position at time $t \ge 0$ of a particle moving along a coordinate line is

$$s = 10\cos(t + \pi/4).$$

a. What is the particle's starting position $(t = 0)$?

b. What are the points farthest to the left and right of the origin reached by the particle?

c. Find the particle's velocity and acceleration at the points in part (b).

d. When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

11. **Shooting a paper clip** On Earth, you can easily shoot a paper clip 64 ft straight up into the air with a rubber band. In $t$ sec after firing, the paper clip is $s = 64t - 16t^2$ ft above your hand.

a. How long does it take the paper clip to reach its maximum height? With what velocity does it leave your hand?

b. On the moon, the same acceleration will send the paper clip to a height of $s = 64t - 2.6t^2$ ft in $t$ sec. About how long will it take the paper clip to reach its maximum height, and how high will it go?

12. **Velocities of two particles** At time $t$ sec, the positions of two particles on a coordinate line are $s_1 = 3t^3 - 12t^2 + 18t + 5$ m and $s_2 = -t^3 + 9t^2 - 12t$ m. When do the particles have the same velocities?

13. **Velocity of a particle** A particle of constant mass $m$ moves along the $x$-axis. Its velocity $v$ and position $x$ satisfy the equation

$$\frac{1}{2}m(v^2 - {v_0}^2) = \frac{1}{2}k({x_0}^2 - x^2),$$

where $k$, $v_0$, and $x_0$ are constants. Show that whenever $v \ne 0$,

$$m\frac{dv}{dt} = -kx.$$

14. **Average and instantaneous velocity**

a. Show that if the position $x$ of a moving point is given by a quadratic function of $t$, $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1, t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.

b. What is the geometric significance of the result in part (a)?

15. Find all values of the constants $m$ and $b$ for which the function

$$y = \begin{cases} \sin x, & x < \pi \\ mx + b, & x \ge \pi \end{cases}$$

is

a. continuous at $x = \pi$.

b. differentiable at $x = \pi$.

16. Does the function

$$f(x) = \begin{cases} \dfrac{1 - \cos x}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

have a derivative at $x = 0$? Explain.

17. a. For what values of $a$ and $b$ will

$$f(x) = \begin{cases} ax, & x < 2 \\ ax^2 - bx + 3, & x \ge 2 \end{cases}$$

be differentiable for all values of $x$?

b. Discuss the geometry of the resulting graph of $f$.

18. a. For what values of $a$ and $b$ will

$$g(x) = \begin{cases} ax + b, & x \le -1 \\ ax^3 + x + 2b, & x > -1 \end{cases}$$

be differentiable for all values of $x$?

b. Discuss the geometry of the resulting graph of $g$.

19. **Odd differentiable functions** Is there anything special about the derivative of an odd differentiable function of $x$? Give reasons for your answer.

20. **Even differentiable functions** Is there anything special about the derivative of an even differentiable function of $x$? Give reasons for your answer.

21. Suppose that the functions $f$ and $g$ are defined throughout an open interval containing the point $x_0$, that $f$ is differentiable at $x_0$, that $f(x_0) = 0$, and that $g$ is continuous at $x_0$. Show that the product $fg$ is differentiable at $x_0$. This process shows, for example, that although $|x|$ is not differentiable at $x = 0$, the product $x|x|$ *is* differentiable at $x = 0$.

**22.** (*Continuation of Exercise 21.*) Use the result of Exercise 21 to show that the following functions are differentiable at $x = 0$.

**a.** $|x| \sin x$ **b.** $x^{2/3} \sin x$ **c.** $\sqrt[3]{x}(1 - \cos x)$

**d.** $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

**23.** Is the derivative of

$$h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answers.

**24.** Suppose that a function $f$ satisfies the following conditions for all real values of $x$ and $y$:

**i)** $f(x + y) = f(x) \cdot f(y)$.

**ii)** $f(x) = 1 + xg(x)$, where $\lim_{x \to 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists at every value of $x$ and that $f'(x) = f(x)$.

**25. The generalized product rule** Use mathematical induction to prove that if $y = u_1 u_2 \cdots u_n$ is a finite product of differentiable functions, then $y$ is differentiable on their common domain and

$$\frac{dy}{dx} = \frac{du_1}{dx} u_2 \cdots u_n + u_1 \frac{du_2}{dx} \cdots u_n + \cdots + u_1 u_2 \cdots u_{n-1} \frac{du_n}{dx}.$$

**26. Leibniz's rule for higher-order derivatives of products** Leibniz's rule for higher-order derivatives of products of differentiable functions says that

**a.** $\dfrac{d^2(uv)}{dx^2} = \dfrac{d^2u}{dx^2} v + 2 \dfrac{du}{dx} \dfrac{dv}{dx} + u \dfrac{d^2v}{dx^2}$.

**b.** $\dfrac{d^3(uv)}{dx^3} = \dfrac{d^3u}{dx^3} v + 3 \dfrac{d^2u}{dx^2} \dfrac{dv}{dx} + 3 \dfrac{du}{dx} \dfrac{d^2v}{dx^2} + u \dfrac{d^3v}{dx^3}$.

**c.** $$\frac{d^n(uv)}{dx^n} = \frac{d^nu}{dx^n} v + n \frac{d^{n-1}u}{dx^{n-1}} \frac{dv}{dx} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{d^{n-k}u}{dx^{n-k}} \frac{d^kv}{dx^k} + \cdots + u \frac{d^nv}{dx^n}.$$

The equations in parts (a) and (b) are special cases of the equation in part (c). Derive the equation in part (c) by mathematical induction, using

$$\binom{m}{k} + \binom{m}{k+1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k+1)!(m-k-1)!}.$$

**27. The period of a clock pendulum** The period $T$ of a clock pendulum (time for one full swing and back) is given by the formula $T^2 = 4\pi^2 L/g$, where $T$ is measured in seconds, $g = 32.2$ ft/sec$^2$, and $L$, the length of the pendulum, is measured in feet. Find approximately

**a.** the length of a clock pendulum whose period is $T = 1$ sec.

**b.** the change $dT$ in $T$ if the pendulum in part (a) is lengthened 0.01 ft.

**c.** the amount the clock gains or loses in a day as a result of the period's changing by the amount $dT$ found in part (b).

**28. The melting ice cube** Assume that an ice cube retains its cubical shape as it melts. If we call its edge length $s$, its volume is $V = s^3$ and its surface area is $6s^2$. We assume that $V$ and $s$ are differentiable functions of time $t$. We assume also that the cube's volume decreases at a rate that is proportional to its surface area. (This latter assumption seems reasonable enough when we think that the melting takes place at the surface: Changing the amount of surface changes the amount of ice exposed to melt.) In mathematical terms,

$$\frac{dV}{dt} = -k(6s^2), \qquad k > 0.$$

The minus sign indicates that the volume is decreasing. We assume that the proportionality factor $k$ is constant. (It probably depends on many things, such as the relative humidity of the surrounding air, the air temperature, and the incidence or absence of sunlight, to name only a few.) Assume a particular set of conditions in which the cube lost $1/4$ of its volume during the first hour, and that the volume is $V_0$ when $t = 0$. How long will it take the ice cube to melt?

# Chapter 3 Technology Application Projects

## Mathematica/Maple Modules:

***Convergence of Secant Slopes to the Derivative Function***
You will visualize the secant line between successive points on a curve and observe what happens as the distance between them becomes small. The function, sample points, and secant lines are plotted on a single graph, while a second graph compares the slopes of the secant lines with the derivative function.

***Derivatives, Slopes, Tangent Lines, and Making Movies***
**Parts I–III**. You will visualize the derivative at a point, the linearization of a function, and the derivative of a function. You learn how to plot the function and selected tangents on the same graph.
**Part IV (Plotting Many Tangents)**
**Part V (Making Movies)**. Parts IV and V of the module can be used to animate tangent lines as one moves along the graph of a function.

***Convergence of Secant Slopes to the Derivative Function***
You will visualize right-hand and left-hand derivatives.

***Motion Along a Straight Line: Position → Velocity → Acceleration***
Observe dramatic animated visualizations of the derivative relations among the position, velocity, and acceleration functions. Figures in the text can be animated.

# 4

# Applications of Derivatives

**OVERVIEW** In this chapter we use derivatives to find extreme values of functions, to determine and analyze the shapes of graphs, and to find numerically where a function equals zero. We also introduce the idea of recovering a function from its derivative. The key to many of these applications is the Mean Value Theorem, which paves the way to integral calculus in Chapter 5.

## 4.1 Extreme Values of Functions

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative. Once we can do this, we can solve a variety of problems in which we find the optimal (best) way to do something in a given situation (see Section 4.6). Finding maximum and minimum values is one of the most important applications of the derivative.

**DEFINITIONS** Let $f$ be a function with domain $D$. Then $f$ has an **absolute maximum** value on $D$ at a point $c$ if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on $D$ at $c$ if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

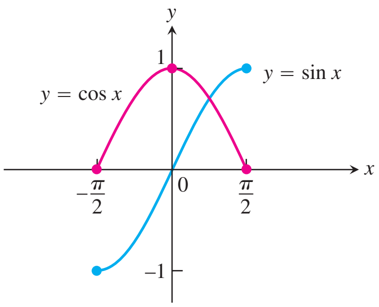



**FIGURE 4.1** Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Maximum and minimum values are called **extreme values** of the function $f$. Absolute maxima or minima are also referred to as **global** maxima or minima.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of $-1$ (Figure 4.1).

Functions with the same defining rule or formula can have different extrema (maximum or minimum values), depending on the domain. We see this in the following example.

**EXAMPLE 1** The absolute extrema of the following functions on their domains can be seen in Figure 4.2. Notice that a function might not have a maximum or minimum if the domain is unbounded or fails to contain an endpoint.

| **Function rule** | **Domain *D*** | **Absolute extrema on *D*** |
|---|---|---|
| **(a)** $y = x^2$ | $(-\infty, \infty)$ | No absolute maximum.<br>Absolute minimum of 0 at $x = 0$. |
| **(b)** $y = x^2$ | $[0, 2]$ | Absolute maximum of 4 at $x = 2$.<br>Absolute minimum of 0 at $x = 0$. |
| **(c)** $y = x^2$ | $(0, 2]$ | Absolute maximum of 4 at $x = 2$.<br>No absolute minimum. |
| **(d)** $y = x^2$ | $(0, 2)$ | No absolute extrema. |

■

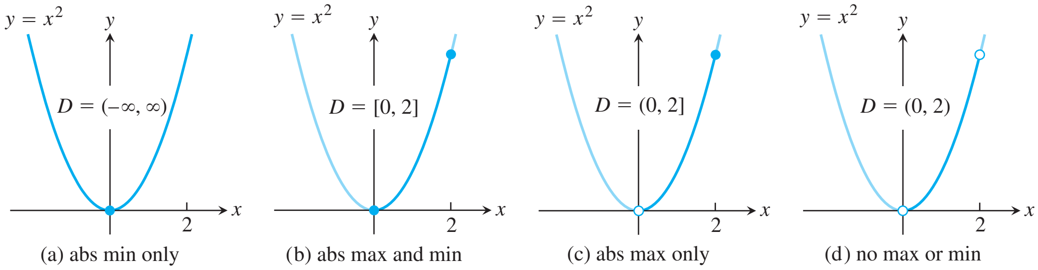



**FIGURE 4.2** Graphs for Example 1.

HISTORICAL BIOGRAPHY

Daniel Bernoulli
(1700–1789)

Some of the functions in Example 1 did not have a maximum or a minimum value. The following theorem asserts that a function which is *continuous* at every point of a *closed* interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval. We look for these extreme values when we graph a function.

> **THEOREM 1—The Extreme Value Theorem** If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains both an absolute maximum value $M$ and an absolute minimum value $m$ in $[a, b]$. That is, there are numbers $x_1$ and $x_2$ in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other $x$ in $[a, b]$.

The proof of the Extreme Value Theorem requires a detailed knowledge of the real number system (see Appendix 6) and we will not give it here. Figure 4.3 illustrates possible locations for the absolute extrema of a continuous function on a closed interval $[a, b]$. As we observed for the function $y = \cos x$, it is possible that an absolute minimum (or absolute maximum) may occur at two or more different points of the interval.

The requirements in Theorem 1 that the interval be closed and finite, and that the function be continuous, are key ingredients. Without them, the conclusion of the theorem

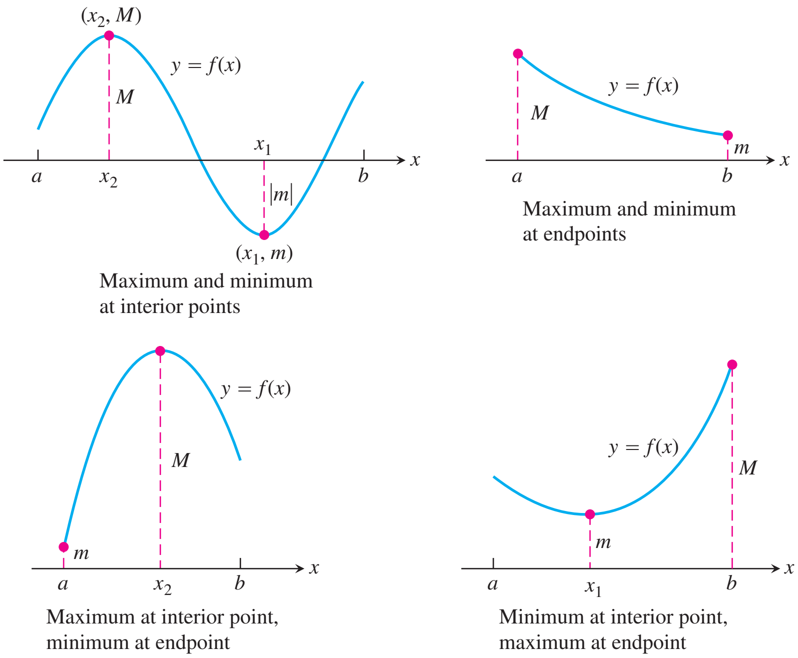



**FIGURE 4.3** Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

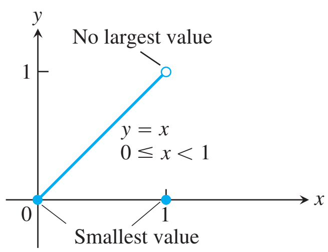



**FIGURE 4.4** Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1]$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

need not hold. Example 1 shows that an absolute extreme value may not exist if the interval fails to be both closed and finite. Figure 4.4 shows that the continuity requirement cannot be omitted.

## Local (Relative) Extreme Values

Figure 4.5 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at $a$ even though at $e$ the function's value is smaller than at any other point *nearby*. The curve rises to the left and falls to the right around $c$, making $f(c)$ a maximum locally. The function attains its absolute maximum at $d$. We now define what we mean by local extrema.

**DEFINITIONS** A function $f$ has a **local maximum** value at a point $c$ within its domain $D$ if $f(x) \le f(c)$ for all $x \in D$ lying in some open interval containing $c$.

A function $f$ has a **local minimum** value at a point $c$ within its domain $D$ if $f(x) \ge f(c)$ for all $x \in D$ lying in some open interval containing $c$.

If the domain of $f$ is the closed interval $[a, b]$, then $f$ has a local maximum at the endpoint $x = a$, if $f(x) \le f(a)$ for all $x$ in some half-open interval $[a, a + \delta)$, $\delta > 0$. Likewise, $f$ has a local maximum at an interior point $x = c$ if $f(x) \le f(c)$ for all $x$ in some open interval $(c - \delta, c + \delta)$, $\delta > 0$, and a local maximum at the endpoint $x = b$ if $f(x) \le f(b)$ for all $x$ in some half-open interval $(b - \delta, b]$, $\delta > 0$. The inequalities are reversed for local minimum values. In Figure 4.5, the function $f$ has local maxima at $c$ and $d$ and local minima at $a$, $e$, and $b$. Local extrema are also called **relative extrema**. Some functions can have infinitely many local extrema, even over a finite interval. One example is the function $f(x) = \sin(1/x)$ on the interval $(0, 1]$. (We graphed this function in Figure 2.40.)

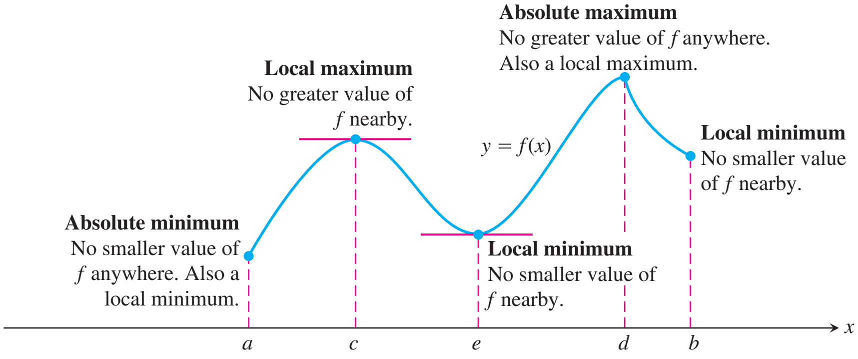



**FIGURE 4.5** How to identify types of maxima and minima for a function with domain $a \le x \le b$.

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will automatically include the absolute maximum if there is one*. Similarly, *a list of all local minima will include the absolute minimum if there is one*.

## Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

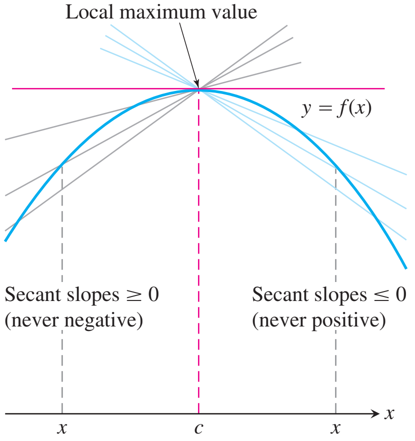



**FIGURE 4.6** A curve with a local maximum value. The slope at $c$, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

> **THEOREM 2—The First Derivative Theorem for Local Extreme Values** If $f$ has a local maximum or minimum value at an interior point $c$ of its domain, and if $f'$ is defined at $c$, then
>
> $$f'(c) = 0.$$

**Proof** To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that $f$ has a local maximum value at $x = c$ (Figure 4.6) so that $f(x) - f(c) \le 0$ for all values of $x$ near enough to $c$. Since $c$ is an interior point of $f$'s domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

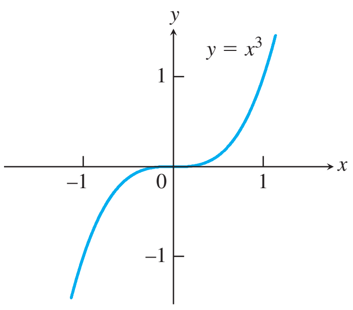
$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0. \qquad \text{Because } (x - c) > 0 \text{ and } f(x) \le f(c) \qquad (1)$$

Similarly,

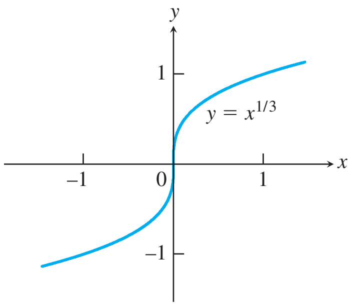
$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0. \qquad \text{Because } (x - c) < 0 \text{ and } f(x) \le f(c) \qquad (2)$$

Together, Equations (1) and (2) imply $f'(c) = 0$.

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \ge f(c)$, which reverses the inequalities in Equations (1) and (2). ■




(a)




(b)

**FIGURE 4.7** Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function $f$ can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$,
2. interior points where $f'$ is undefined,
3. endpoints of the domain of $f$.

The following definition helps us to summarize.

**DEFINITION** An interior point of the domain of a function $f$ where $f'$ is zero or undefined is a **critical point** of $f$.

Thus the only domain points where a function can assume extreme values are critical points and endpoints. However, be careful not to misinterpret what is being said here. A function may have a critical point at $x = c$ without having a local extreme value there. For instance, both of the functions $y = x^3$ and $y = x^{1/3}$ have critical points at the origin and a zero value there, but each function is positive to the right of the origin and negative to the left. So neither function has a local extreme value at the origin. Instead, each function has a *point of inflection* there (see Figure 4.7). We define and explore inflection points in Section 4.4.

Most problems that ask for extreme values call for finding the absolute extrema of a continuous function on a closed and finite interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints. Often we can simply list these points and calculate the corresponding function values to find what the largest and smallest values are, and where they are located. Of course, if the interval is not closed or not finite (such as $a < x < b$ or $a < x < \infty$), we have seen that absolute extrema need not exist. If an absolute maximum or minimum value does exist, it must occur at a critical point or at an included right- or left-hand endpoint of the interval.

**How to Find the Absolute Extrema of a Continuous Function $f$ on a Finite Closed Interval**

1. Evaluate $f$ at all critical points and endpoints.
2. Take the largest and smallest of these values.

**EXAMPLE 2** Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

**Solution** The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = 4$

$f(1) = 1$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

**EXAMPLE 3** Find the absolute maximum and minimum values of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

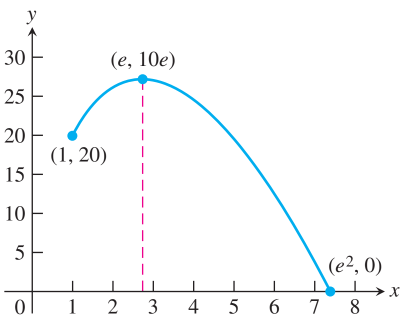



**FIGURE 4.8** The extreme values of $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$ occur at $x = e$ and $x = e^2$ (Example 3).

**Solution** Figure 4.8 suggests that $f$ has its absolute maximum value near $x = 3$ and its absolute minimum value of 0 at $x = e^2$. Let's verify this observation.

We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative is

$$f'(x) = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 10(1 - \ln x).$$

The only critical point in the domain $[1, e^2]$ is the point $x = e$, where $\ln x = 1$. The values of $f$ at this one critical point and at the endpoints are

Critical point value: $f(e) = 10e$

Endpoint values: $f(1) = 10(2 - \ln 1) = 20$

$f(e^2) = 10e^2(2 - 2\ln e) = 0.$

We can see from this list that the function's absolute maximum value is $10e \approx 27.2$; it occurs at the critical interior point $x = e$. The absolute minimum value is 0 and occurs at the right endpoint $x = e^2$. ■

**EXAMPLE 4** Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

**Solution** We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of $f$ at this one critical point and at the endpoints are

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

$f(3) = (3)^{2/3} = \sqrt[3]{9}.$

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp (Figure 4.9). ■

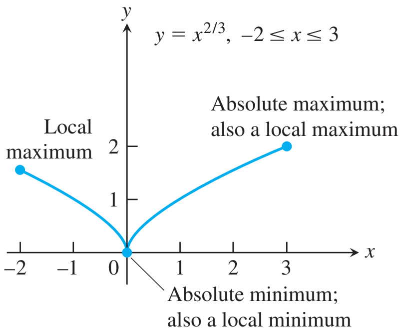


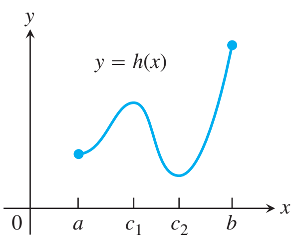

**FIGURE 4.9** The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

## Exercises 4.1

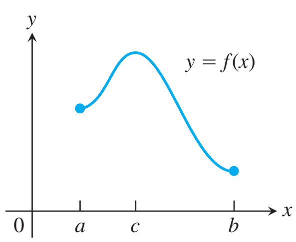
### Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Theorem 1.

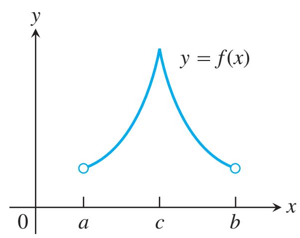
**1.**



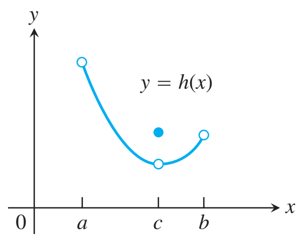

**2.**



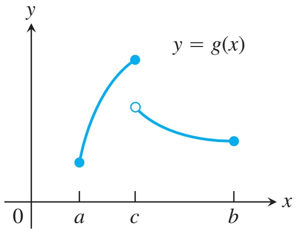

**3.**



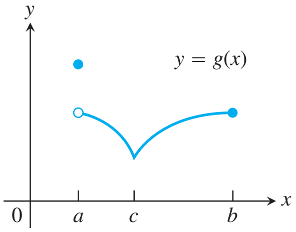

**4.**



**5.**

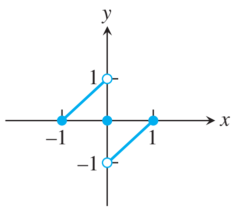



**6.**

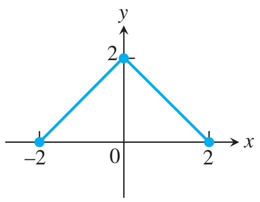



In Exercises 7–10, find the absolute extreme values and where they occur.

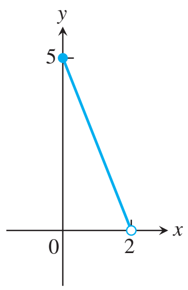
**7.**



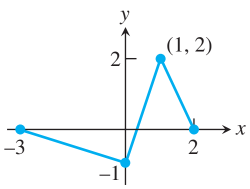

**8.**




**9.**




**10.**




In Exercises 11–14, match the table with a graph.

**11.**

| $x$ | $f'(x)$ |
|---|---|
| $a$ | 0 |
| $b$ | 0 |
| $c$ | 5 |

**12.**

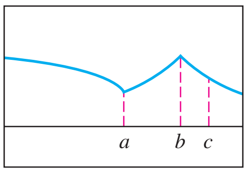
| $x$ | $f'(x)$ |
|---|---|
| $a$ | 0 |
| $b$ | 0 |
| $c$ | $-5$ |

**13.**

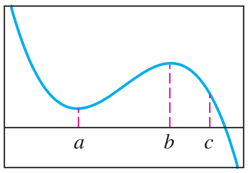
| $x$ | $f'(x)$ |
|---|---|
| $a$ | does not exist |
| $b$ | 0 |
| $c$ | $-2$ |

**14.**

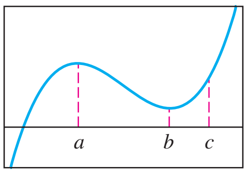
| $x$ | $f'(x)$ |
|---|---|
| $a$ | does not exist |
| $b$ | does not exist |
| $c$ | $-1.7$ |



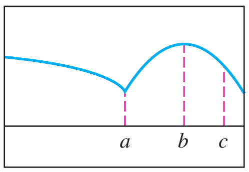

(a)




(b)




(c)




(d)

In Exercises 15–20, sketch the graph of each function and determine whether the function has any absolute extreme values on its domain. Explain how your answer is consistent with Theorem 1.

**15.** $f(x) = |x|, \quad -1 < x < 2$

**16.** $y = \dfrac{6}{x^2 + 2}, \quad -1 < x < 1$

**17.** $g(x) = \begin{cases} -x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2 \end{cases}$

**18.** $h(x) = \begin{cases} \dfrac{1}{x}, & -1 \le x < 0 \\ \sqrt{x}, & 0 \le x \le 4 \end{cases}$

**19.** $y = 3 \sin x, \quad 0 < x < 2\pi$

**20.** $f(x) = \begin{cases} x + 1, & -1 \le x < 0 \\ \cos x, & 0 \le x \le \dfrac{\pi}{2} \end{cases}$

## Absolute Extrema on Finite Closed Intervals

In Exercises 21–40, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

**21.** $f(x) = \dfrac{2}{3}x - 5, \quad -2 \le x \le 3$

**22.** $f(x) = -x - 4, \quad -4 \le x \le 1$

**23.** $f(x) = x^2 - 1, \quad -1 \le x \le 2$

**24.** $f(x) = 4 - x^2, \quad -3 \le x \le 1$

**25.** $F(x) = -\dfrac{1}{x^2}, \quad 0.5 \le x \le 2$

**26.** $F(x) = -\dfrac{1}{x}, \quad -2 \le x \le -1$

**27.** $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$

**28.** $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

**29.** $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$

**30.** $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

**31.** $f(\theta) = \sin \theta, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{5\pi}{6}$

**32.** $f(\theta) = \tan \theta, \quad -\dfrac{\pi}{3} \le \theta \le \dfrac{\pi}{4}$

**33.** $g(x) = \csc x, \quad \dfrac{\pi}{3} \le x \le \dfrac{2\pi}{3}$

**34.** $g(x) = \sec x, \quad -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{6}$

**35.** $f(t) = 2 - |t|, \quad -1 \le t \le 3$

**36.** $f(t) = |t - 5|, \quad 4 \le t \le 7$

**37.** $g(x) = xe^{-x}, \quad -1 \le x \le 1$

**38.** $h(x) = \ln(x + 1), \quad 0 \le x \le 3$

**39.** $f(x) = \dfrac{1}{x} + \ln x, \quad 0.5 \le x \le 4$

**40.** $g(x) = e^{-x^2}, \quad -2 \le x \le 1$

In Exercises 41–44, find the function's absolute maximum and minimum values and say where they are assumed.

**41.** $f(x) = x^{4/3}, \quad -1 \le x \le 8$

**42.** $f(x) = x^{5/3}, \quad -1 \le x \le 8$

**43.** $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

**44.** $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

## Finding Critical Points

In Exercises 45–52, determine all critical points for each function.

**45.** $y = x^2 - 6x + 7$

**46.** $f(x) = 6x^2 - x^3$

**47.** $f(x) = x(4 - x)^3$

**48.** $g(x) = (x - 1)^2(x - 3)^2$

**49.** $y = x^2 + \dfrac{2}{x}$

**50.** $f(x) = \dfrac{x^2}{x - 2}$

**51.** $y = x^2 - 32\sqrt{x}$

**52.** $g(x) = \sqrt{2x - x^2}$

## Finding Extreme Values

In Exercises 53–68, find the extreme values (absolute and local) of the function and where they occur.

**53.** $y = 2x^2 - 8x + 9$

**54.** $y = x^3 - 2x + 4$

**55.** $y = x^3 + x^2 - 8x + 5$

**56.** $y = x^3(x - 5)^2$

**57.** $y = \sqrt{x^2 - 1}$

**58.** $y = x - 4\sqrt{x}$

**59.** $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$

**60.** $y = \sqrt{3 + 2x - x^2}$

**61.** $y = \dfrac{x}{x^2 + 1}$

**62.** $y = \dfrac{x + 1}{x^2 + 2x + 2}$

**63.** $y = e^x + e^{-x}$

**64.** $y = e^x - e^{-x}$

**65.** $y = x \ln x$

**66.** $y = x^2 \ln x$

**67.** $y = \cos^{-1}(x^2)$

**68.** $y = \sin^{-1}(e^x)$

## Local Extrema and Critical Points

In Exercises 69–76, find the critical points, domain endpoints, and extreme values (absolute and local) for each function.

**69.** $y = x^{2/3}(x + 2)$

**70.** $y = x^{2/3}(x^2 - 4)$

**71.** $y = x\sqrt{4 - x^2}$

**72.** $y = x^2\sqrt{3 - x}$

**73.** $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

**74.** $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

**75.** $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

**76.** $y = \begin{cases} -\dfrac{1}{4}x^2 - \dfrac{1}{2}x + \dfrac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

In Exercises 77 and 78, give reasons for your answers.

**77.** Let $f(x) = (x - 2)^{2/3}$.

**a.** Does $f'(2)$ exist?

**b.** Show that the only local extreme value of $f$ occurs at $x = 2$.

**c.** Does the result in part (b) contradict the Extreme Value Theorem?

**d.** Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by $a$.

**78.** Let $f(x) = |x^3 - 9x|$.

**a.** Does $f'(0)$ exist?

**b.** Does $f'(3)$ exist?

**c.** Does $f'(-3)$ exist?

**d.** Determine all extrema of $f$.

## Theory and Examples

**79. A minimum with no derivative** The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though $f$ is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

**80. Even functions** If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of $f$ at $x = -c$? Give reasons for your answer.

**81. Odd functions** If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of $g$ at $x = -c$? Give reasons for your answer.

**82.** We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

**83.** The function

$$V(x) = x(10 - 2x)(16 - 2x), \qquad 0 < x < 5,$$

models the volume of a box.

**a.** Find the extreme values of $V$.

**b.** Interpret any values found in part (a) in terms of the volume of the box.

**84. Cubic functions** Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

**a.** Show that $f$ can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

**b.** How many local extreme values can $f$ have?

**85. Maximum height of a vertically moving body** The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \qquad g > 0,$$

with $s$ in meters and $t$ in seconds. Find the body's maximum height.

**86. Peak alternating current** Suppose that at any given time $t$ (in seconds) the current $i$ (in amperes) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak current for this circuit (largest magnitude)?

**T** Graph the functions in Exercises 87–90. Then find the extreme values of the function on the interval and say where they occur.

**87.** $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

**88.** $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

**89.** $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

**90.** $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

## COMPUTER EXPLORATIONS

In Exercises 91–98, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps.

**a.** Plot the function over the interval to see its general behavior there.

**b.** Find the interior points where $f' = 0$. (In some exercises, you may have to use the numerical equation solver to approximate a solution.) You may want to plot $f'$ as well.

**c.** Find the interior points where $f'$ does not exist.

**d.** Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.

**e.** Find the function's absolute extreme values on the interval and identify where they occur.

**91.** $f(x) = x^4 - 8x^2 + 4x + 2, \quad [-20/25, 64/25]$

**92.** $f(x) = -x^4 + 4x^3 - 4x + 1, \quad [-3/4, 3]$

**93.** $f(x) = x^{2/3}(3 - x), \quad [-2, 2]$

**94.** $f(x) = 2 + 2x - 3x^{2/3}, \quad [-1, 10/3]$

**95.** $f(x) = \sqrt{x} + \cos x, \quad [0, 2\pi]$

**96.** $f(x) = x^{3/4} - \sin x + \frac{1}{2}, \quad [0, 2\pi]$

**97.** $f(x) = \pi x^2 e^{-3x/2}, \quad [0, 5]$

**98.** $f(x) = \ln(2x + x \sin x), \quad [1, 15]$

# 4.2 The Mean Value Theorem

We know that constant functions have zero derivatives, but could there be a more complicated function whose derivative is always zero? If two functions have identical derivatives over an interval, how are the functions related? We answer these and other questions in this chapter by applying the Mean Value Theorem. First we introduce a special case, known as Rolle's Theorem, which is used to prove the Mean Value Theorem.

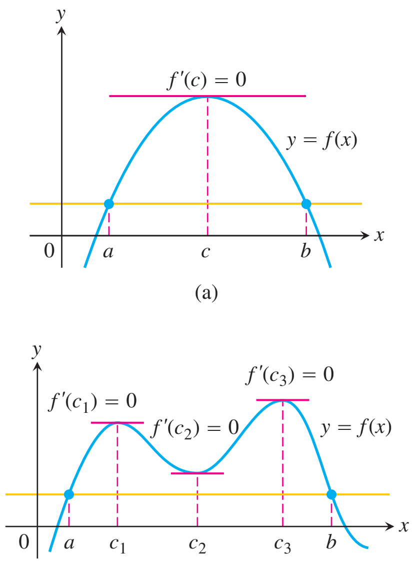



**FIGURE 4.10** Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

## Rolle's Theorem

As suggested by its graph, if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero (Figure 4.10). We now state and prove this result.

> **THEOREM 3—Rolle's Theorem** Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$. If $f(a) = f(b)$, then there is at least one number $c$ in $(a, b)$ at which $f'(c) = 0$.

**Proof** Being continuous, $f$ assumes absolute maximum and minimum values on $[a, b]$ by Theorem 1. These can occur only

1. at interior points where $f'$ is zero,
2. at interior points where $f'$ does not exist,
3. at the endpoints of the function's domain, in this case $a$ and $b$.

By hypothesis, $f$ has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints $a$ and $b$.

If either the maximum or the minimum occurs at a point $c$ between $a$ and $b$, then $f'(c) = 0$ by Theorem 2 in Section 4.1, and we have found a point for Rolle's Theorem.

If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that $f$ is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point $c$ can be taken anywhere in the interior $(a, b)$. ■

HISTORICAL BIOGRAPHY

Michel Rolle
(1652–1719)

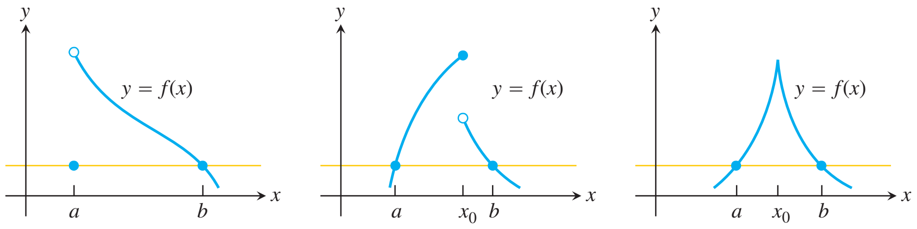
The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Figure 4.11).

Rolle's Theorem may be combined with the Intermediate Value Theorem to show when there is only one real solution of an equation $f(x) = 0$, as we illustrate in the next example.

**EXAMPLE 1** Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.




(a) Discontinuous at an endpoint of $[a, b]$ (b) Discontinuous at an interior point of $[a, b]$ (c) Continuous on $[a, b]$ but not differentiable at an interior point

**FIGURE 4.11** There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

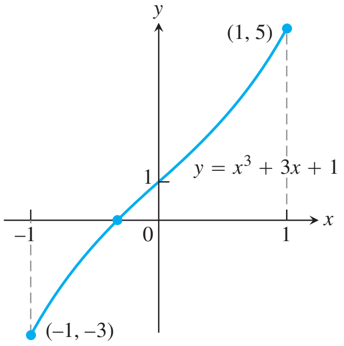



**FIGURE 4.12** The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the $x$-axis between $-1$ and 0 (Example 1).

**Solution** We define the continuous function

$$f(x) = x^3 + 3x + 1.$$

Since $f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem tells us that the graph of $f$ crosses the $x$-axis somewhere in the open interval $(-1, 0)$. (See Figure 4.12.) The derivative

$$f'(x) = 3x^2 + 3$$

is never zero (because it is always positive). Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where $f'$ was zero. Therefore, $f$ has no more than one zero. ■

Our main use of Rolle's Theorem is in proving the Mean Value Theorem.

## The Mean Value Theorem

The Mean Value Theorem, which was first stated by Joseph-Louis Lagrange, is a slanted version of Rolle's Theorem (Figure 4.13). The Mean Value Theorem guarantees that there is a point where the tangent line is parallel to the chord $AB$.

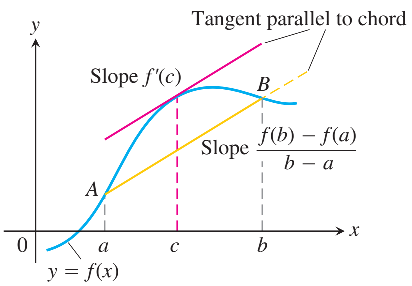



**FIGURE 4.13** Geometrically, the Mean Value Theorem says that somewhere between $a$ and $b$ the curve has at least one tangent parallel to chord $AB$.

**THEOREM 4—The Mean Value Theorem** Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. Then there is at least one point $c$ in $(a, b)$ at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

**Proof** We picture the graph of $f$ and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$. (See Figure 4.14.) The line is the graph of the function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \tag{2}$$

(point-slope equation). The vertical difference between the graphs of $f$ and $g$ at $x$ is

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned} \tag{3}$$

Figure 4.15 shows the graphs of $f$, $g$, and $h$ together.

HISTORICAL BIOGRAPHY

Joseph-Louis Lagrange
(1736–1813)

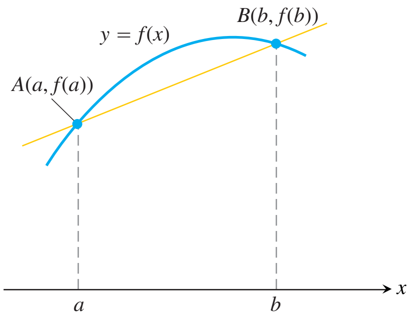



FIGURE 4.14 The graph of $f$ and the chord $AB$ over the interval $[a, b]$.

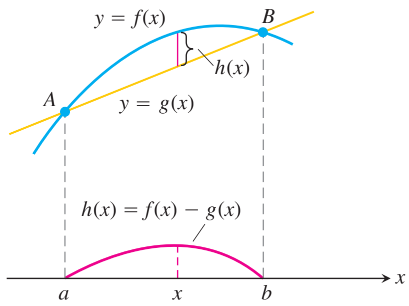



FIGURE 4.15 The chord $AB$ is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of $f$ and $g$ at $x$.

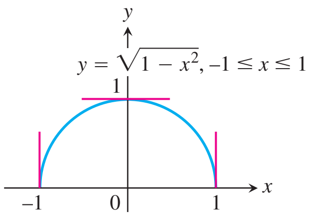



FIGURE 4.16 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though $f$ is not differentiable at $-1$ and 1.

The function $h$ satisfies the hypotheses of Rolle's Theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on $(a, b)$ because both $f$ and $g$ are. Also, $h(a) = h(b) = 0$ because the graphs of $f$ and $g$ both pass through $A$ and $B$. Therefore $h'(c) = 0$ at some point $c \in (a, b)$. This is the point we want for Equation (1).

To verify Equation (1), we differentiate both sides of Equation (3) with respect to $x$ and then set $x = c$:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad \text{Derivative of Eq. (3) . . .}$$

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \text{. . . with } x = c$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad h'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad \text{Rearranged}$$

which is what we set out to prove. ■

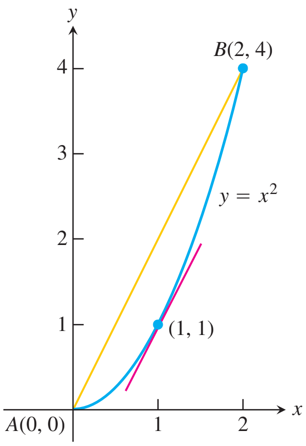



FIGURE 4.17 As we find in Example 2, $c = 1$ is where the tangent is parallel to the chord.

The hypotheses of the Mean Value Theorem do not require $f$ to be differentiable at either $a$ or $b$. Continuity at $a$ and $b$ is enough (Figure 4.16).

**EXAMPLE 2** The function $f(x) = x^2$ (Figure 4.17) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point $c$ in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this case we can identify $c$ by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find $c$ algebraically, even though we know it always exists. ■

## A Physical Interpretation

We can think of the number $(f(b) - f(a))/(b - a)$ as the average change in $f$ over $[a, b]$ and $f'(c)$ as an instantaneous change. Then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

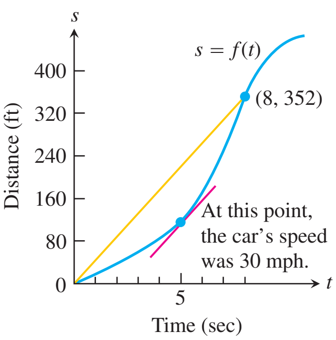



FIGURE 4.18 Distance versus elapsed time for the car in Example 3.

**EXAMPLE 3** If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. The Mean Value Theorem says that at some point during the acceleration the speedometer must read exactly 30 mph (44 ft/sec) (Figure 4.18). ■

## Mathematical Consequences

At the beginning of the section, we asked what kind of function has a zero derivative over an interval. The first corollary of the Mean Value Theorem provides the answer that only constant functions have zero derivatives.

**COROLLARY 1** If $f'(x) = 0$ at each point $x$ of an open interval $(a, b)$, then $f(x) = C$ for all $x \in (a, b)$, where $C$ is a constant.

**Proof** We want to show that $f$ has a constant value on the interval $(a, b)$. We do so by showing that if $x_1$ and $x_2$ are any two points in $(a, b)$ with $x_1 < x_2$, then $f(x_1) = f(x_2)$. Now $f$ satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$: It is differentiable at every point of $[x_1, x_2]$ and hence continuous at every point as well. Therefore,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

at some point $c$ between $x_1$ and $x_2$. Since $f' = 0$ throughout $(a, b)$, this equation implies successively that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0, \qquad f(x_2) - f(x_1) = 0, \qquad \text{and} \qquad f(x_1) = f(x_2). \quad \blacksquare$$

At the beginning of this section, we also asked about the relationship between two functions that have identical derivatives over an interval. The next corollary tells us that their values on the interval have a constant difference.

**COROLLARY 2** If $f'(x) = g'(x)$ at each point $x$ in an open interval $(a, b)$, then there exists a constant $C$ such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on $(a, b)$.

**Proof** At each point $x \in (a, b)$ the derivative of the difference function $h = f - g$ is

$$h'(x) = f'(x) - g'(x) = 0.$$

Thus, $h(x) = C$ on $(a, b)$ by Corollary 1. That is, $f(x) - g(x) = C$ on $(a, b)$, so $f(x) = g(x) + C$. $\blacksquare$

Corollaries 1 and 2 are also true if the open interval $(a, b)$ fails to be finite. That is, they remain true if the interval is $(a, \infty)$, $(-\infty, b)$, or $(-\infty, \infty)$.

Corollary 2 plays an important role when we discuss antiderivatives in Section 4.8. It tells us, for instance, that since the derivative of $f(x) = x^2$ on $(-\infty, \infty)$ is $2x$, any other function with derivative $2x$ on $(-\infty, \infty)$ must have the formula $x^2 + C$ for some value of $C$ (Figure 4.19).

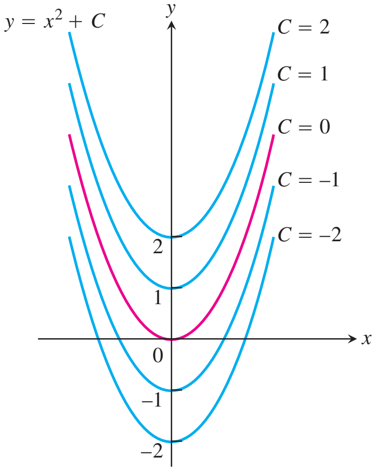



**FIGURE 4.19** From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift there. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of $C$.

**EXAMPLE 4** Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

**Solution** Since the derivative of $g(x) = -\cos x$ is $g'(x) = \sin x$, we see that $f$ and $g$ have the same derivative. Corollary 2 then says that $f(x) = -\cos x + C$ for some constant $C$. Since the graph of $f$ passes through the point (0, 2), the value of $C$ is determined from the condition that $f(0) = 2$:

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The function is $f(x) = -\cos x + 3$. ■

## Finding Velocity and Position from Acceleration

We can use Corollary 2 to find the velocity and position functions of an object moving along a vertical line. Assume the object or body is falling freely from rest with acceleration $9.8 \text{ m/sec}^2$. We assume the position $s(t)$ of the body is measured positive downward from the rest position (so the vertical coordinate line points *downward*, in the direction of the motion, with the rest position at 0).

We know that the velocity $v(t)$ is some function whose derivative is 9.8. We also know that the derivative of $g(t) = 9.8t$ is 9.8. By Corollary 2,

$$v(t) = 9.8t + C$$

for some constant $C$. Since the body falls from rest, $v(0) = 0$. Thus

$$9.8(0) + C = 0, \qquad \text{and} \qquad C = 0.$$

The velocity function must be $v(t) = 9.8t$. What about the position function $s(t)$?

We know that $s(t)$ is some function whose derivative is $9.8t$. We also know that the derivative of $f(t) = 4.9t^2$ is $9.8t$. By Corollary 2,

$$s(t) = 4.9t^2 + C$$

for some constant $C$. Since $s(0) = 0$,

$$4.9(0)^2 + C = 0, \qquad \text{and} \qquad C = 0.$$

The position function is $s(t) = 4.9t^2$ until the body hits the ground.

The ability to find functions from their rates of change is one of the very powerful tools of calculus. As we will see, it lies at the heart of the mathematical developments in Chapter 5.

## Proofs of the Laws of Logarithms

The algebraic properties of logarithms were stated in Section 1.6. We can prove those properties by applying Corollary 2 of the Mean Value Theorem to each of them. The steps in the proofs are similar to those used in solving problems involving logarithms.

**Proof that $\ln bx = \ln b + \ln x$** The argument starts by observing that $\ln bx$ and $\ln x$ have the same derivative:

$$\frac{d}{dx}\ln(bx) = \frac{b}{bx} = \frac{1}{x} = \frac{d}{dx}\ln x.$$

According to Corollary 2 of the Mean Value Theorem, then, the functions must differ by a constant, which means that

$$\ln bx = \ln x + C$$

for some $C$.

Since this last equation holds for all positive values of $x$, it must hold for $x = 1$. Hence,

$$\begin{aligned} \ln(b \cdot 1) &= \ln 1 + C \\ \ln b &= 0 + C \qquad \ln 1 = 0 \\ C &= \ln b. \end{aligned}$$

By substituting we conclude,

$$\ln bx = \ln b + \ln x. \qquad \blacksquare$$

**Proof that $\ln x^r = r \ln x$** We use the same-derivative argument again. For all positive values of $x$,

$$\begin{aligned}\frac{d}{dx}\ln x^r &= \frac{1}{x^r}\frac{d}{dx}(x^r) && \text{Chain Rule}\\ &= \frac{1}{x^r} r x^{r-1} && \text{Derivative Power Rule}\\ &= r \cdot \frac{1}{x} = \frac{d}{dx}(r \ln x).\end{aligned}$$

Since $\ln x^r$ and $r \ln x$ have the same derivative,

$$\ln x^r = r \ln x + C$$

for some constant $C$. Taking $x$ to be 1 identifies $C$ as zero, and we're done. ■

You are asked to prove the Quotient Rule for logarithms,

$$\ln\left(\frac{b}{x}\right) = \ln b - \ln x,$$

in Exercise 75. The Reciprocal Rule, $\ln(1/x) = -\ln x$, is a special case of the Quotient Rule, obtained by taking $b = 1$ and noting that $\ln 1 = 0$.

## Laws of Exponents

The laws of exponents for the natural exponential $e^x$ are consequences of the algebraic properties of $\ln x$. They follow from the inverse relationship between these functions.

**Laws of Exponents for $e^x$**

For all numbers $x$, $x_1$, and $x_2$, the natural exponential $e^x$ obeys the following laws:

**1.** $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

**2.** $e^{-x} = \dfrac{1}{e^x}$

**3.** $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

**4.** $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

**Proof of Law 1** Let

$$y_1 = e^{x_1} \quad \text{and} \quad y_2 = e^{x_2}. \tag{4}$$

Then

$$\begin{aligned} x_1 = \ln y_1 \quad &\text{and} \quad x_2 = \ln y_2 && \text{Take logs of both sides of Eqs. (4).}\\ x_1 + x_2 &= \ln y_1 + \ln y_2 \\ &= \ln y_1 y_2 && \text{Product Rule for logarithms}\\ e^{x_1 + x_2} &= e^{\ln y_1 y_2} && \text{Exponentiate.}\\ &= y_1 y_2 && e^{\ln u} = u\\ &= e^{x_1} e^{x_2}. \end{aligned} \qquad \blacksquare$$

The proof of Law 4 is similar. Laws 2 and 3 follow from Law 1 (Exercises 77 and 78).

# Exercises 4.2

## Checking the Mean Value Theorem

Find the value or values of $c$ that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–8.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
2. $f(x) = x^{2/3}, \quad [0, 1]$
3. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$
4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$
5. $f(x) = \sin^{-1} x, \quad [-1, 1]$
6. $f(x) = \ln(x - 1), \quad [2, 4]$
7. $f(x) = x^3 - x^2, \quad [-1, 2]$
8. $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

Which of the functions in Exercises 9–14 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

9. $f(x) = x^{2/3}, \quad [-1, 8]$
10. $f(x) = x^{4/5}, \quad [0, 1]$
11. $f(x) = \sqrt{x(1 - x)}, \quad [0, 1]$
12. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$
13. $f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1 \\ 2x^2 - 3x - 3, & -1 < x \le 0 \end{cases}$
14. $f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$
15. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

16. For what values of $a$, $m$, and $b$ does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

## Roots (Zeros)

17. **a.** Plot the zeros of each polynomial on a line together with the zeros of its first derivative.
    **i)** $y = x^2 - 4$
    **ii)** $y = x^2 + 8x + 15$
    **iii)** $y = x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2$
    **iv)** $y = x^3 - 33x^2 + 216x = x(x - 9)(x - 24)$
    **b.** Use Rolle's Theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ there lies a zero of
    $$nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \cdots + a_1.$$
18. Suppose that $f''$ is continuous on $[a, b]$ and that $f$ has three zeros in the interval. Show that $f''$ has at least one zero in $(a, b)$. Generalize this result.
19. Show that if $f'' > 0$ throughout an interval $[a, b]$, then $f'$ has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?
20. Show that a cubic polynomial can have at most three real zeros.

Show that the functions in Exercises 21–28 have exactly one zero in the given interval.

21. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$
22. $f(x) = x^3 + \frac{4}{x^2} + 7, \quad (-\infty, 0)$
23. $g(t) = \sqrt{t} + \sqrt{1 + t} - 4, \quad (0, \infty)$
24. $g(t) = \frac{1}{1 - t} + \sqrt{1 + t} - 3.1, \quad (-1, 1)$
25. $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$
26. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$
27. $r(\theta) = \sec\theta - \frac{1}{\theta^3} + 5, \quad (0, \pi/2)$
28. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

## Finding Functions from Derivatives

29. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all $x$. Must $f(x) = 3$ for all $x$? Give reasons for your answer.
30. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all $x$. Must $f(x) = 2x + 5$ for all $x$? Give reasons for your answer.
31. Suppose that $f'(x) = 2x$ for all $x$. Find $f(2)$ if
    **a.** $f(0) = 0$ **b.** $f(1) = 0$ **c.** $f(-2) = 3$.
32. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 33–38, find all possible functions with the given derivative.

33. **a.** $y' = x$ **b.** $y' = x^2$ **c.** $y' = x^3$
34. **a.** $y' = 2x$ **b.** $y' = 2x - 1$ **c.** $y' = 3x^2 + 2x - 1$
35. **a.** $y' = -\frac{1}{x^2}$ **b.** $y' = 1 - \frac{1}{x^2}$ **c.** $y' = 5 + \frac{1}{x^2}$

**36.** **a.** $y' = \dfrac{1}{2\sqrt{x}}$ **b.** $y' = \dfrac{1}{\sqrt{x}}$ **c.** $y' = 4x - \dfrac{1}{\sqrt{x}}$

**37.** **a.** $y' = \sin 2t$ **b.** $y' = \cos \dfrac{t}{2}$ **c.** $y' = \sin 2t + \cos \dfrac{t}{2}$

**38.** **a.** $y' = \sec^2 \theta$ **b.** $y' = \sqrt{\theta}$ **c.** $y' = \sqrt{\theta} - \sec^2 \theta$

In Exercises 39–42, find the function with the given derivative whose graph passes through the point $P$.

**39.** $f'(x) = 2x - 1, \quad P(0, 0)$

**40.** $g'(x) = \dfrac{1}{x^2} + 2x, \quad P(-1, 1)$

**41.** $f'(x) = e^{2x}, \quad P\left(0, \dfrac{3}{2}\right)$

**42.** $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

## Finding Position from Velocity or Acceleration

Exercises 43–46 give the velocity $v = ds/dt$ and initial position of a body moving along a coordinate line. Find the body's position at time $t$.

**43.** $v = 9.8t + 5, \quad s(0) = 10$

**44.** $v = 32t - 2, \quad s(0.5) = 4$

**45.** $v = \sin \pi t, \quad s(0) = 0$

**46.** $v = \dfrac{2}{\pi} \cos \dfrac{2t}{\pi}, \quad s(\pi^2) = 1$

Exercises 47–50 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of a body moving on a coordinate line. Find the body's position at time $t$.

**47.** $a = e^t, \quad v(0) = 20, \quad s(0) = 5$

**48.** $a = 9.8, \quad v(0) = -3, \quad s(0) = 0$

**49.** $a = -4 \sin 2t, \quad v(0) = 2, \quad s(0) = -3$

**50.** $a = \dfrac{9}{\pi^2} \cos \dfrac{3t}{\pi}, \quad v(0) = 0, \quad s(0) = -1$

## Applications

**51.** **Temperature change** It took 14 sec for a mercury thermometer to rise from −19°C to 100°C when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at the rate of 8.5°C/sec.

**52.** A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

**53.** Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 hours. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea miles per hour).

**54.** A marathoner ran the 26.2-mi New York City Marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 11 mph, assuming the initial and final speeds are zero.

**55.** Show that at some instant during a 2-hour automobile trip the car's speedometer reading will equal the average speed for the trip.

**56.** **Free fall on the moon** On our moon, the acceleration of gravity is $1.6 \text{ m/sec}^2$. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

## Theory and Examples

**57.** **The geometric mean of *a* and *b*** The *geometric mean* of two positive numbers $a$ and $b$ is the number $\sqrt{ab}$. Show that the value of $c$ in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a, b]$ is $c = \sqrt{ab}$.

**58.** **The arithmetic mean of *a* and *b*** The *arithmetic mean* of two numbers $a$ and $b$ is the number $(a + b)/2$. Show that the value of $c$ in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

T **59.** Graph the function

$$f(x) = \sin x \sin (x + 2) - \sin^2 (x + 1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

**60.** **Rolle's Theorem**

**a.** Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1$, and $2$.

**b.** Graph $f$ and its derivative $f'$ together. How is what you see related to Rolle's Theorem?

**c.** Do $g(x) = \sin x$ and its derivative $g'$ illustrate the same phenomenon as $f$ and $f'$?

**61.** **Unique solution** Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Also assume that $f(a)$ and $f(b)$ have opposite signs and that $f' \neq 0$ between $a$ and $b$. Show that $f(x) = 0$ exactly once between $a$ and $b$.

**62.** **Parallel tangents** Assume that $f$ and $g$ are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between $a$ and $b$ where the tangents to the graphs of $f$ and $g$ are parallel or the same line. Illustrate with a sketch.

**63.** Suppose that $f'(x) \leq 1$ for $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$.

**64.** Suppose that $0 < f'(x) < 1/2$ for all $x$-values. Show that $f(-1) < f(1) < 2 + f(-1)$.

**65.** Show that $|\cos x - 1| \leq |x|$ for all $x$-values. (*Hint:* Consider $f(t) = \cos t$ on $[0, x]$.)

**66.** Show that for any numbers $a$ and $b$, the sine inequality $|\sin b - \sin a| \leq |b - a|$ is true.

**67.** If the graphs of two differentiable functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.

**68.** If $|f(w) - f(x)| \leq |w - x|$ for all values $w$ and $x$ and $f$ is a differentiable function, show that $-1 \leq f'(x) \leq 1$ for all $x$-values.

**69.** Assume that $f$ is differentiable on $a \leq x \leq b$ and that $f(b) < f(a)$. Show that $f'$ is negative at some point between $a$ and $b$.

**70.** Let $f$ be a function defined on an interval $[a, b]$. What conditions could you place on $f$ to guarantee that

$$\min f' \leq \frac{f(b) - f(a)}{b - a} \leq \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of $f'$ on $[a, b]$? Give reasons for your answers.

T **71.** Use the inequalities in Exercise 70 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \le x \le 0.1$ and $f(0) = 1$.

T **72.** Use the inequalities in Exercise 70 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \le x \le 0.1$ and $f(0) = 2$.

**73.** Let $f$ be differentiable at every value of $x$ and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.

**a.** Show that $f(x) \ge 1$ for all $x$.

**b.** Must $f'(1) = 0$? Explain.

**74.** Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point $c$ in $(a, b)$ at which $f$ satisfies the conclusion of the Mean Value Theorem.

**75.** Use the same-derivative argument, as was done to prove the Product and Power Rules for logarithms, to prove the Quotient Rule property.

**76.** Use the same-derivative argument to prove the identities

**a.** $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$ **b.** $\sec^{-1} x + \csc^{-1} x = \dfrac{\pi}{2}$

**77.** Starting with the equation $e^{x_1}e^{x_2} = e^{x_1 + x_2}$, derived in the text, show that $e^{-x} = 1/e^x$ for any real number $x$. Then show that $e^{x_1}/e^{x_2} = e^{x_1 - x_2}$ for any numbers $x_1$ and $x_2$.

**78.** Show that $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$ for any numbers $x_1$ and $x_2$.

# 4.3 Monotonic Functions and the First Derivative Test

In sketching the graph of a differentiable function it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

## Increasing Functions and Decreasing Functions

As another corollary to the Mean Value Theorem, we show that functions with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions. A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

**COROLLARY 3** Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

If $f'(x) > 0$ at each point $x \in (a, b)$, then $f$ is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then $f$ is decreasing on $[a, b]$.

**Proof** Let $x_1$ and $x_2$ be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to $f$ on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some $c$ between $x_1$ and $x_2$. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if $f'$ is positive on $(a, b)$ and $f(x_2) < f(x_1)$ if $f'$ is negative on $(a, b)$. ■

Corollary 3 is valid for infinite as well as finite intervals. To find the intervals where a function $f$ is increasing or decreasing, we first find all of the critical points of $f$. If $a < b$ are two critical points for $f$, and if the derivative $f'$ is continuous but never zero on the interval $(a, b)$, then by the Intermediate Value Theorem applied to $f'$, the derivative must be everywhere positive on $(a, b)$, or everywhere negative there. One way we can determine the sign of $f'$ on $(a, b)$ is simply by evaluating the derivative at a single point $c$ in $(a, b)$. If $f'(c) > 0$, then $f'(x) > 0$ for all $x$ in $(a, b)$ so $f$ is increasing on $[a, b]$ by Corollary 3; if $f'(c) < 0$, then $f$ is decreasing on $[a, b]$. The next example illustrates how we use this procedure.

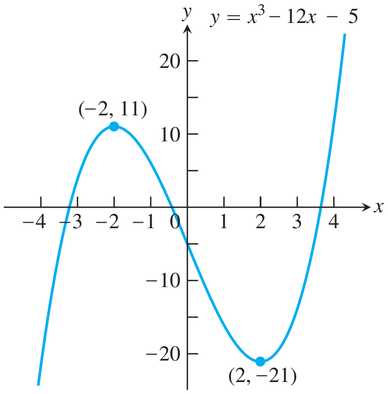



**FIGURE 4.20** The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

**EXAMPLE 1** Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which $f$ is increasing and on which $f$ is decreasing.

**Solution** The function $f$ is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of $f$ to create nonoverlapping open intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which $f'$ is either positive or negative. We determine the sign of $f'$ by evaluating $f'$ at a convenient point in each subinterval. The behavior of $f$ is determined by then applying Corollary 3 to each subinterval. The results are summarized in the following table, and the graph of $f$ is given in Figure 4.20.

| | | | |
|---|---|---|---|
| **Interval** | $-\infty < x < -2$ | $-2 < x < 2$ | $2 < x < \infty$ |
| **$f'$ evaluated** | $f'(-3) = 15$ | $f'(0) = -12$ | $f'(3) = 15$ |
| **Sign of $f'$** | $+$ | $-$ | $+$ |
| **Behavior of $f$** | increasing | decreasing | increasing |

■

We used "strict" less-than inequalities to specify the intervals in the summary table for Example 1. Corollary 3 says that we could use $\le$ inequalities as well. That is, the function $f$ in the example is increasing on $-\infty < x \le -2$, decreasing on $-2 \le x \le 2$, and increasing on $2 \le x < \infty$. We do not talk about whether a function is increasing or decreasing at a single point.

HISTORICAL BIOGRAPHY

Edmund Halley
(1656–1742)

## First Derivative Test for Local Extrema

In Figure 4.21, at the points where $f$ has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.) Thus, the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where $f$ has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus, the function is increasing on the left of the maximum value and decreasing on its right. In summary, at a local extreme point, the sign of $f'(x)$ changes.

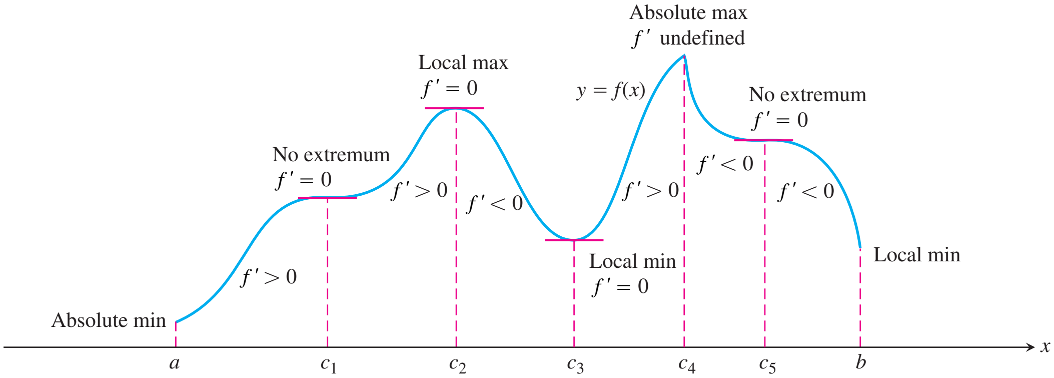



**FIGURE 4.21** The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

These observations lead to a test for the presence and nature of local extreme values of differentiable functions.

**First Derivative Test for Local Extrema**

Suppose that $c$ is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing $c$ except possibly at $c$ itself. Moving across this interval from left to right,

1. if $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$;
2. if $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$;
3. if $f'$ does not change sign at $c$ (that is, $f'$ is positive on both sides of $c$ or negative on both sides), then $f$ has no local extremum at $c$.

The test for local extrema at endpoints is similar, but there is only one side to consider.

**Proof of the First Derivative Test** Part (1). Since the sign of $f'$ changes from negative to positive at $c$, there are numbers $a$ and $b$ such that $a < c < b$, $f' < 0$ on $(a, c)$, and $f' > 0$ on $(c, b)$. If $x \in (a, c)$, then $f(c) < f(x)$ because $f' < 0$ implies that $f$ is decreasing on $[a, c]$. If $x \in (c, b)$, then $f(c) < f(x)$ because $f' > 0$ implies that $f$ is increasing on $[c, b]$. Therefore, $f(x) \geq f(c)$ for every $x \in (a, b)$. By definition, $f$ has a local minimum at $c$.

Parts (2) and (3) are proved similarly. ∎

**EXAMPLE 2** Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the intervals on which $f$ is increasing and decreasing. Find the function's local and absolute extreme values.

**Solution** The function $f$ is continuous at all $x$ since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}\left(x - 1\right) = \frac{4(x - 1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where $f$ might have an extreme value.

The critical points partition the $x$-axis into intervals on which $f'$ is either positive or negative. The sign pattern of $f'$ reveals the behavior of $f$ between and at the critical points, as summarized in the following table.

| **Interval** | $x < 0$ | $0 < x < 1$ | $x > 1$ |
|---|---|---|---|
| **Sign of $f'$** | $-$ | $-$ | $+$ |
| **Behavior of $f$** | decreasing | decreasing | increasing |

Corollary 3 to the Mean Value Theorem tells us that $f$ decreases on $(-\infty, 0]$, decreases on $[0, 1]$, and increases on $[1, \infty)$. The First Derivative Test for Local Extrema tells us that $f$ does not have an extreme value at $x = 0$ ($f'$ does not change sign) and that $f$ has a local minimum at $x = 1$ ($f'$ changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum since $f$ is decreasing on $(-\infty, 1]$ and increasing on $[1, \infty)$. Figure 4.22 shows this value in relation to the function's graph.

Note that $\lim_{x\to 0} f'(x) = -\infty$, so the graph of $f$ has a vertical tangent at the origin. ∎

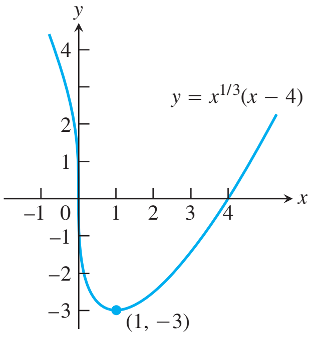



**FIGURE 4.22** The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

**EXAMPLE 3** Find the critical points of

$$f(x) = (x^2 - 3)e^x.$$

Identify the intervals on which $f$ is increasing and decreasing. Find the function's local and absolute extreme values.

**Solution** The function $f$ is continuous and differentiable for all real numbers, so the critical points occur only at the zeros of $f'$.

Using the Derivative Product Rule, we find the derivative

$$\begin{aligned} f'(x) &= (x^2 - 3) \cdot \frac{d}{dx} e^x + \frac{d}{dx}(x^2 - 3) \cdot e^x \\ &= (x^2 - 3) \cdot e^x + (2x) \cdot e^x \\ &= (x^2 + 2x - 3)e^x. \end{aligned}$$

Since $e^x$ is never zero, the first derivative is zero if and only if

$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0.$$

The zeros $x = -3$ and $x = 1$ partition the $x$-axis into intervals as follows.

| **Interval** | $x < -3$ | $-3 < x < 1$ | $1 < x$ |
|---|---|---|---|
| **Sign of $f'$** | $+$ | $-$ | $+$ |
| **Behavior of $f$** | increasing | decreasing | increasing |

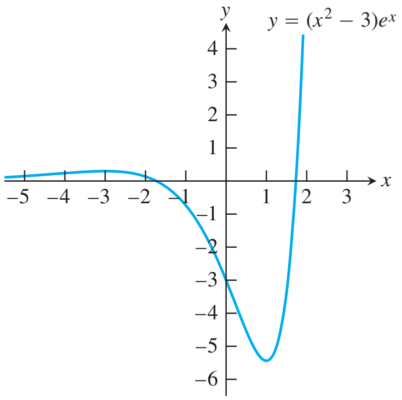



FIGURE 4.23 The graph of $f(x) = (x^2 - 3)e^x$ (Example 3).

We can see from the table that there is a local maximum (about 0.299) at $x = -3$ and a local minimum (about $-5.437$) at $x = 1$. The local minimum value is also an absolute minimum because $f(x) > 0$ for $|x| > \sqrt{3}$. There is no absolute maximum. The function increases on $(-\infty, -3)$ and $(1, \infty)$ and decreases on $(-3, 1)$. Figure 4.23 shows the graph. ■

## Exercises 4.3

### Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

**a.** What are the critical points of $f$?

**b.** On what intervals is $f$ increasing or decreasing?

**c.** At what points, if any, does $f$ assume local maximum and minimum values?

**1.** $f'(x) = x(x - 1)$ **2.** $f'(x) = (x - 1)(x + 2)$

**3.** $f'(x) = (x - 1)^2(x + 2)$ **4.** $f'(x) = (x - 1)^2(x + 2)^2$

**5.** $f'(x) = (x - 1)e^{-x}$

**6.** $f'(x) = (x - 7)(x + 1)(x + 5)$

**7.** $f'(x) = \dfrac{x^2(x - 1)}{x + 2}, \quad x \neq -2$

**8.** $f'(x) = \dfrac{(x - 2)(x + 4)}{(x + 1)(x - 3)}, \quad x \neq -1, 3$

**9.** $f'(x) = 1 - \dfrac{4}{x^2}, \quad x \neq 0$ **10.** $f'(x) = 3 - \dfrac{6}{\sqrt{x}}, \quad x \neq 0$

**11.** $f'(x) = x^{-1/3}(x + 2)$ **12.** $f'(x) = x^{-1/2}(x - 3)$

**13.** $f'(x) = (\sin x - 1)(2 \cos x + 1), 0 \le x \le 2\pi$

**14.** $f'(x) = (\sin x + \cos x)(\sin x - \cos x), 0 \le x \le 2\pi$

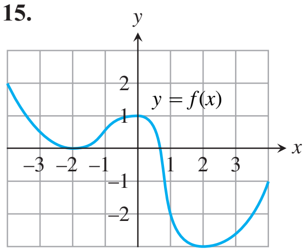
### Identifying Extrema

In Exercises 15–44:

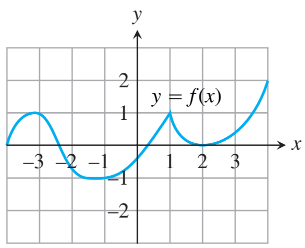
**a.** Find the open intervals on which the function is increasing and decreasing.

**b.** Identify the function's local and absolute extreme values, if any, saying where they occur.

**15.**




**16.**



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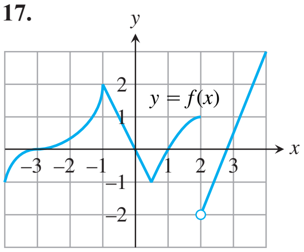


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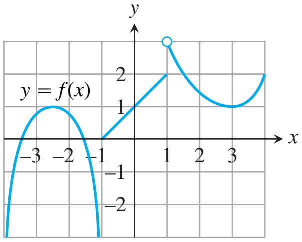



19. $g(t) = -t^2 - 3t + 3$

20. $g(t) = -3t^2 + 9t + 5$

21. $h(x) = -x^3 + 2x^2$

22. $h(x) = 2x^3 - 18x$

23. $f(\theta) = 3\theta^2 - 4\theta^3$

24. $f(\theta) = 6\theta - \theta^3$

25. $f(r) = 3r^3 + 16r$

26. $h(r) = (r + 7)^3$

27. $f(x) = x^4 - 8x^2 + 16$

28. $g(x) = x^4 - 4x^3 + 4x^2$

29. $H(t) = \frac{3}{2}t^4 - t^6$

30. $K(t) = 15t^3 - t^5$

31. $f(x) = x - 6\sqrt{x - 1}$

32. $g(x) = 4\sqrt{x} - x^2 + 3$

33. $g(x) = x\sqrt{8 - x^2}$

34. $g(x) = x^2\sqrt{5 - x}$

35. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$

36. $f(x) = \dfrac{x^3}{3x^2 + 1}$

37. $f(x) = x^{1/3}(x + 8)$

38. $g(x) = x^{2/3}(x + 5)$

39. $h(x) = x^{1/3}(x^2 - 4)$

40. $k(x) = x^{2/3}(x^2 - 4)$

41. $f(x) = e^{2x} + e^{-x}$

42. $f(x) = e^{\sqrt{x}}$

43. $f(x) = x \ln x$

44. $f(x) = x^2 \ln x$

In Exercises 45–56:

**a.** Identify the function's local extreme values in the given domain, and say where they occur.

**b.** Which of the extreme values, if any, are absolute?

T **c.** Support your findings with a graphing calculator or computer grapher.

45. $f(x) = 2x - x^2, \quad -\infty < x \leq 2$

46. $f(x) = (x + 1)^2, \quad -\infty < x \leq 0$

47. $g(x) = x^2 - 4x + 4, \quad 1 \leq x < \infty$

48. $g(x) = -x^2 - 6x - 9, \quad -4 \leq x < \infty$

49. $f(t) = 12t - t^3, \quad -3 \leq t < \infty$

50. $f(t) = t^3 - 3t^2, \quad -\infty < t \leq 3$

51. $h(x) = \dfrac{x^3}{3} - 2x^2 + 4x, \quad 0 \leq x < \infty$

52. $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \leq 0$

53. $f(x) = \sqrt{25 - x^2}, \quad -5 \leq x \leq 5$

54. $f(x) = \sqrt{x^2 - 2x - 3}, \quad 3 \leq x < \infty$

55. $g(x) = \dfrac{x - 2}{x^2 - 1}, \quad 0 \leq x < 1$

56. $g(x) = \dfrac{x^2}{4 - x^2}, \quad -2 < x \leq 1$

In Exercises 57–64:

**a.** Find the local extrema of each function on the given interval, and say where they occur.

T **b.** Graph the function and its derivative together. Comment on the behavior of $f$ in relation to the signs and values of $f'$.

57. $f(x) = \sin 2x, \quad 0 \leq x \leq \pi$

58. $f(x) = \sin x - \cos x, \quad 0 \leq x \leq 2\pi$

59. $f(x) = \sqrt{3}\cos x + \sin x, \quad 0 \leq x \leq 2\pi$

60. $f(x) = -2x + \tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

61. $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}, \quad 0 \leq x \leq 2\pi$

62. $f(x) = -2\cos x - \cos^2 x, \quad -\pi \leq x \leq \pi$

63. $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$

64. $f(x) = \sec^2 x - 2\tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

## Theory and Examples

Show that the functions in Exercises 65 and 66 have local extreme values at the given values of $\theta$, and say which kind of local extreme the function has.

65. $h(\theta) = 3\cos\dfrac{\theta}{2}, \quad 0 \leq \theta \leq 2\pi, \quad$ at $\theta = 0$ and $\theta = 2\pi$

66. $h(\theta) = 5\sin\dfrac{\theta}{2}, \quad 0 \leq \theta \leq \pi, \quad$ at $\theta = 0$ and $\theta = \pi$

67. Sketch the graph of a differentiable function $y = f(x)$ through the point $(1, 1)$ if $f'(1) = 0$ and

**a.** $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;

**b.** $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;

**c.** $f'(x) > 0$ for $x \neq 1$;

**d.** $f'(x) < 0$ for $x \neq 1$.

68. Sketch the graph of a differentiable function $y = f(x)$ that has

**a.** a local minimum at $(1, 1)$ and a local maximum at $(3, 3)$;

**b.** a local maximum at $(1, 1)$ and a local minimum at $(3, 3)$;

**c.** local maxima at $(1, 1)$ and $(3, 3)$;

**d.** local minima at $(1, 1)$ and $(3, 3)$.

69. Sketch the graph of a continuous function $y = g(x)$ such that

**a.** $g(2) = 2$, $0 < g' < 1$ for $x < 2$, $g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;

**b.** $g(2) = 2$, $g' < 0$ for $x < 2$, $g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

70. Sketch the graph of a continuous function $y = h(x)$ such that

**a.** $h(0) = 0$, $-2 \leq h(x) \leq 2$ for all $x$, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to \infty$ as $x \to 0^+$;

**b.** $h(0) = 0$, $-2 \leq h(x) \leq 0$ for all $x$, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

71. Discuss the extreme-value behavior of the function $f(x) = x\sin(1/x)$, $x \neq 0$. How many critical points does this function have? Where are they located on the $x$-axis? Does $f$ have an absolute minimum? An absolute maximum? (See Exercise 49 in Section 2.3.)

72. Find the intervals on which the function $f(x) = ax^2 + bx + c$, $a \neq 0$, is increasing and decreasing. Describe the reasoning behind your answer.

73. Determine the values of constants $a$ and $b$ so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

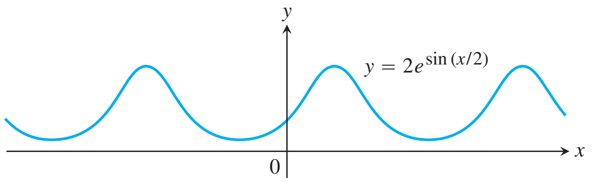
74. Determine the values of constants $a$, $b$, $c$, and $d$ so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(0, 0)$ and a local minimum at the point $(1, -1)$.

**75.** Locate and identify the absolute extreme values of
  **a.** $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,
  **b.** $\cos(\ln x)$ on $[1/2, 2]$.

**76.** **a.** Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.
  **b.** Using part (a), show that $\ln x < x$ if $x > 1$.

**77.** Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

**78.** Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?




**79.** Find the absolute maximum value of $f(x) = x^2 \ln(1/x)$ and say where it is assumed.

**80.** **a.** Prove that $e^x \geq 1 + x$ if $x \geq 0$.
  **b.** Use the result in part (a) to show that

$$e^x \geq 1 + x + \frac{1}{2}x^2.$$

**81.** Show that increasing functions and decreasing functions are one-to-one. That is, show that for any $x_1$ and $x_2$ in $I$, $x_2 \neq x_1$ implies $f(x_2) \neq f(x_1)$.

Use the results of Exercise 81 to show that the functions in Exercises 82–86 have inverses over their domains. Find a formula for $df^{-1}/dx$ using Theorem 3, Section 3.8.

**82.** $f(x) = (1/3)x + (5/6)$

**83.** $f(x) = 27x^3$

**84.** $f(x) = 1 - 8x^3$

**85.** $f(x) = (1 - x)^3$

**86.** $f(x) = x^{5/3}$

# 4.4 Concavity and Curve Sketching

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of asymptotic behavior and symmetry studied in Sections 2.6 and 1.1, we can now draw an accurate graph of a function. By organizing all of these ideas into a coherent procedure, we give a method for sketching graphs and revealing visually the key features of functions. Identifying and knowing the locations of these features is of major importance in mathematics and its applications to science and engineering, especially in the graphical analysis and interpretation of data.

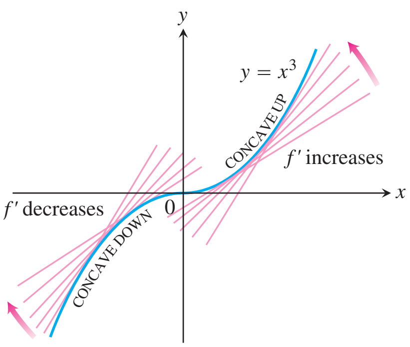



**FIGURE 4.24** The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

## Concavity

As you can see in Figure 4.24, the curve $y = x^3$ rises as $x$ increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the *concavity* of the curve.

**DEFINITION** The graph of a differentiable function $y = f(x)$ is

**(a)** **concave up** on an open interval $I$ if $f'$ is increasing on $I$;

**(b)** **concave down** on an open interval $I$ if $f'$ is decreasing on $I$.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to the first derivative function. We conclude that $f'$ increases if $f'' > 0$ on $I$, and decreases if $f'' < 0$.

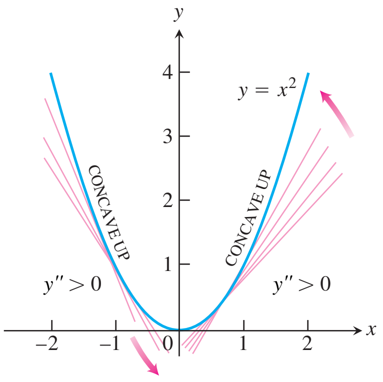



**FIGURE 4.25** The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

**The Second Derivative Test for Concavity**

Let $y = f(x)$ be twice-differentiable on an interval $I$.

1. If $f'' > 0$ on $I$, the graph of $f$ over $I$ is concave up.
2. If $f'' < 0$ on $I$, the graph of $f$ over $I$ is concave down.

If $y = f(x)$ is twice-differentiable, we will use the notations $f''$ and $y''$ interchangeably when denoting the second derivative.

## EXAMPLE 1

**(a)** The curve $y = x^3$ (Figure 4.24) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

**(b)** The curve $y = x^2$ (Figure 4.25) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

**EXAMPLE 2** Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

**Solution** The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.26). ■

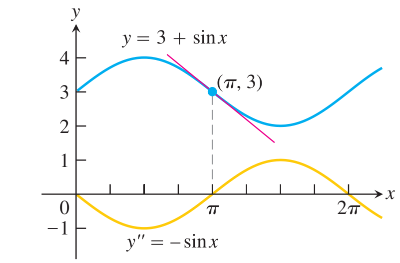



**FIGURE 4.26** Using the sign of $y''$ to determine the concavity of $y$ (Example 2).

## Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. Since the first derivative $y' = \cos x$ exists for all $x$, we see that the curve has a tangent line of slope $-1$ at the point $(\pi, 3)$. This point is called a *point of inflection* of the curve. Notice from Figure 4.26 that the graph crosses its tangent line at this point and that the second derivative $y'' = -\sin x$ has value 0 when $x = \pi$. In general, we have the following definition.

**DEFINITION** A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

We observed that the second derivative of $f(x) = 3 + \sin x$ is equal to zero at the inflection point $(\pi, 3)$. Generally, if the second derivative exists at a point of inflection $(c, f(c))$, then $f''(c) = 0$. This follows immediately from the Intermediate Value Theorem whenever $f''$ is continuous over an interval containing $x = c$ because the second derivative changes sign moving across this interval. Even if the continuity assumption is dropped, it is still true that $f''(c) = 0$, provided the second derivative exists (although a more advanced agrument is required in this noncontinuous case). Since a tangent line must exist at the point of inflection, either the first derivative $f'(c)$ exists (is finite) or a vertical tangent exists at the point. At a vertical tangent neither the first nor second derivative exists. In summary, we conclude the following result.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

The next example illustrates a function having a point of inflection where the first derivative exists, but the second derivative fails to exist.

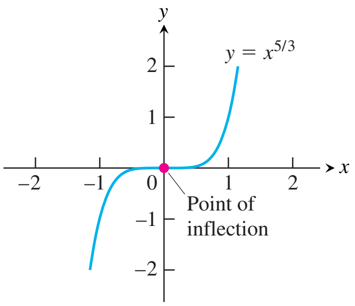



**FIGURE 4.27** The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although $f''$ does not exist at $x = 0$ (Example 3).

**EXAMPLE 3** The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 4.27. ■

Here is an example showing that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

**EXAMPLE 4** The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.28). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. ■

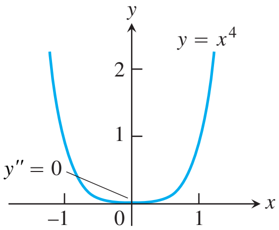



**FIGURE 4.28** The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 4).

As our final illustration, we show a situation in which a point of inflection occurs at a vertical tangent to the curve where neither the first nor the second derivative exists.

**EXAMPLE 5** The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and $y''$ fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 4.29. ■

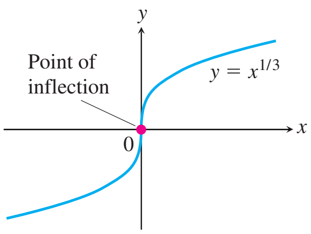



**FIGURE 4.29** A point of inflection where $y'$ and $y''$ fail to exist (Example 5).

To study the motion of an object moving along a line as a function of time, we often are interested in knowing when the object's acceleration, given by the second derivative, is positive or negative. The points of inflection on the graph of the object's position function reveal where the acceleration changes sign.

**EXAMPLE 6** A particle is moving along a horizontal coordinate line (positive to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

**Solution** The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Notice that the first derivative ($v = s'$) is zero at the critical points $t = 1$ and $t = 11/3$.

| **Interval** | $0 < t < 1$ | $1 < t < 11/3$ | $11/3 < t$ |
|---|---|---|---|
| **Sign of $v = s'$** | + | − | + |
| **Behavior of $s$** | increasing | decreasing | increasing |
| **Particle motion** | right | left | right |

The particle is moving to the right in the time intervals [0, 1) and $(11/3, \infty)$, and moving to the left in (1, 11/3). It is momentarily stationary (at rest) at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

| | | |
|---|---|---|
| **Interval** | $0 < t < 7/3$ | $7/3 < t$ |
| **Sign of** $a = s''$ | $-$ | $+$ |
| **Graph of** $s$ | concave down | concave up |

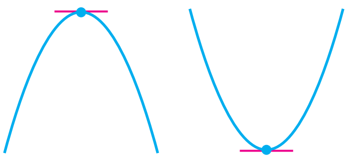
The particle starts out moving to the right while slowing down, and then reverses and begins moving to the left at $t = 1$ under the influence of the leftward acceleration over the time interval [0, 7/3). The acceleration then changes direction at $t = 7/3$ but the particle continues moving leftward, while slowing down under the rightward acceleration. At $t = 11/3$ the particle reverses direction again: moving to the right in the same direction as the acceleration. ■

## Second Derivative Test for Local Extrema

Instead of looking for sign changes in $f'$ at critical points, we can sometimes use the following test to determine the presence and nature of local extrema.

**THEOREM 5—Second Derivative Test for Local Extrema** Suppose $f''$ is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function $f$ may have a local maximum, a local minimum, or neither.




**Proof** Part (1). If $f''(c) < 0$, then $f''(x) < 0$ on some open interval $I$ containing the point $c$, since $f''$ is continuous. Therefore, $f'$ is decreasing on $I$. Since $f'(c) = 0$, the sign of $f'$ changes from positive to negative at $c$ so $f$ has a local maximum at $c$ by the First Derivative Test.

The proof of Part (2) is similar.

For Part (3), consider the three functions $y = x^4$, $y = -x^4$, and $y = x^3$. For each function, the first and second derivatives are zero at $x = 0$. Yet the function $y = x^4$ has a local minimum there, $y = -x^4$ has a local maximum, and $y = x^3$ is increasing in any open interval containing $x = 0$ (having neither a maximum nor a minimum there). Thus the test fails. ■

This test requires us to know $f''$ *only at c itself* and not in an interval about $c$. This makes the test easy to apply. That's the good news. The bad news is that the test is inconclusive if $f'' = 0$ or if $f''$ does not exist at $x = c$. When this happens, use the First Derivative Test for local extreme values.

Together $f'$ and $f''$ tell us the shape of the function's graph—that is, where the critical points are located and what happens at a critical point, where the function is increasing and where it is decreasing, and how the curve is turning or bending as defined by its concavity. We use this information to sketch a graph of the function that captures its key features.

**EXAMPLE 7** Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

**(a)** Identify where the extrema of $f$ occur.

**(b)** Find the intervals on which $f$ is increasing and the intervals on which $f$ is decreasing.

**(c)** Find where the graph of $f$ is concave up and where it is concave down.

**(d)** Sketch the general shape of the graph for $f$.

**(e)** Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

**Solution** The function $f$ is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of $f$ is $(-\infty, \infty)$, and the domain of $f'$ is also $(-\infty, \infty)$. Thus, the critical points of $f$ occur only at the zeros of $f'$. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where $f$ is increasing or decreasing.

| **Interval** | $x < 0$ | $0 < x < 3$ | $3 < x$ |
|---|---|---|---|
| **Sign of $f'$** | $-$ | $-$ | $+$ |
| **Behavior of $f$** | decreasing | decreasing | increasing |

**(a)** Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

**(b)** Using the table above, we see that $f$ is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

**(c)** $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where $f$ is concave up or concave down.

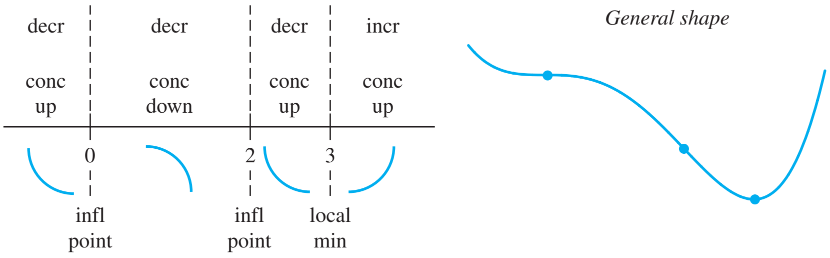
| **Interval** | $x < 0$ | $0 < x < 2$ | $2 < x$ |
|---|---|---|---|
| **Sign of $f''$** | $+$ | $-$ | $+$ |
| **Behavior of $f$** | concave up | concave down | concave up |

We see that $f$ is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

**(d)** Summarizing the information in the last two tables, we obtain the following.

| $x < 0$ | $0 < x < 2$ | $2 < x < 3$ | $3 < x$ |
|---|---|---|---|
| decreasing<br>concave up | decreasing<br>concave down | decreasing<br>concave up | increasing<br>concave up |

The general shape of the curve is shown in the accompanying figure.




**(e)** Plot the curve's intercepts (if possible) and the points where $y'$ and $y''$ are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 4.30 shows the graph of $f$. ■

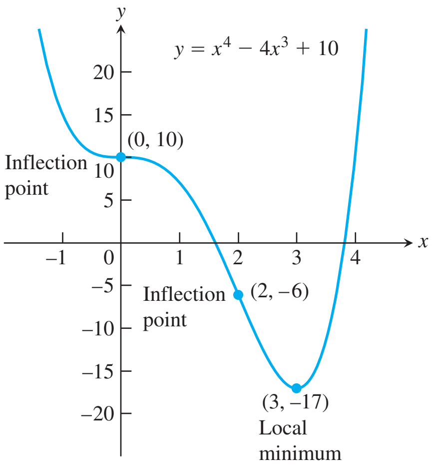



**FIGURE 4.30** The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 7).

The steps in Example 7 give a procedure for graphing the key features of a function.

### Procedure for Graphing $y = f(x)$

1. Identify the domain of $f$ and any symmetries the curve may have.
2. Find the derivatives $y'$ and $y''$.
3. Find the critical points of $f$, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist (see Section 2.6).
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

**EXAMPLE 8** Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

**Solution**

1. The domain of $f$ is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
2. *Find $f'$ and $f''$.*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

$x$-intercept at $x = -1$, $y$-intercept ($y = 1$) at $x = 0$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2\cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f''(x) = \frac{(1+x^2)^2\cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

After some algebra

3. *Behavior at critical points*. The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since $f'$ exists everywhere over the domain of $f$. At $x = -1$, $f''(-1) = 1 > 0$ yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$ yielding a relative maximum by the Second Derivative test.
4. *Increasing and decreasing*. We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.

5. *Inflection points*. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative $f''$ is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $\left(-\infty, -\sqrt{3}\right)$, positive on $\left(-\sqrt{3}, 0\right)$, negative on $\left(0, \sqrt{3}\right)$, and positive again on $\left(\sqrt{3}, \infty\right)$. Thus each point is a point of inflection. The curve is concave down on the interval $\left(-\infty, -\sqrt{3}\right)$, concave up on $\left(-\sqrt{3}, 0\right)$, concave down on $\left(0, \sqrt{3}\right)$, and concave up again on $\left(\sqrt{3}, \infty\right)$.

6. *Asymptotes*. Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by $x^2$ gives

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} \qquad \text{Expanding numerator}$$

$$= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}. \qquad \text{Dividing by } x^2$$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since $f$ decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although $f$ decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of $f$ is $0 \le y \le 2$).

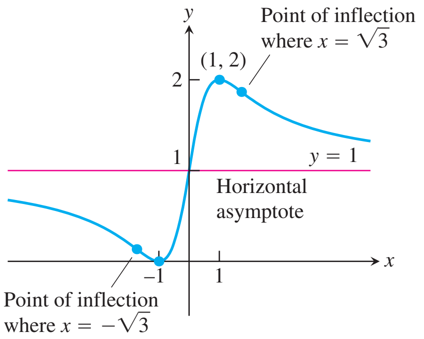



**FIGURE 4.31** The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 8).

7. The graph of $f$ is sketched in Figure 4.31. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$. ■

**EXAMPLE 9** Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$.

**Solution**

1. The domain of $f$ is all nonzero real numbers. There are no intercepts because neither $x$ nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that $f$ is an odd function, so the graph of $f$ is symmetric about the origin.

2. We calculate the derivatives of the function, but first rewrite it in order to simplify our computations:

$$f(x) = \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x} \qquad \text{Function simplified for differentiation}$$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2} \qquad \text{Combine fractions to solve easily } f'(x) = 0.$$

$$f''(x) = \frac{4}{x^3} \qquad \text{Exists throughout the entire domain of } f$$

3. The critical points occur at $x = \pm 2$ where $f'(x) = 0$. Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.

4. On the interval $(-\infty, -2)$ the derivative $f'$ is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.
5. There are no points of inflection because $f''(x) < 0$ whenever $x < 0$, $f''(x) > 0$ whenever $x > 0$, and $f''$ exists everywhere and is never zero throughout the domain of $f$. The graph is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.
6. From the rewritten formula for $f(x)$, we see that

$$\lim_{x\to 0^+}\left(\frac{x}{2}+\frac{2}{x}\right) = +\infty \quad \text{and} \quad \lim_{x\to 0^-}\left(\frac{x}{2}+\frac{2}{x}\right) = -\infty,$$

so the $y$-axis is a vertical asymptote. Also, as $x \to \infty$ or as $x \to -\infty$, the graph of $f(x)$ approaches the line $y = x/2$. Thus $y = x/2$ is an oblique asymptote.
7. The graph of $f$ is sketched in Figure 4.32. ■

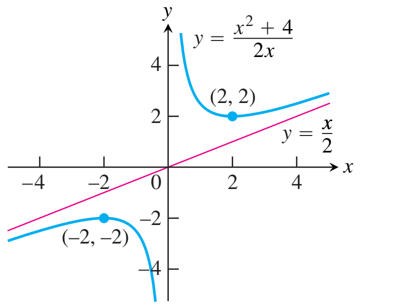



**FIGURE 4.32** The graph of $y = \dfrac{x^2+4}{2x}$ (Example 9).

**EXAMPLE 10** Sketch the graph of $f(x) = e^{2/x}$.

**Solution** The domain of $f$ is $(-\infty, 0) \cup (0, \infty)$ and there are no symmetries about either axis or the origin. The derivatives of $f$ are

$$f'(x) = e^{2/x}\left(-\frac{2}{x^2}\right) = -\frac{2e^{2/x}}{x^2}$$

and

$$f''(x) = \frac{x^2(2e^{2/x})(-2/x^2) - 2e^{2/x}(2x)}{x^4} = \frac{4e^{2/x}(1+x)}{x^4}.$$

Both derivatives exist everywhere over the domain of $f$. Moreover, since $e^{2/x}$ and $x^2$ are both positive for all $x \neq 0$, we see that $f' < 0$ everywhere over the domain and the graph is everywhere decreasing. Examining the second derivative, we see that $f''(x) = 0$ at $x = -1$. Since $e^{2/x} > 0$ and $x^4 > 0$, we have $f'' < 0$ for $x < -1$ and $f'' > 0$ for $x > -1, x \neq 0$. Therefore, the point $(-1, e^{-2})$ is a point of inflection. The curve is concave down on the interval $(-\infty, -1)$ and concave up over $(-1, 0) \cup (0, \infty)$.

From Example 7, Section 2.6, we see that $\lim_{x\to 0^-} f(x) = 0$. As $x \to 0^+$, we see that $2/x \to \infty$, so $\lim_{x\to 0^+} f(x) = \infty$ and the $y$-axis is a vertical asymptote. Also, as $x \to -\infty$, $2/x \to 0^-$ and so $\lim_{x\to -\infty} f(x) = e^0 = 1$. Therefore, $y = 1$ is a horizontal asymptote. There are no absolute extrema since $f$ never takes on the value 0. The graph of $f$ is sketched in Figure 4.33. ■

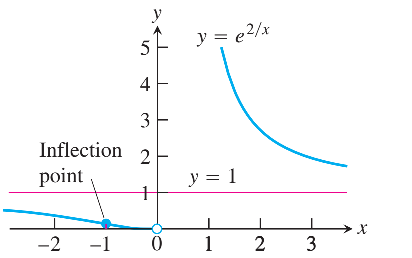
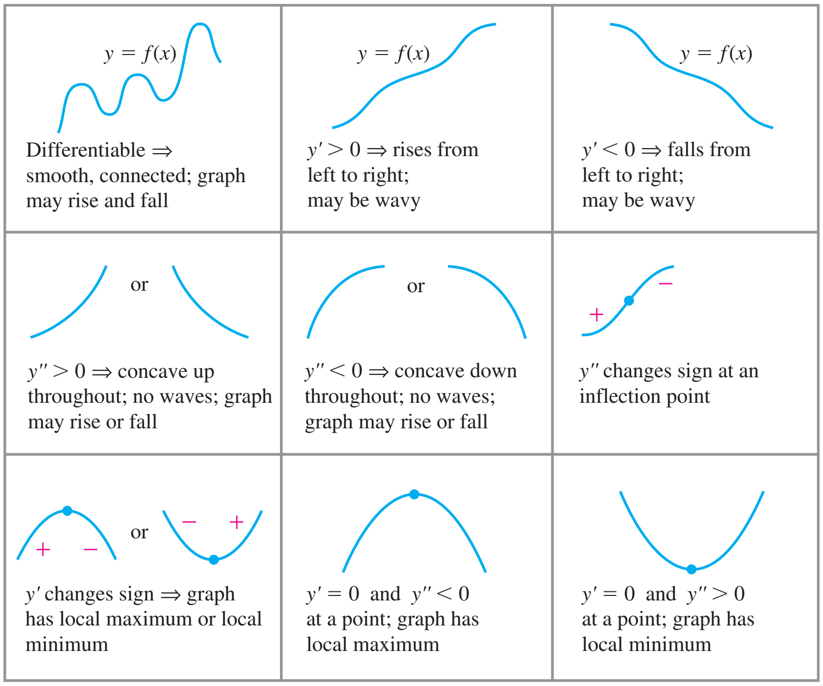



**FIGURE 4.33** The graph of $y = e^{2/x}$ has a point of inflection at $(-1, e^{-2})$. The line $y = 1$ is a horizontal asymptote and $x = 0$ is a vertical asymptote (Example 10).

## Graphical Behavior of Functions from Derivatives

As we saw in Examples 7–10, we can learn much about a twice-differentiable function $y = f(x)$ by examining its first derivative. We can find where the function's graph rises and falls and where any local extrema are located. We can differentiate $y'$ to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. Information we cannot get from the derivative is how to place the graph in the $xy$-plane. But, as we discovered in Section 4.2, the only additional information we need to position the graph is the value of $f$ at one point. Information about the asymptotes is found using limits (Section 2.6). The following figure summarizes how the derivative and second derivative affect the shape of a graph.

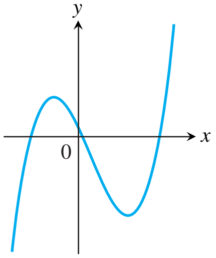



## Exercises 4.4

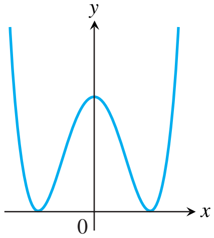
### Analyzing Functions from Graphs

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

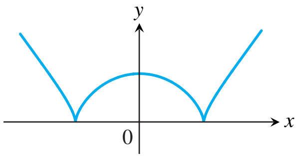
**1.** $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



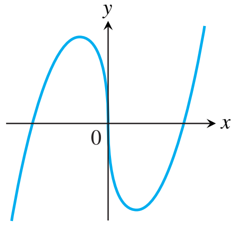

**2.** $y = \frac{x^4}{4} - 2x^2 + 4$



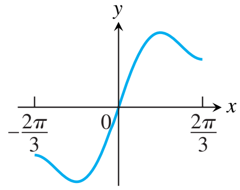

**3.** $y = \frac{3}{4}(x^2 - 1)^{2/3}$



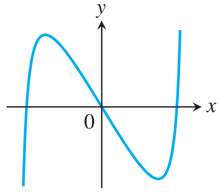

**4.** $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



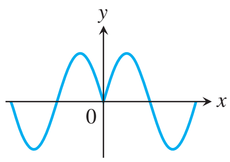

**5.** $y = x + \sin 2x, \; -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$




**6.** $y = \tan x - 4x, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$

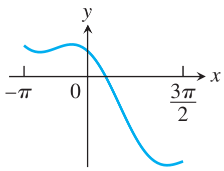



**7.** $y = \sin |x|, \; -2\pi \le x \le 2\pi$




NOT TO SCALE

**8.** $y = 2\cos x - \sqrt{2}\,x, \; -\pi \le x \le \frac{3\pi}{2}$




### Graphing Equations

Use the steps of the graphing procedure on page 248 to graph the equations in Exercises 9–58. Include the coordinates of any local and absolute extreme points and inflection points.

**9.** $y = x^2 - 4x + 3$

**10.** $y = 6 - 2x - x^2$

**11.** $y = x^3 - 3x + 3$

**12.** $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

19. $y = 4x^3 - x^4 = x^3(4 - x)$

20. $y = x^4 + 2x^3 = x^3(x + 2)$

21. $y = x^5 - 5x^4 = x^4(x - 5)$

22. $y = x\left(\frac{x}{2} - 5\right)^4$

23. $y = x + \sin x, \quad 0 \le x \le 2\pi$

24. $y = x - \sin x, \quad 0 \le x \le 2\pi$

25. $y = \sqrt{3}x - 2\cos x, \quad 0 \le x \le 2\pi$

26. $y = \frac{4}{3}x - \tan x, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

27. $y = \sin x \cos x, \quad 0 \le x \le \pi$

28. $y = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$

29. $y = x^{1/5}$

30. $y = x^{2/5}$

31. $y = \frac{x}{\sqrt{x^2 + 1}}$

32. $y = \frac{\sqrt{1 - x^2}}{2x + 1}$

33. $y = 2x - 3x^{2/3}$

34. $y = 5x^{2/5} - 2x$

35. $y = x^{2/3}\left(\frac{5}{2} - x\right)$

36. $y = x^{2/3}(x - 5)$

37. $y = x\sqrt{8 - x^2}$

38. $y = (2 - x^2)^{3/2}$

39. $y = \sqrt{16 - x^2}$

40. $y = x^2 + \frac{2}{x}$

41. $y = \frac{x^2 - 3}{x - 2}$

42. $y = \sqrt[3]{x^3 + 1}$

43. $y = \frac{8x}{x^2 + 4}$

44. $y = \frac{5}{x^4 + 5}$

45. $y = |x^2 - 1|$

46. $y = |x^2 - 2x|$

47. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

48. $y = \sqrt{|x - 4|}$

49. $y = xe^{1/x}$

50. $y = \frac{e^x}{x}$

51. $y = \ln(3 - x^2)$

52. $y = x(\ln x)^2$

53. $y = e^x - 2e^{-x} - 3x$

54. $y = xe^{-x}$

55. $y = \ln(\cos x)$

56. $y = \frac{\ln x}{\sqrt{x}}$

57. $y = \frac{1}{1 + e^{-x}}$

58. $y = \frac{e^x}{1 + e^x}$

### Sketching the General Shape, Knowing $y'$

Each of Exercises 59–80 gives the first derivative of a continuous function $y = f(x)$. Find $y''$ and then use steps 2–4 of the graphing procedure on page 248 to sketch the general shape of the graph of $f$.

59. $y' = 2 + x - x^2$

60. $y' = x^2 - x - 6$

61. $y' = x(x - 3)^2$

62. $y' = x^2(2 - x)$

63. $y' = x(x^2 - 12)$

64. $y' = (x - 1)^2(2x + 3)$

65. $y' = (8x - 5x^2)(4 - x)^2$

66. $y' = (x^2 - 2x)(x - 5)^2$

67. $y' = \sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

68. $y' = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

69. $y' = \cot\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

70. $y' = \csc^2\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

71. $y' = \tan^2\theta - 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

72. $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$

73. $y' = \cos t, \quad 0 \le t \le 2\pi$

74. $y' = \sin t, \quad 0 \le t \le 2\pi$

75. $y' = (x + 1)^{-2/3}$

76. $y' = (x - 2)^{-1/3}$

77. $y' = x^{-2/3}(x - 1)$

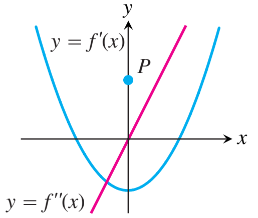
78. $y' = x^{-4/5}(x + 1)$

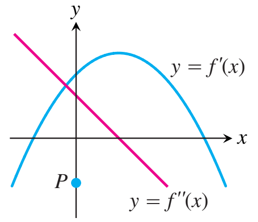
79. $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

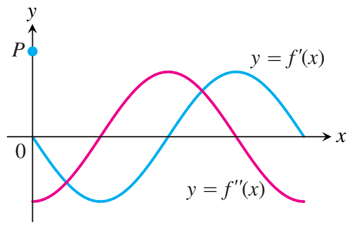
80. $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

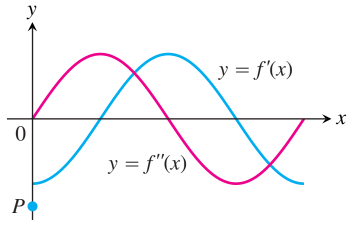
### Sketching $y$ from Graphs of $y'$ and $y''$

Each of Exercises 81–84 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of $f$, given that the graph passes through the point $P$.

81.



82.



83.



84.



### Graphing Rational Functions

Graph the rational functions in Exercises 85–102.

85. $y = \frac{2x^2 + x - 1}{x^2 - 1}$

86. $y = \frac{x^2 - 49}{x^2 + 5x - 14}$

87. $y = \frac{x^4 + 1}{x^2}$

88. $y = \frac{x^2 - 4}{2x}$

89. $y = \frac{1}{x^2 - 1}$

90. $y = \frac{x^2}{x^2 - 1}$

**91.** $y = -\dfrac{x^2 - 2}{x^2 - 1}$

**92.** $y = \dfrac{x^2 - 4}{x^2 - 2}$

**93.** $y = \dfrac{x^2}{x + 1}$

**94.** $y = -\dfrac{x^2 - 4}{x + 1}$

**95.** $y = \dfrac{x^2 - x + 1}{x - 1}$

**96.** $y = -\dfrac{x^2 - x + 1}{x - 1}$

**97.** $y = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$

**98.** $y = \dfrac{x^3 + x - 2}{x - x^2}$

**99.** $y = \dfrac{x}{x^2 - 1}$

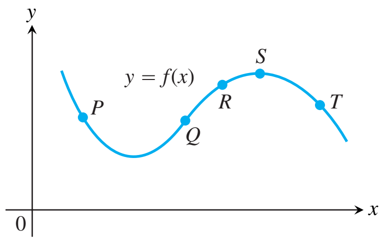
**100.** $y = \dfrac{x - 1}{x^2(x - 2)}$

**101.** $y = \dfrac{8}{x^2 + 4}$ (Agnesi's witch)

**102.** $y = \dfrac{4x}{x^2 + 4}$ (Newton's serpentine)

## Theory and Examples

**103.** The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify $y'$ and $y''$ as positive, negative, or zero.




**104.** Sketch a smooth connected curve $y = f(x)$ with

$f(-2) = 8$, $\quad f'(2) = f'(-2) = 0$,

$f(0) = 4$, $\quad f'(x) < 0$ for $|x| < 2$,

$f(2) = 0$, $\quad f''(x) < 0$ for $x < 0$,

$f'(x) > 0$ for $|x| > 2$, $\quad f''(x) > 0$ for $x > 0$.

**105.** Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

| $x$ | $y$ | Derivatives |
|---|---|---|
| $x < 2$ | | $y' < 0,\ y'' > 0$ |
| 2 | 1 | $y' = 0,\ y'' > 0$ |
| $2 < x < 4$ | | $y' > 0,\ y'' > 0$ |
| 4 | 4 | $y' > 0,\ y'' = 0$ |
| $4 < x < 6$ | | $y' > 0,\ y'' < 0$ |
| 6 | 7 | $y' = 0,\ y'' < 0$ |
| $x > 6$ | | $y' < 0,\ y'' < 0$ |

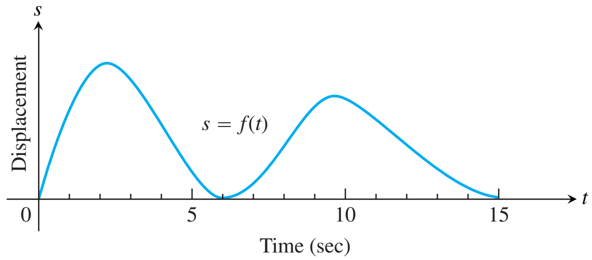
**106.** Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 2)$ and whose first two derivatives have the following sign patterns.

$y'$: $+$ | $-2$ | $-$ | $0$ | $+$ | $2$ | $-$

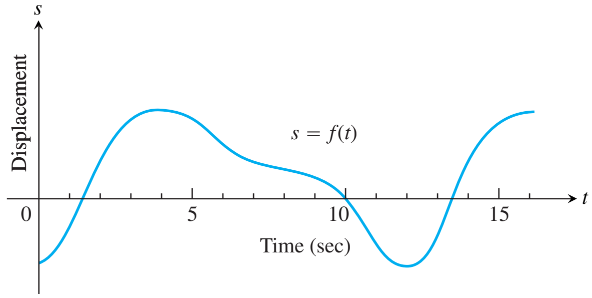
$y''$: $-$ | $-1$ | $+$ | $1$ | $-$

**Motion Along a Line** The graphs in Exercises 107 and 108 show the position $s = f(t)$ of an object moving up and down on a coordinate line. **(a)** When is the object moving away from the origin? toward the origin? At approximately what times is the **(b)** velocity equal to zero? **(c)** acceleration equal to zero? **(d)** When is the acceleration positive? negative?

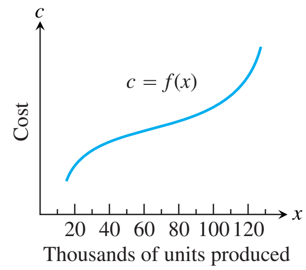
**107.**



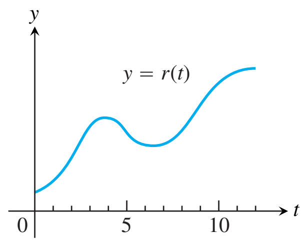

**108.**




**109. Marginal cost** The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing $x$ items. At approximately what production level does the marginal cost change from decreasing to increasing?




**110.** The accompanying graph shows the monthly revenue of the Widget Corporation for the last 12 years. During approximately what time intervals was the marginal revenue increasing? Decreasing?




**111.** Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2).$$

At what points, if any, does the graph of $f$ have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for $y'$.)

**112.** Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of $f$ have a local minimum, local maximum, or point of inflection?

**113.** For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

**114.** Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

**115.** If $b$, $c$, and $d$ are constants, for what value of $b$ will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

**116. Parabolas**

**a.** Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c, a \neq 0$.

**b.** When is the parabola concave up? Concave down? Give reasons for your answers.

**117. Quadratic curves** What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c, a \neq 0$? Give reasons for your answer.

**118. Cubic curves** What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

**119.** Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = (x + 1)(x - 2).$$

For what $x$-values does the graph of $f$ have an inflection point?

**120.** Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = x^2(x - 2)^3(x + 3).$$

For what $x$-values does the graph of $f$ have an inflection point?

**121.** Find the values of constants $a$, $b$, and $c$ so that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.

**122.** Find the values of constants $a$, $b$, and $c$ so that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

### COMPUTER EXPLORATIONS

In Exercises 123–126, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the $x$-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

**123.** $y = x^5 - 5x^4 - 240$ **124.** $y = x^3 - 12x^2$

**125.** $y = \frac{4}{5}x^5 + 16x^2 - 25$

**126.** $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

**127.** Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of $f$ in relation to the signs and values of $f'$ and $f''$.

**128.** Graph $f(x) = x \cos x$ and its second derivative together for $0 \leq x \leq 2\pi$. Comment on the behavior of the graph of $f$ in relation to the signs and values of $f''$.

# 4.5 Indeterminate Forms and L'Hôpital's Rule

John (Johann) Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as **l'Hôpital's Rule**, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print. Limits involving transcendental functions often require some use of the rule for their calculation.

HISTORICAL BIOGRAPHY

Guillaume François Antoine de l'Hôpital (1661–1704)

Johann Bernoulli (1667–1748)

## Indeterminate Form 0/0

If we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $F(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits (Theorem 1 of Chapter 2) because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l'Hôpital's Rule, as we will see in Example 1d.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as $\infty/\infty$, $\infty \cdot 0$, $\infty - \infty$, $0^0$, and $1^\infty$, which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancellation, rearrangement of terms, or other algebraic manipulations. This was our experience in Chapter 2. It took considerable analysis in Section 2.4 to find $\lim_{x \to 0} (\sin x)/x$. But we have had success with the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

from which we calculate derivatives and which produces the indeterminant form $0/0$ when we substitute $x = a$. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

**THEOREM 6— L'Hôpital's Rule** Suppose that $f(a) = g(a) = 0$, that $f$ and $g$ are differentiable on an open interval $I$ containing $a$, and that $g'(x) \neq 0$ on $I$ if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

We give a proof of Theorem 6 at the end of this section.

**Caution**
To apply l'Hôpital's Rule to $f/g$, divide the derivative of $f$ by the derivative of $g$. Do not fall into the trap of taking the derivative of $f/g$. The quotient to use is $f'/g'$, not $(f/g)'$.

**EXAMPLE 1** The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

**(a)** $$\lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = \left.\frac{3 - \cos x}{1}\right|_{x=0} = 2$$

**(b)** $$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1 + x}}}{1} = \frac{1}{2}$$

**(c)** $$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - x/2}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{(1/2)(1 + x)^{-1/2} - 1/2}{2x} \qquad \text{Still } \frac{0}{0}\text{; differentiate again.}$$

$$= \lim_{x \to 0} \frac{-(1/4)(1 + x)^{-3/2}}{2} = -\frac{1}{8} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

**(d)** $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$ $\qquad \frac{0}{0}$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

Here is a summary of the procedure we followed in Example 1.

> **Using L'Hôpital's Rule**
>
> To find
>
> $$\lim_{x \to a} \frac{f(x)}{g(x)}$$
>
> by l'Hôpital's Rule, continue to differentiate $f$ and $g$, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

**EXAMPLE 2** Be careful to apply l'Hôpital's Rule correctly:

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0. \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

Up to now the calculation is correct, but if we continue to differentiate in an attempt to apply l'Hôpital's Rule once more, we get

$$\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2},$$

which is not the correct limit. L'Hôpital's Rule can only be applied to limits that give indeterminate forms, and $0/1$ is not an indeterminate form. ■

L'Hôpital's Rule applies to one-sided limits as well.

**EXAMPLE 3** In this example the one-sided limits are different.

**(a)** $\lim_{x \to 0^+} \dfrac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$$= \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty \qquad \text{Positive for } x > 0$$

Recall that $\infty$ and $+\infty$ mean the same thing.

**(b)** $\lim_{x \to 0^-} \dfrac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$$= \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty \qquad \text{Negative for } x < 0$$

■

## Indeterminate Forms $\infty/\infty$, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an indeterminant form like $\infty/\infty$, $\infty \cdot 0$, or $\infty - \infty$, instead of $0/0$. We first consider the form $\infty/\infty$.

In more advanced treatments of calculus it is proved that l'Hôpital's Rule applies to the indeterminate form $\infty/\infty$ as well as to $0/0$. If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. In the notation $x \to a$, $a$ may be either finite or infinite. Moreover, $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

**EXAMPLE 4** Find the limits of these $\infty/\infty$ forms:

**(a)** $\lim_{x\to\pi/2} \frac{\sec x}{1 + \tan x}$ **(b)** $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x\to\infty} \frac{e^x}{x^2}$.

**Solution**

**(a)** The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose $I$ to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty} \text{ from the left}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

**(b)** $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0 \qquad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

**(c)** $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$ ■

Next we turn our attention to the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$. Sometimes these forms can be handled by using algebra to convert them to a $0/0$ or $\infty/\infty$ form. Here again we do not mean to suggest that $\infty \cdot 0$ or $\infty - \infty$ is a number. They are only notations for functional behaviors when considering limits. Here are examples of how we might work with these indeterminate forms.

**EXAMPLE 5** Find the limits of these $\infty \cdot 0$ forms:

**(a)** $\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right)$ **(b)** $\lim_{x\to 0^+} \sqrt{x} \ln x$

**Solution**

**(a)** $$\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right) = \lim_{h\to 0^+} \left(\frac{1}{h} \sin h\right) = \lim_{h\to 0^+} \frac{\sin h}{h} = 1 \qquad \infty \cdot 0; \text{ Let } h = 1/x.$$

**(b)** $$\lim_{x\to 0^+} \sqrt{x} \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}} \qquad \infty \cdot 0 \text{ converted to } \infty/\infty$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/2x^{3/2}} \qquad \text{l'Hôpital's Rule}$$

$$= \lim_{x\to 0^+} \left(-2\sqrt{x}\right) = 0$$ ■

**EXAMPLE 6** Find the limit of this $\infty - \infty$ form:

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

**Solution** If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} \qquad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x\to 0} \frac{x - \sin x}{x \sin x} && \frac{0}{0} \\
&= \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x \cos x} && \text{Still } \frac{0}{0} \\
&= \lim_{x\to 0} \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0.
\end{aligned}$$

■

## Indeterminate Powers

Limits that lead to the indeterminate forms $1^\infty$, $0^0$, and $\infty^0$ can sometimes be handled by first taking the logarithm of the function. We use l'Hôpital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit. This procedure is justified by the continuity of the exponential function and Theorem 10 in Section 2.5, and it is formulated as follows. (The formula is also valid for one-sided limits.)

> If $\lim_{x\to a} \ln f(x) = L$, then
>
> $$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$
>
> Here $a$ may be either finite or infinite.

**EXAMPLE 7** Apply l'Hôpital's Rule to show that $\lim_{x\to 0^+} (1 + x)^{1/x} = e$.

**Solution** The limit leads to the indeterminate form $1^\infty$. We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x\to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln (1 + x)^{1/x} = \frac{1}{x}\ln(1 + x),$$

l'Hôpital's Rule now applies to give

$$\begin{aligned}\lim_{x\to 0^+} \ln f(x) &= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \qquad \frac{0}{0}\\ &= \lim_{x\to 0^+} \frac{\dfrac{1}{1+x}}{1}\\ &= \frac{1}{1} = 1.\end{aligned}$$

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e.$ ■

**EXAMPLE 8** Find $\lim_{x\to\infty} x^{1/x}$.

**Solution** The limit leads to the indeterminate form $\infty^0$. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\begin{aligned}\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} \qquad \frac{\infty}{\infty}\\ &= \lim_{x\to\infty} \frac{1/x}{1}\\ &= \frac{0}{1} = 0.\end{aligned}$$

Therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$ ■

## Proof of L'Hôpital's Rule

The proof of l'Hôpital's Rule is based on Cauchy's Mean Value Theorem, an extension of the Mean Value Theorem that involves two functions instead of one. We prove Cauchy's Theorem first and then show how it leads to l'Hôpital's Rule.

HISTORICAL BIOGRAPHY

Augustin-Louis Cauchy
(1789–1857)

**THEOREM 7—Cauchy's Mean Value Theorem** Suppose functions $f$ and $g$ are continuous on $[a, b]$ and differentiable throughout $(a, b)$ and also suppose $g'(x) \neq 0$ throughout $(a, b)$. Then there exists a number $c$ in $(a, b)$ at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

**Proof** We apply the Mean Value Theorem of Section 4.2 twice. First we use it to show that $g(a) \neq g(b)$. For if $g(b)$ did equal $g(a)$, then the Mean Value Theorem would give

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some $c$ between $a$ and $b$, which cannot happen because $g'(x) \neq 0$ in $(a, b)$.

We next apply the Mean Value Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)].$$

This function is continuous and differentiable where $f$ and $g$ are, and $F(b) = F(a) = 0$. Therefore, there is a number $c$ between $a$ and $b$ for which $F'(c) = 0$. When expressed in terms of $f$ and $g$, this equation becomes

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}[g'(c)] = 0$$

so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \blacksquare$$

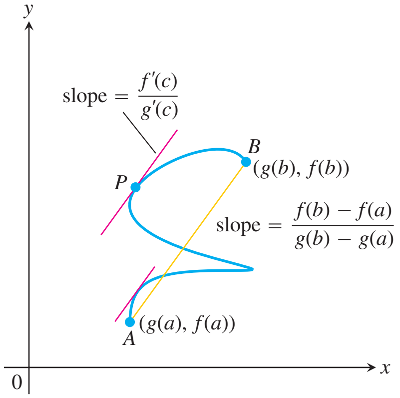



**FIGURE 4.34** There is at least one point $P$ on the curve $C$ for which the slope of the tangent to the curve at $P$ is the same as the slope of the secant line joining the points $A(g(a), f(a))$ and $B(g(b), f(b))$.

Notice that the Mean Value Theorem in Section 4.2 is Theorem 7 with $g(x) = x$.

Cauchy's Mean Value Theorem has a geometric interpretation for a general winding curve $C$ in the plane joining the two points $A = (g(a), f(a))$ and $B = (g(b), f(b))$. In Chapter 11 you will learn how the curve $C$ can be formulated so that there is at least one point $P$ on the curve for which the tangent to the curve at $P$ is parallel to the secant line joining the points $A$ and $B$. The slope of that tangent line turns out to be the quotient $f'/g'$ evaluated at the number $c$ in the interval $(a, b)$, which is the left-hand side of the equation in Theorem 7. Because the slope of the secant line joining $A$ and $B$ is

$$\frac{f(b) - f(a)}{g(b) - g(a)},$$

the equation in Cauchy's Mean Value Theorem says that the slope of the tangent line equals the slope of the secant line. This geometric interpretation is shown in Figure 4.34. Notice from the figure that it is possible for more than one point on the curve $C$ to have a tangent line that is parallel to the secant line joining $A$ and $B$.

**Proof of l'Hôpital's Rule** We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that $x$ lies to the right of $a$. Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from $a$ to $x$. This step produces a number $c$ between $a$ and $x$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As $x$ approaches $a$, $c$ approaches $a$ because it always lies between $a$ and $x$. Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)},$$

which establishes l'Hôpital's Rule for the case where $x$ approaches $a$ from above. The case where $x$ approaches $a$ from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, a]$, $x < a$. $\blacksquare$

# Exercises 4.5

## Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x\to -2} \dfrac{x+2}{x^2-4}$

2. $\lim_{x\to 0} \dfrac{\sin 5x}{x}$

3. $\lim_{x\to \infty} \dfrac{5x^2-3x}{7x^2+1}$

4. $\lim_{x\to 1} \dfrac{x^3-1}{4x^3-x-3}$

5. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

6. $\lim_{x\to \infty} \dfrac{2x^2+3x}{x^3+x+1}$

## Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$

8. $\lim_{x\to -5} \dfrac{x^2-25}{x+5}$

9. $\lim_{t\to -3} \dfrac{t^3-4t+15}{t^2-t-12}$

10. $\lim_{t\to 1} \dfrac{3t^3-3}{4t^3-t-3}$

11. $\lim_{x\to \infty} \dfrac{5x^3-2x}{7x^3+3}$

12. $\lim_{x\to \infty} \dfrac{x-8x^2}{12x^2+5x}$

13. $\lim_{t\to 0} \dfrac{\sin t^2}{t}$

14. $\lim_{t\to 0} \dfrac{\sin 5t}{2t}$

15. $\lim_{x\to 0} \dfrac{8x^2}{\cos x-1}$

16. $\lim_{x\to 0} \dfrac{\sin x-x}{x^3}$

17. $\lim_{\theta\to \pi/2} \dfrac{2\theta-\pi}{\cos(2\pi-\theta)}$

18. $\lim_{\theta\to -\pi/3} \dfrac{3\theta+\pi}{\sin(\theta+(\pi/3))}$

19. $\lim_{\theta\to \pi/2} \dfrac{1-\sin\theta}{1+\cos 2\theta}$

20. $\lim_{x\to 1} \dfrac{x-1}{\ln x-\sin \pi x}$

21. $\lim_{x\to 0} \dfrac{x^2}{\ln(\sec x)}$

22. $\lim_{x\to \pi/2} \dfrac{\ln(\csc x)}{(x-(\pi/2))^2}$

23. $\lim_{t\to 0} \dfrac{t(1-\cos t)}{t-\sin t}$

24. $\lim_{t\to 0} \dfrac{t\sin t}{1-\cos t}$

25. $\lim_{x\to (\pi/2)^-} \left(x-\dfrac{\pi}{2}\right)\sec x$

26. $\lim_{x\to (\pi/2)^-} \left(\dfrac{\pi}{2}-x\right)\tan x$

27. $\lim_{\theta\to 0} \dfrac{3^{\sin\theta}-1}{\theta}$

28. $\lim_{\theta\to 0} \dfrac{(1/2)^{\theta}-1}{\theta}$

29. $\lim_{x\to 0} \dfrac{x2^x}{2^x-1}$

30. $\lim_{x\to 0} \dfrac{3^x-1}{2^x-1}$

31. $\lim_{x\to \infty} \dfrac{\ln(x+1)}{\log_2 x}$

32. $\lim_{x\to \infty} \dfrac{\log_2 x}{\log_3(x+3)}$

33. $\lim_{x\to 0^+} \dfrac{\ln(x^2+2x)}{\ln x}$

34. $\lim_{x\to 0^+} \dfrac{\ln(e^x-1)}{\ln x}$

35. $\lim_{y\to 0} \dfrac{\sqrt{5y+25}-5}{y}$

36. $\lim_{y\to 0} \dfrac{\sqrt{ay+a^2}-a}{y}, \quad a>0$

37. $\lim_{x\to \infty} (\ln 2x-\ln(x+1))$

38. $\lim_{x\to 0^+} (\ln x-\ln\sin x)$

39. $\lim_{x\to 0^+} \dfrac{(\ln x)^2}{\ln(\sin x)}$

40. $\lim_{x\to 0^+} \left(\dfrac{3x+1}{x}-\dfrac{1}{\sin x}\right)$

41. $\lim_{x\to 1^+} \left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right)$

42. $\lim_{x\to 0^+} (\csc x-\cot x+\cos x)$

43. $\lim_{\theta\to 0} \dfrac{\cos\theta-1}{e^{\theta}-\theta-1}$

44. $\lim_{h\to 0} \dfrac{e^h-(1+h)}{h^2}$

45. $\lim_{t\to \infty} \dfrac{e^t+t^2}{e^t-t}$

46. $\lim_{x\to \infty} x^2e^{-x}$

47. $\lim_{x\to 0} \dfrac{x-\sin x}{x\tan x}$

48. $\lim_{x\to 0} \dfrac{(e^x-1)^2}{x\sin x}$

49. $\lim_{\theta\to 0} \dfrac{\theta-\sin\theta\cos\theta}{\tan\theta-\theta}$

50. $\lim_{x\to 0} \dfrac{\sin 3x-3x+x^2}{\sin x\sin 2x}$

## Indeterminate Powers and Products

Find the limits in Exercise 51–66.

51. $\lim_{x\to 1^+} x^{1/(1-x)}$

52. $\lim_{x\to 1^+} x^{1/(x-1)}$

53. $\lim_{x\to \infty} (\ln x)^{1/x}$

54. $\lim_{x\to e^+} (\ln x)^{1/(x-e)}$

55. $\lim_{x\to 0^+} x^{-1/\ln x}$

56. $\lim_{x\to \infty} x^{1/\ln x}$

57. $\lim_{x\to \infty} (1+2x)^{1/(2\ln x)}$

58. $\lim_{x\to 0} (e^x+x)^{1/x}$

59. $\lim_{x\to 0^+} x^x$

60. $\lim_{x\to 0^+} \left(1+\dfrac{1}{x}\right)^x$

61. $\lim_{x\to \infty} \left(\dfrac{x+2}{x-1}\right)^x$

62. $\lim_{x\to \infty} \left(\dfrac{x^2+1}{x+2}\right)^{1/x}$

63. $\lim_{x\to 0^+} x^2\ln x$

64. $\lim_{x\to 0^+} x(\ln x)^2$

65. $\lim_{x\to 0^+} x\tan\left(\dfrac{\pi}{2}-x\right)$

66. $\lim_{x\to 0^+} \sin x\cdot\ln x$

## Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 67–74. Try it—you just keep on cycling. Find the limits some other way.

67. $\lim_{x\to \infty} \dfrac{\sqrt{9x+1}}{\sqrt{x+1}}$

68. $\lim_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{\sin x}}$

69. $\lim_{x\to (\pi/2)^-} \dfrac{\sec x}{\tan x}$

70. $\lim_{x\to 0^+} \dfrac{\cot x}{\csc x}$

71. $\lim_{x\to \infty} \dfrac{2^x-3^x}{3^x+4^x}$

72. $\lim_{x\to -\infty} \dfrac{2^x+4^x}{5^x-2^x}$

73. $\lim_{x\to \infty} \dfrac{e^{x^2}}{xe^x}$

74. $\lim_{x\to 0^+} \dfrac{x}{e^{-1/x}}$

75. Which one is correct, and which one is wrong? Give reasons for your answers.

 **a.** $\lim_{x\to 3} \dfrac{x-3}{x^2-3} = \lim_{x\to 3} \dfrac{1}{2x} = \dfrac{1}{6}$ **b.** $\lim_{x\to 3} \dfrac{x-3}{x^2-3} = \dfrac{0}{6} = 0$

76. Which one is correct, and which one is wrong? Give reasons for your answers.

 **a.** $\lim_{x\to 0} \dfrac{x^2-2x}{x^2-\sin x} = \lim_{x\to 0} \dfrac{2x-2}{2x-\cos x}$

 $= \lim_{x\to 0} \dfrac{2}{2+\sin x} = \dfrac{2}{2+0} = 1$

 **b.** $\lim_{x\to 0} \dfrac{x^2-2x}{x^2-\sin x} = \lim_{x\to 0} \dfrac{2x-2}{2x-\cos x} = \dfrac{-2}{0-1} = 2$

**77.** Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

**a.** $\lim_{x\to 0^+} x \ln x = 0\cdot(-\infty) = 0$

**b.** $\lim_{x\to 0^+} x \ln x = 0\cdot(-\infty) = -\infty$

**c.** $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{(1/x)} = \dfrac{-\infty}{\infty} = -1$

**d.** $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{(1/x)}$

$$= \lim_{x\to 0^+} \frac{(1/x)}{(-1/x^2)} = \lim_{x\to 0^+} (-x) = 0$$

**78.** Find all values of $c$ that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

**a.** $f(x) = x, \quad g(x) = x^2, \quad (a, b) = (-2, 0)$

**b.** $f(x) = x, \quad g(x) = x^2, \quad (a, b)$ arbitrary

**c.** $f(x) = x^3/3 - 4x, \quad g(x) = x^2, \quad (a, b) = (0, 3)$

**79. Continuous extension** Find a value of $c$ that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of $c$ works.

**80.** For what values of $a$ and $b$ is

$$\lim_{x\to 0}\left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x}\right) = 0?$$

T **81. $\infty - \infty$ Form**

**a.** Estimate the value of

$$\lim_{x\to\infty}\left(x - \sqrt{x^2 + x}\right)$$

by graphing $f(x) = x - \sqrt{x^2 + x}$ over a suitably large interval of $x$-values.

**b.** Now confirm your estimate by finding the limit with l'Hôpital's Rule. As the first step, multiply $f(x)$ by the fraction $(x + \sqrt{x^2 + x})/(x + \sqrt{x^2 + x})$ and simplify the new numerator.

**82.** Find $\lim_{x\to\infty}\left(\sqrt{x^2 + 1} - \sqrt{x}\right)$.

T **83. 0/0 Form** Estimate the value of

$$\lim_{x\to 1} \frac{2x^2 - (3x + 1)\sqrt{x} + 2}{x - 1}$$

by graphing. Then confirm your estimate with l'Hôpital's Rule.

**84.** This exercise explores the difference between the limit

$$\lim_{x\to\infty}\left(1 + \frac{1}{x^2}\right)^x$$

and the limit

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

**a.** Use l'Hôpital's Rule to show that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

T **b.** Graph

$$f(x) = \left(1 + \frac{1}{x^2}\right)^x \quad \text{and} \quad g(x) = \left(1 + \frac{1}{x}\right)^x$$

together for $x \geq 0$. How does the behavior of $f$ compare with that of $g$? Estimate the value of $\lim_{x\to\infty} f(x)$.

**c.** Confirm your estimate of $\lim_{x\to\infty} f(x)$ by calculating it with l'Hôpital's Rule.

**85.** Show that

$$\lim_{k\to\infty}\left(1 + \frac{r}{k}\right)^k = e^r.$$

**86.** Given that $x > 0$, find the maximum value, if any, of

**a.** $x^{1/x}$

**b.** $x^{1/x^2}$

**c.** $x^{1/x^n}$ ($n$ a positive integer)

**d.** Show that $\lim_{x\to\infty} x^{1/x^n} = 1$ for every positive integer $n$.

**87.** Use limits to find horizontal asymptotes for each function.

**a.** $y = x \tan\left(\dfrac{1}{x}\right)$ **b.** $y = \dfrac{3x + e^{2x}}{2x + e^{3x}}$

**88.** Find $f'(0)$ for $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

T **89. The continuous extension of $(\sin x)^x$ to $[0, \pi]$**

**a.** Graph $f(x) = (\sin x)^x$ on the interval $0 \leq x \leq \pi$. What value would you assign to $f$ to make it continuous at $x = 0$?

**b.** Verify your conclusion in part (a) by finding $\lim_{x\to 0^+} f(x)$ with l'Hôpital's Rule.

**c.** Returning to the graph, estimate the maximum value of $f$ on $[0, \pi]$. About where is max $f$ taken on?

**d.** Sharpen your estimate in part (c) by graphing $f'$ in the same window to see where its graph crosses the $x$-axis. To simplify your work, you might want to delete the exponential factor from the expression for $f'$ and graph just the factor that has a zero.

T **90. The function $(\sin x)^{\tan x}$** (*Continuation of Exercise 89.*)

**a.** Graph $f(x) = (\sin x)^{\tan x}$ on the interval $-7 \leq x \leq 7$. How do you account for the gaps in the graph? How wide are the gaps?

**b.** Now graph $f$ on the interval $0 \leq x \leq \pi$. The function is not defined at $x = \pi/2$, but the graph has no break at this point. What is going on? What value does the graph appear to give for $f$ at $x = \pi/2$? (*Hint:* Use l'Hôpital's Rule to find lim $f$ as $x \to (\pi/2)^-$ and $x \to (\pi/2)^+$.)

**c.** Continuing with the graphs in part (b), find max $f$ and min $f$ as accurately as you can and estimate the values of $x$ at which they are taken on.

# 4.6 Applied Optimization

What are the dimensions of a rectangle with fixed perimeter having *maximum area*? What are the dimensions for the *least expensive* cylindrical can of a given volume? How many items should be produced for the *most profitable* production run? Each of these questions asks for the best, or optimal, value of a given function. In this section we use derivatives to solve a variety of optimization problems in business, mathematics, physics, and economics.

### Solving Applied Optimization Problems

1. *Read the problem*. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture*. Label any part that may be important to the problem.
3. *Introduce variables*. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity*. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

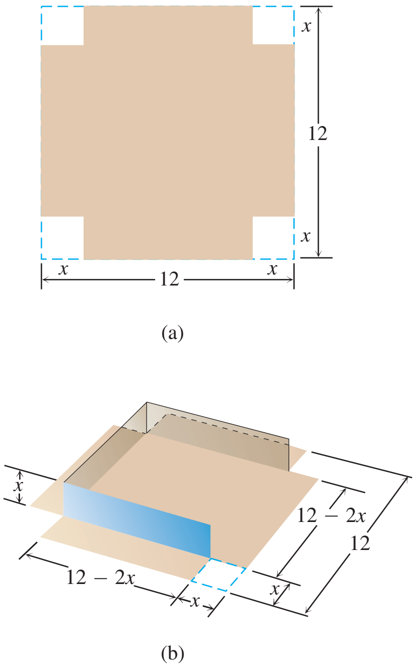



**FIGURE 4.35** An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

**EXAMPLE 1** An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

**Solution** We start with a picture (Figure 4.35). In the figure, the corner squares are $x$ in. on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3. \qquad V = hlw$$

Since the sides of the sheet of tin are only 12 in. long, $x \le 6$ and the domain of $V$ is the interval $0 \le x \le 6$.

A graph of $V$ (Figure 4.36) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of $V$ with respect to $x$:

$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of $V$ at this one critical point and two endpoints are

Critical-point value: $V(2) = 128$

Endpoint values: $V(0) = 0, \qquad V(6) = 0.$

The maximum volume is 128 in$^3$. The cutout squares should be 2 in. on a side. ■

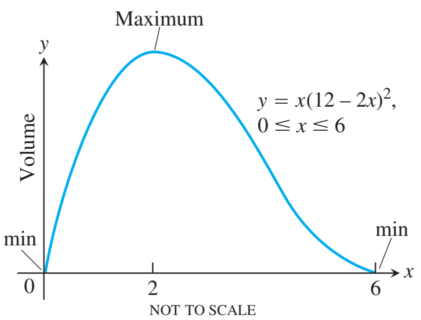



**FIGURE 4.36** The volume of the box in Figure 4.35 graphed as a function of $x$.

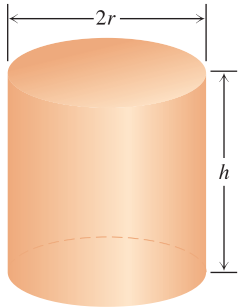



**FIGURE 4.37** This one-liter can uses the least material when $h = 2r$ (Example 2).

**EXAMPLE 2** You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 4.37). What dimensions will use the least material?

**Solution** *Volume of can:* If $r$ and $h$ are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000. \qquad \text{1 liter} = 1000\ \text{cm}^3$$

*Surface area of can:*

$$A = \underbrace{2\pi r^2}_{\substack{\text{circular}\\ \text{ends}}} + \underbrace{2\pi rh}_{\substack{\text{cylindrical}\\ \text{wall}}}$$

How can we interpret the phrase "least material"? For a first approximation we can ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions $r$ and $h$ that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000$.

To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for $h$ is easier:

$$h = \frac{1000}{\pi r^2}.$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{2000}{r}. \end{aligned}$$

Our goal is to find a value of $r > 0$ that minimizes the value of $A$. Figure 4.38 suggests that such a value exists.

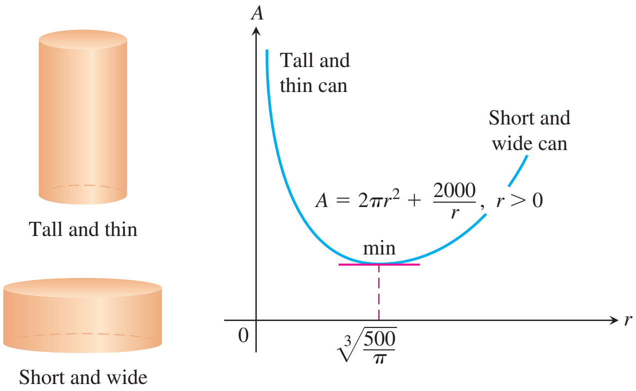



**FIGURE 4.38** The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

Notice from the graph that for small $r$ (a tall, thin cylindrical container), the term $2000/r$ dominates (see Section 2.6) and $A$ is large. For large $r$ (a short, wide cylindrical container), the term $2\pi r^2$ dominates and $A$ again is large.

Since $A$ is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - \frac{2000}{r^2} \qquad \text{Set } dA/dr = 0.$$

$$4\pi r^3 = 2000 \qquad \text{Multiply by } r^2.$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \qquad \text{Solve for } r.$$

What happens at $r = \sqrt[3]{500/\pi}$?

The second derivative

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of $A$. The graph is therefore everywhere concave up and the value of $A$ at $r = \sqrt[3]{500/\pi}$ is an absolute minimum.

The corresponding value of $h$ (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

The one-liter can that uses the least material has height equal to twice the radius, here with $r \approx 5.42$ cm and $h \approx 10.84$ cm. ■

## Examples from Mathematics and Physics

**EXAMPLE 3** A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

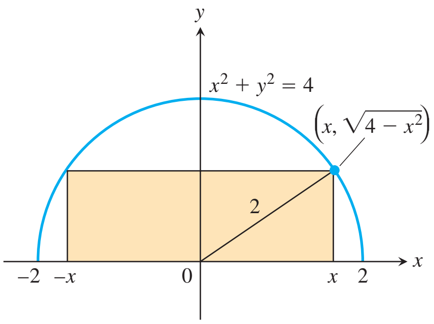



**FIGURE 4.39** The rectangle inscribed in the semicircle in Example 3.

**Solution** Let $(x, \sqrt{4 - x^2})$ be the coordinates of the corner of the rectangle obtained by placing the circle and rectangle in the coordinate plane (Figure 4.39). The length, height, and area of the rectangle can then be expressed in terms of the position $x$ of the lower right-hand corner:

$$\text{Length: } 2x, \qquad \text{Height: } \sqrt{4 - x^2}, \qquad \text{Area: } 2x\sqrt{4 - x^2}.$$

Notice that the values of $x$ are to be found in the interval $0 \le x \le 2$, where the selected corner of the rectangle lies.

Our goal is to find the absolute maximum value of the function

$$A(x) = 2x\sqrt{4 - x^2}$$

on the domain [0, 2].

The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

is not defined when $x = 2$ and is equal to zero when

$$\frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} = 0$$

$$-2x^2 + 2(4 - x^2) = 0$$

$$8 - 4x^2 = 0$$

$$x^2 = 2 \text{ or } x = \pm\sqrt{2}.$$

Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of $A$'s domain and makes the critical-point list. The values of $A$ at the endpoints and at this one critical point are

$$\text{Critical-point value:} \quad A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4-2} = 4$$
$$\text{Endpoint values:} \quad A(0) = 0, \qquad A(2) = 0.$$

The area has a maximum value of 4 when the rectangle is $\sqrt{4-x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long. ■

HISTORICAL BIOGRAPHY

Willebrord Snell van Royen (1580–1626)

**EXAMPLE 4** The speed of light depends on the medium through which it travels, and is generally slower in denser media.

**Fermat's principle in optics** states that light travels from one point to another along a path for which the time of travel is a minimum. Describe the path that a ray of light will follow in going from a point $A$ in a medium where the speed of light is $c_1$ to a point $B$ in a second medium where its speed is $c_2$.

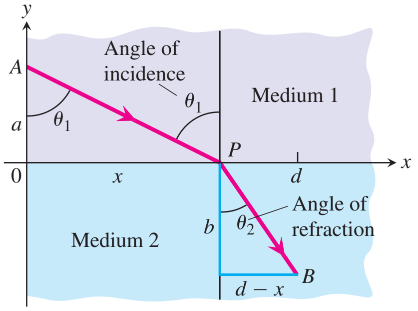



**FIGURE 4.40** A light ray refracted (deflected from its path) as it passes from one medium to a denser medium (Example 4).

**Solution** Since light traveling from $A$ to $B$ follows the quickest route, we look for a path that will minimize the travel time. We assume that $A$ and $B$ lie in the $xy$-plane and that the line separating the two media is the $x$-axis (Figure 4.40).

In a uniform medium, where the speed of light remains constant, "shortest time" means "shortest path," and the ray of light will follow a straight line. Thus the path from $A$ to $B$ will consist of a line segment from $A$ to a boundary point $P$, followed by another line segment from $P$ to $B$. Distance traveled equals rate times time, so

$$\text{Time} = \frac{\text{distance}}{\text{rate}}.$$

From Figure 4.40, the time required for light to travel from $A$ to $P$ is

$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2 + x^2}}{c_1}.$$

From $P$ to $B$, the time is

$$t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2 + (d-x)^2}}{c_2}.$$

The time from $A$ to $B$ is the sum of these:

$$t = t_1 + t_2 = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c_2}.$$

This equation expresses $t$ as a differentiable function of $x$ whose domain is $[0, d]$. We want to find the absolute minimum value of $t$ on this closed interval. We find the derivative

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d-x}{c_2\sqrt{b^2 + (d-x)^2}}$$

and observe that it is continuous. In terms of the angles $\theta_1$ and $\theta_2$ in Figure 4.40,

$$\frac{dt}{dx} = \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}.$$

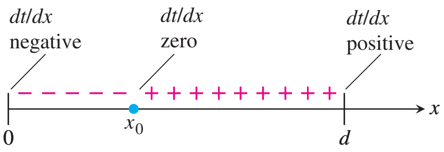



**FIGURE 4.41** The sign pattern of $dt/dx$ in Example 4.

The function $t$ has a negative derivative at $x = 0$ and a positive derivative at $x = d$. Since $dt/dx$ is continuous over the interval $[0, d]$, by the Intermediate Value Theorem for continuous functions (Section 2.5), there is a point $x_0 \in [0, d]$ where $dt/dx = 0$ (Figure 4.41).

There is only one such point because $dt/dx$ is an increasing function of $x$ (Exercise 62). At this unique point we then have

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}.$$

This equation is **Snell's Law** or the **Law of Refraction**, and is an important principle in the theory of optics. It describes the path the ray of light follows. ■

## Examples from Economics

Suppose that

$$r(x) = \text{the revenue from selling } x \text{ items}$$
$$c(x) = \text{the cost of producing the } x \text{ items}$$
$$p(x) = r(x) - c(x) = \text{the profit from producing and selling } x \text{ items}.$$

Although $x$ is usually an integer in many applications, we can learn about the behavior of these functions by defining them for all nonzero real numbers and by assuming they are differentiable functions. Economists use the terms **marginal revenue**, **marginal cost**, and **marginal profit** to name the derivatives $r'(x)$, $c'(x)$, and $p'(x)$ of the revenue, cost, and profit functions. Let's consider the relationship of the profit $p$ to these derivatives.

If $r(x)$ and $c(x)$ are differentiable for $x$ in some interval of production possibilities, and if $p(x) = r(x) - c(x)$ has a maximum value there, it occurs at a critical point of $p(x)$ or at an endpoint of the interval. If it occurs at a critical point, then $p'(x) = r'(x) - c'(x) = 0$ and we see that $r'(x) = c'(x)$. In economic terms, this last equation means that

> At a production level yielding maximum profit, marginal revenue equals marginal cost (Figure 4.42).

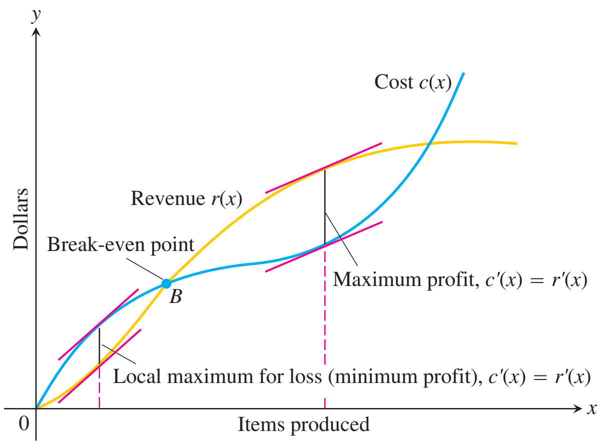



**FIGURE 4.42** The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point $B$. To the left of $B$, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and material costs and market saturation) and production levels become unprofitable again.

**EXAMPLE 5** Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where $x$ represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

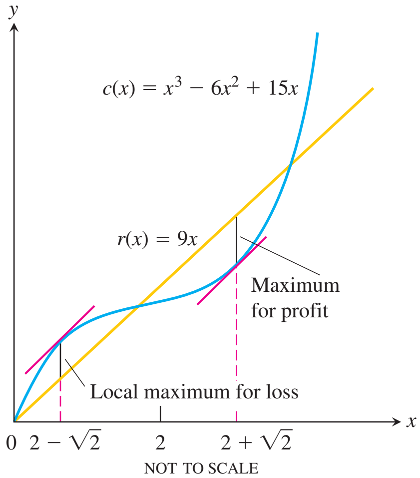



**FIGURE 4.43** The cost and revenue curves for Example 5.

**Solution** Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$3x^2 - 12x + 15 = 9 \qquad \text{Set } c'(x) = r'(x).$$
$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \qquad \text{and}$$

$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

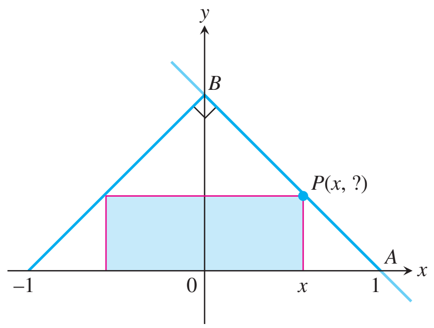
The possible production levels for maximum profit are $x \approx 0.586$ million MP3 players or $x \approx 3.414$ million. The second derivative of $p(x) = r(x) - c(x)$ is $p''(x) = -c''(x)$ since $r''(x)$ is everywhere zero. Thus, $p''(x) = 6(2 - x)$, which is negative at $x = 2 + \sqrt{2}$ and positive at $x = 2 - \sqrt{2}$. By the Second Derivative Test, a maximum profit occurs at about $x = 3.414$ (where revenue exceeds costs) and maximum loss occurs at about $x = 0.586$. The graphs of $r(x)$ and $c(x)$ are shown in Figure 4.43. ■

# Exercises 4.6

### Mathematical Applications

Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph it over the domain that is appropriate to the problem you are solving. The graph will provide insight before you calculate and will furnish a visual context for understanding your answer.

1. **Minimizing perimeter** What is the smallest perimeter possible for a rectangle whose area is 16 in$^2$, and what are its dimensions?
2. Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.
3. The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
   a. Express the $y$-coordinate of $P$ in terms of $x$. (*Hint:* Write an equation for the line $AB$.)
   b. Express the area of the rectangle in terms of $x$.
   c. What is the largest area the rectangle can have, and what are its dimensions?




4. A rectangle has its base on the $x$-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?
5. You are planning to make an open rectangular box from an 8-in.-by-15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?
6. You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.
7. **The best fencing plan** A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?
8. **The shortest fence** A 216 m$^2$ rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?
9. **Designing a tank** Your iron works has contracted to design and build a 500 ft$^3$, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.

**a.** What dimensions do you tell the shop to use?

**b.** Briefly describe how you took weight into account.

**10. Catching rainwater** A 1125 $\text{ft}^3$ open-top rectangular tank with a square base $x$ ft on a side and $y$ ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product $xy$.

**a.** If the total cost is

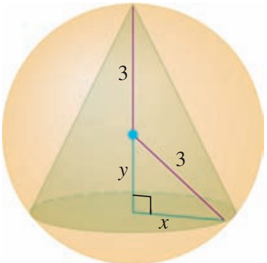
$$c = 5(x^2 + 4xy) + 10xy,$$

what values of $x$ and $y$ will minimize it?

**b.** Give a possible scenario for the cost function in part (a).

**11. Designing a poster** You are designing a rectangular poster to contain 50 $\text{in}^2$ of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?

**12.** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.




**13.** Two sides of a triangle have lengths $a$ and $b$, and the angle between them is $\theta$. What value of $\theta$ will maximize the triangle's area? (*Hint:* $A = (1/2)ab\sin\theta$.)

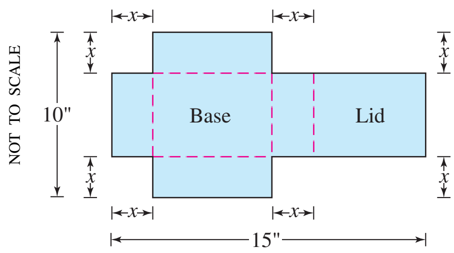
**14. Designing a can** What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 $\text{cm}^3$? Compare the result here with the result in Example 2.

**15. Designing a can** You are designing a 1000 $\text{cm}^3$ right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius $r$ will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 2. In Example 2, the ratio of $h$ to $r$ for the most economical can was 2 to 1. What is the ratio now?

T **16. Designing a box with a lid** A piece of cardboard measures 10 in. by 15 in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.




**a.** Write a formula $V(x)$ for the volume of the box.

**b.** Find the domain of $V$ for the problem situation and graph $V$ over this domain.

**c.** Use a graphical method to find the maximum volume and the value of $x$ that gives it.

**d.** Confirm your result in part (c) analytically.

T **17. Designing a suitcase** A 24-in.-by-36-in. sheet of cardboard is folded in half to form a 24-in.-by-18-in. rectangle as shown in the accompanying figure. Then four congruent squares of side length $x$ are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.

**a.** Write a formula $V(x)$ for the volume of the box.

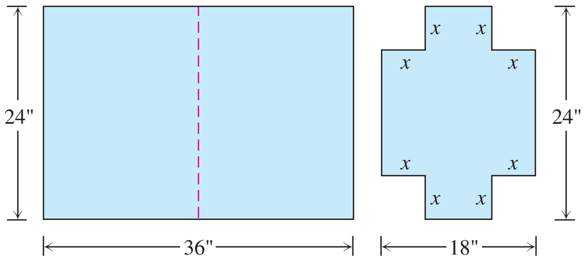
**b.** Find the domain of $V$ for the problem situation and graph $V$ over this domain.

**c.** Use a graphical method to find the maximum volume and the value of $x$ that gives it.

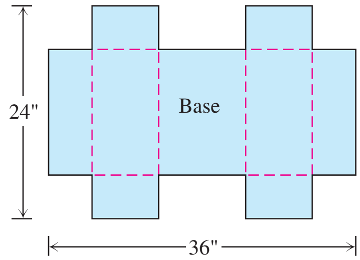
**d.** Confirm your result in part (c) analytically.

**e.** Find a value of $x$ that yields a volume of 1120 $\text{in}^3$.

**f.** Write a paragraph describing the issues that arise in part (b).



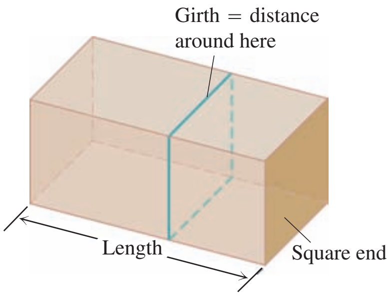

The sheet is then unfolded.




**18.** A rectangle is to be inscribed under the arch of the curve $y = 4\cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

**19.** Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

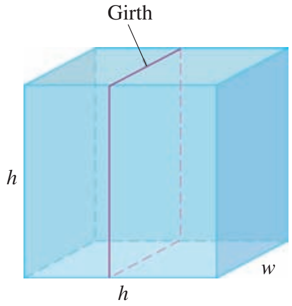
**20. a.** The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?




T **b.** Graph the volume of a 108-in. box (length plus girth equals 108 in.) as a function of its length and compare what you see with your answer in part (a).

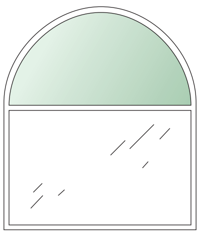
**21.** (*Continuation of Exercise 20.*)

**a.** Suppose that instead of having a box with square ends you have a box with square sides so that its dimensions are $h$ by $h$ by $w$ and the girth is $2h + 2w$. What dimensions will give the box its largest volume now?



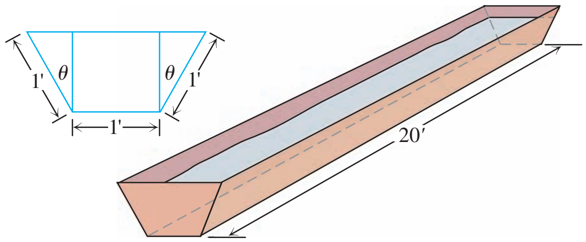

T **b.** Graph the volume as a function of $h$ and compare what you see with your answer in part (a).

**22.** A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



**23.** A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.

**24.** The trough in the figure is to be made to the dimensions shown. Only the angle $\theta$ can be varied. What value of $\theta$ will maximize the trough's volume?

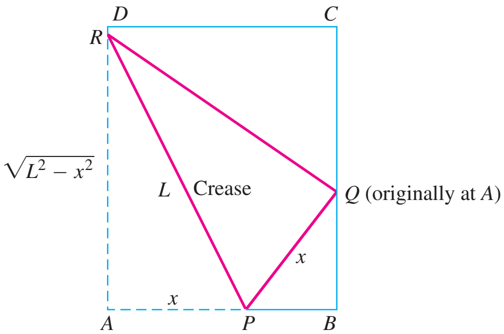



**25. Paper folding** A rectangular sheet of 8.5-in.-by-11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length $L$. Try it with paper.

**a.** Show that $L^2 = 2x^3/(2x - 8.5)$.

**b.** What value of $x$ minimizes $L^2$?

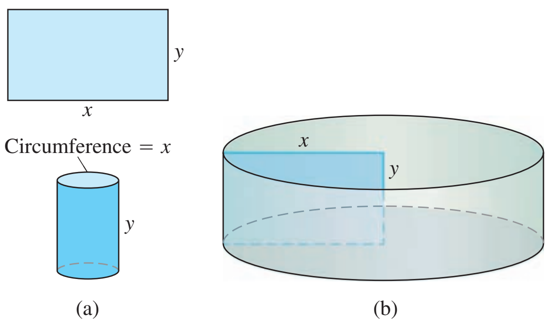
**c.** What is the minimum value of $L$?



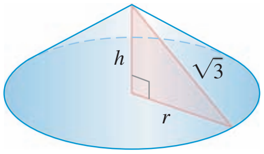

**26. Constructing cylinders** Compare the answers to the following two construction problems.

**a.** A rectangular sheet of perimeter 36 cm and dimensions $x$ cm by $y$ cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of $x$ and $y$ give the largest volume?

**b.** The same sheet is to be revolved about one of the sides of length $y$ to sweep out the cylinder as shown in part (b) of the figure. What values of $x$ and $y$ give the largest volume?



**27. Constructing cones** A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.




**28.** Find the point on the line $\frac{x}{a} + \frac{y}{b} = 1$ that is closest to the origin.

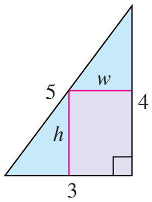
**29.** Find a positive number for which the sum of it and its reciprocal is the smallest (least) possible.

**30.** Find a postitive number for which the sum of its reciprocal and four times its square is the smallest possible.

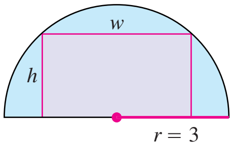
**31.** A wire $b$ m long is cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a circle. If the sum of the areas enclosed by each part is a minimum, what is the length of each part?

**32.** Answer Exercise 31 if one piece is bent into a square and the other into a circle.

**33.** Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle shown in the accompanying figure.




**34.** Determine the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 3. (See accompanying figure.)




**35.** What value of $a$ makes $f(x) = x^2 + (a/x)$ have

**a.** a local minimum at $x = 2$?

**b.** a point of inflection at $x = 1$?

**36.** What values of $a$ and $b$ make $f(x) = x^3 + ax^2 + bx$ have

**a.** a local maximum at $x = -1$ and a local minimum at $x = 3$?

**b.** a local minimum at $x = 4$ and a point of inflection at $x = 1$?

## Physical Applications

**37. Vertical motion** The height above ground of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

with $s$ in feet and $t$ in seconds. Find

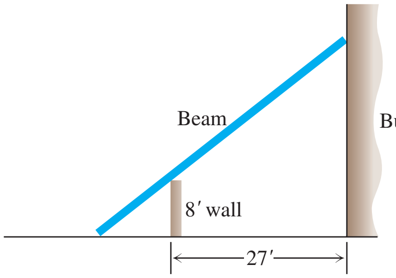
**a.** the object's velocity when $t = 0$;

**b.** its maximum height and when it occurs;

**c.** its velocity when $s = 0$.

**38. Quickest route** Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

**39. Shortest beam** The 8-ft wall shown here stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.




**40. Motion on a line** The positions of two particles on the $s$-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with $s_1$ and $s_2$ in meters and $t$ in seconds.

**a.** At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

**b.** What is the farthest apart that the particles ever get?

**c.** When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

**41.** The intensity of illumination at any point from a light source is proportional to the square of the reciprocal of the distance between the point and the light source. Two lights, one having an intensity eight times that of the other, are 6 m apart. How far from the stronger light is the total illumination least?

**42. Projectile motion** The *range* $R$ of a projectile fired from the origin over horizontal ground is the distance from the origin to the point of impact. If the projectile is fired with an initial velocity $v_0$ at an angle $\alpha$ with the horizontal, then in Chapter 13 we find that

$$R = \frac{v_0^2}{g}\sin 2\alpha,$$

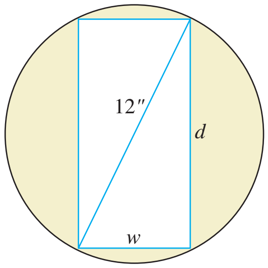
where $g$ is the downward acceleration due to gravity. Find the angle $\alpha$ for which the range $R$ is the largest possible.

T **43. Strength of a beam** The strength $S$ of a rectangular wooden beam is proportional to its width times the square of its depth. (See the accompanying figure.)

**a.** Find the dimensions of the strongest beam that can be cut from a 12-in.-diameter cylindrical log.

**b.** Graph $S$ as a function of the beam's width $w$, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

**c.** On the same screen, graph $S$ as a function of the beam's depth $d$, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of $k$? Try it.



T **44. Stiffness of a beam** The stiffness $S$ of a rectangular beam is proportional to its width times the cube of its depth.

**a.** Find the dimensions of the stiffest beam that can be cut from a 12-in.-diameter cylindrical log.

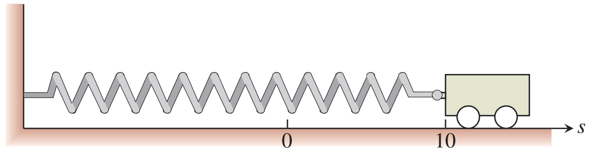
**b.** Graph $S$ as a function of the beam's width $w$, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

**c.** On the same screen, graph $S$ as a function of the beam's depth $d$, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of $k$? Try it.

**45. Frictionless cart** A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time $t$ is $s = 10 \cos \pi t$.

**a.** What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

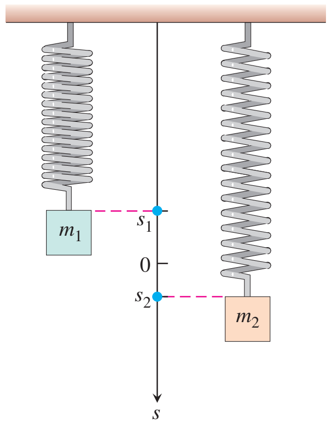
**b.** Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?




**46.** Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively.

**a.** At what times in the interval $0 < t$ do the masses pass each other? (*Hint:* $\sin 2t = 2 \sin t \cos t$.)

**b.** When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2\cos^2 t - 1$.)




**47. Distance between two ships** At noon, ship $A$ was 12 nautical miles due north of ship $B$. Ship $A$ was sailing south at 12 knots (nautical miles per hour; a nautical mile is 2000 yd) and continued to do so all day. Ship $B$ was sailing east at 8 knots and continued to do so all day.

**a.** Start counting time with $t = 0$ at noon and express the distance $s$ between the ships as a function of $t$.

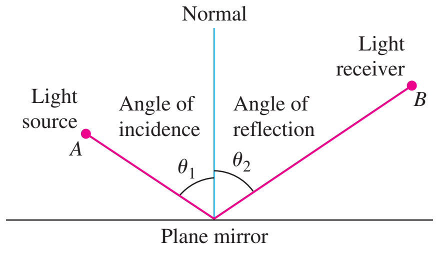
**b.** How rapidly was the distance between the ships changing at noon? One hour later?

**c.** The visibility that day was 5 nautical miles. Did the ships ever sight each other?

T **d.** Graph $s$ and $ds/dt$ together as functions of $t$ for $-1 \le t \le 3$, using different colors if possible. Compare the graphs and reconcile what you see with your answers in parts (b) and (c).

**e.** The graph of $ds/dt$ looks as if it might have a horizontal asymptote in the first quadrant. This in turn suggests that $ds/dt$ approaches a limiting value as $t \to \infty$. What is this value? What is its relation to the ships' individual speeds?

**48. Fermat's principle in optics** Light from a source $A$ is reflected by a plane mirror to a receiver at point $B$, as shown in the accompanying figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)




**49. Tin pest** When metallic tin is kept below 13.2°C, it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious, and indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases, it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, $v$ may be considered to be a function of $x$ alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

$x =$ the amount of product

$a =$ the amount of substance at the beginning

$k =$ a positive constant.

At what value of $x$ does the rate $v$ have a maximum? What is the maximum value of $v$?

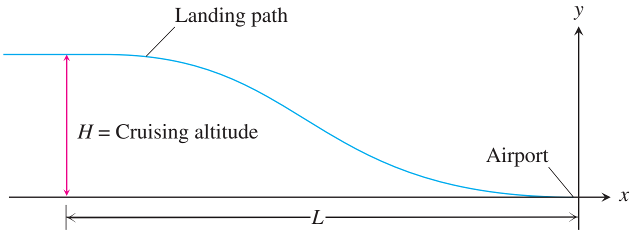
**50. Airplane landing path** An airplane is flying at altitude $H$ when it begins its descent to an airport runway that is at horizontal ground distance $L$ from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$, where $y(-L) = H$ and $y(0) = 0$.

**a.** What is $dy/dx$ at $x = 0$?

**b.** What is $dy/dx$ at $x = -L$?

**c.** Use the values for $dy/dx$ at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$




## Business and Economics

**51.** It costs you $c$ dollars each to manufacture and distribute backpacks. If the backpacks sell at $x$ dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where $a$ and $b$ are positive constants. What selling price will bring a maximum profit?

**52.** You operate a tour service that offers the following rates:

$200 per person if 50 people (the minimum number to book the tour) go on the tour.

For each additional person, up to a maximum of 80 people total, the rate per person is reduced by $2.

It costs $6000 (a fixed cost) plus $32 per person to conduct the tour. How many people does it take to maximize your profit?

**53. Wilson lot size formula** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where $q$ is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), $k$ is the cost of placing an order (the same, no matter how often you order), $c$ is the cost of one item (a constant), $m$ is the number of items sold each week (a constant), and $h$ is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

**a.** Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula*.)

**b.** Shipping costs sometimes depend on order size. When they do, it is more realistic to replace $k$ by $k + bq$, the sum of $k$ and a constant multiple of $q$. What is the most economical quantity to order now?

**54. Production level** Prove that the production level (if any) at which average cost is smallest is a level at which the average cost equals marginal cost.

**55.** Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

**56. Production level** Suppose that $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing $x$ items. Find a production level that will minimize the average cost of making $x$ items.

**57.** You are to construct an open rectangular box with a square base and a volume of 48 ft$^3$. If material for the bottom costs \$6/ft$^2$ and material for the sides costs \$4/ft$^2$, what dimensions will result in the least expensive box? What is the minimum cost?

**58.** The 800-room Mega Motel chain is filled to capacity when the room charge is $50 per night. For each $10 increase in room charge, 40 fewer rooms are filled each night. What charge per room will result in the maximum revenue per night?

## Biology

**59. Sensitivity to medicine** (*Continuation of Exercise 72, Section 3.3.*) Find the amount of medicine to which the body is most sensitive by finding the value of $M$ that maximizes the derivative $dR/dM$, where

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$$

and $C$ is a constant.

**60. How we cough**

**a.** When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity $v$ can be modeled by the equation

$$v = c(r_0 - r)r^2 \text{ cm/sec}, \qquad \frac{r_0}{2} \le r \le r_0,$$

where $r_0$ is the rest radius of the trachea in centimeters and $c$ is a positive constant whose value depends in part on the length of the trachea.

Show that $v$ is greatest when $r = (2/3)r_0$; that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

T **b.** Take $r_0$ to be 0.5 and $c$ to be 1 and graph $v$ over the interval $0 \le r \le 0.5$. Compare what you see with the claim that $v$ is at a maximum when $r = (2/3)r_0$.

## Theory and Examples

**61. An inequality for positive integers** Show that if $a$, $b$, $c$, and $d$ are positive integers, then

$$\frac{(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)}{abcd} \ge 16.$$

**62. The derivative $dt/dx$ in Example 4**

**a.** Show that

$$f(x) = \frac{x}{\sqrt{a^2 + x^2}}$$

is an increasing function of $x$.

**b.** Show that

$$g(x) = \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$$

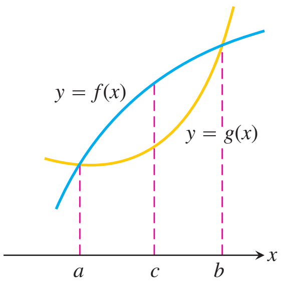
is a decreasing function of $x$.

**c.** Show that

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d - x}{c_2\sqrt{b^2 + (d - x)^2}}$$

is an increasing function of $x$.

**63.** Let $f(x)$ and $g(x)$ be the differentiable functions graphed here. Point $c$ is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at $c$? Give reasons for your answer.




**64.** You have been asked to determine whether the function $f(x) = 3 + 4\cos x + \cos 2x$ is ever negative.

**a.** Explain why you need to consider values of $x$ only in the interval $[0, 2\pi]$.

**b.** Is $f$ ever negative? Explain.

**65. a.** The function $y = \cot x - \sqrt{2}\csc x$ has an absolute maximum value on the interval $0 < x < \pi$. Find it.

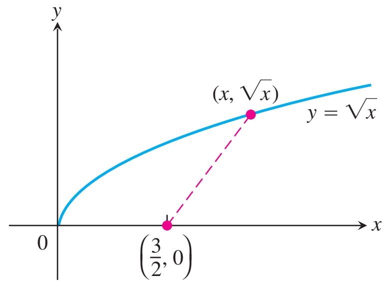
T **b.** Graph the function and compare what you see with your answer in part (a).

**66. a.** The function $y = \tan x + 3\cot x$ has an absolute minimum value on the interval $0 < x < \pi/2$. Find it.

T **b.** Graph the function and compare what you see with your answer in part (a).

**67. a.** How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? (*Hint:* If you minimize the *square* of the distance, you can avoid square roots.)

T **b.** Graph the distance function $D(x)$ and $y = \sqrt{x}$ together and reconcile what you see with your answer in part (a).




**68. a.** How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $\left(1, \sqrt{3}\right)$?

T **b.** Graph the distance function and $y = \sqrt{16 - x^2}$ together and reconcile what you see with your answer in part (a).

# 4.7 Newton's Method

In this section we study a numerical method, called *Newton's method* or the *Newton–Raphson method*, which is a technique to approximate the solution to an equation $f(x) = 0$. Essentially it uses tangent lines in place of the graph of $y = f(x)$ near the points where $f$ is zero. (A value of $x$ where $f$ is zero is a *root* of the function $f$ and a *solution* of the equation $f(x) = 0$.)

## Procedure for Newton's Method

The goal of Newton's method for estimating a solution of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution. We pick the first number $x_0$ of the sequence. Then, under favorable circumstances, the method does the rest by moving step by step toward a point where the graph of $f$ crosses the $x$-axis (Figure 4.44). At each step the method approximates a zero of $f$ with a zero of one of its linearizations. Here is how it works.

The initial estimate, $x_0$, may be found by graphing or just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ to approximate the curve, calling

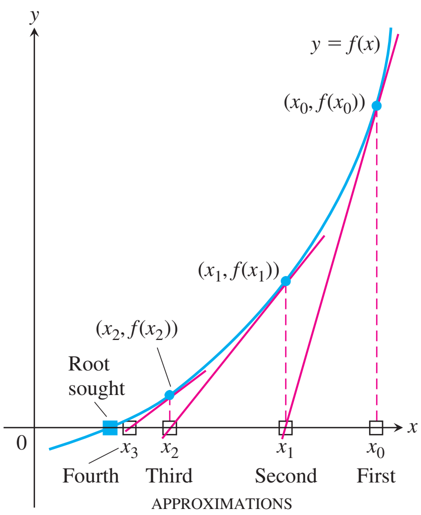



**FIGURE 4.44** Newton's method starts with an initial guess $x_0$ and (under favorable circumstances) improves the guess one step at a time.

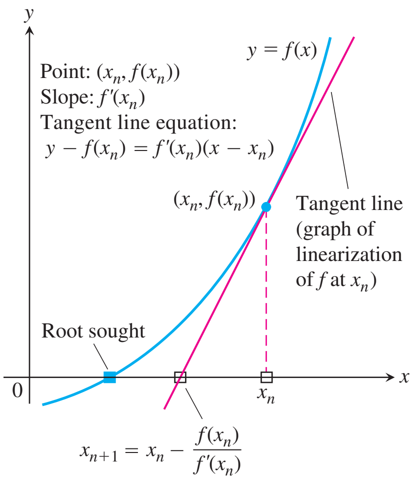



**FIGURE 4.45** The geometry of the successive steps of Newton's method. From $x_n$ we go up to the curve and follow the tangent line down to find $x_{n+1}$.

the point $x_1$ where the tangent meets the $x$-axis (Figure 4.44). The number $x_1$ is usually a better approximation to the solution than is $x_0$. The point $x_2$ where the tangent to the curve at $(x_1, f(x_1))$ crosses the $x$-axis is the next approximation in the sequence. We continue on, using each approximation to generate the next, until we are close enough to the root to stop.

We can derive a formula for generating the successive approximations in the following way. Given the approximation $x_n$, the point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y = f(x_n) + f'(x_n)(x - x_n).$$

We can find where it crosses the $x$-axis by setting $y = 0$ (Figure 4.45):

$$0 = f(x_n) + f'(x_n)(x - x_n)$$

$$-\frac{f(x_n)}{f'(x_n)} = x - x_n$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{If } f'(x_n) \neq 0$$

This value of $x$ is the next approximation $x_{n+1}$. Here is a summary of Newton's method.

> **Newton's Method**
>
> 1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
> 2. Use the first approximation to get a second, the second to get a third, and so on, using the formula
>
> $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0. \qquad (1)$$

## Applying Newton's Method

Applications of Newton's method generally involve many numerical computations, making them well suited for computers or calculators. Nevertheless, even when the calculations are done by hand (which may be very tedious), they give a powerful way to find solutions of equations.

In our first example, we find decimal approximations to $\sqrt{2}$ by estimating the positive root of the equation $f(x) = x^2 - 2 = 0$.

**EXAMPLE 1** Find the positive root of the equation

$$f(x) = x^2 - 2 = 0.$$

**Solution** With $f(x) = x^2 - 2$ and $f'(x) = 2x$, Equation (1) becomes

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^2 - 2}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n}. \end{aligned}$$

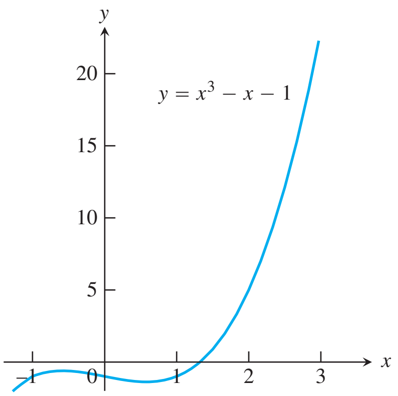



**FIGURE 4.46** The graph of $f(x) = x^3 - x - 1$ crosses the $x$-axis once; this is the root we want to find (Example 2).

The equation

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

enables us to go from each approximation to the next with just a few keystrokes. With the starting value $x_0 = 1$, we get the results in the first column of the following table. (To five decimal places, $\sqrt{2} = 1.41421$.)

| | Error | Number of correct digits |
|---|---|---|
| $x_0 = 1$ | $-0.41421$ | 1 |
| $x_1 = 1.5$ | 0.08579 | 1 |
| $x_2 = 1.41667$ | 0.00246 | 3 |
| $x_3 = 1.41422$ | 0.00001 | 5 |

■

Newton's method is the method used by most calculators to calculate roots because it converges so fast (more about this later). If the arithmetic in the table in Example 1 had been carried to 13 decimal places instead of 5, then going one step further would have given $\sqrt{2}$ correctly to more than 10 decimal places.

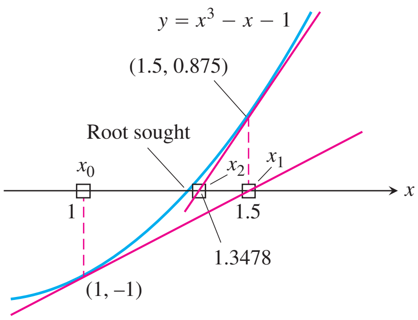



**FIGURE 4.47** The first three $x$-values in Table 4.1 (four decimal places).

**EXAMPLE 2** Find the $x$-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

**Solution** The curve crosses the line when $x^3 - x = 1$ or $x^3 - x - 1 = 0$. When does $f(x) = x^3 - x - 1$ equal zero? Since $f(1) = -1$ and $f(2) = 5$, we know by the Intermediate Value Theorem there is a root in the interval (1, 2) (Figure 4.46).

We apply Newton's method to $f$ with the starting value $x_0 = 1$. The results are displayed in Table 4.1 and Figure 4.47.

At $n = 5$, we come to the result $x_6 = x_5 = 1.3247\,17957$. When $x_{n+1} = x_n$, Equation (1) shows that $f(x_n) = 0$. We have found a solution of $f(x) = 0$ to nine decimals. ■

**TABLE 4.1** The result of applying Newton's method to $f(x) = x^3 - x - 1$ with $x_0 = 1$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|
| 0 | 1 | $-1$ | 2 | 1.5 |
| 1 | 1.5 | 0.875 | 5.75 | 1.3478 26087 |
| 2 | 1.3478 26087 | 0.1006 82173 | 4.4499 05482 | 1.3252 00399 |
| 3 | 1.3252 00399 | 0.0020 58362 | 4.2684 68292 | 1.3247 18174 |
| 4 | 1.3247 18174 | 0.0000 00924 | 4.2646 34722 | 1.3247 17957 |
| 5 | 1.3247 17957 | $-1.8672$E-13 | 4.2646 32999 | 1.3247 17957 |

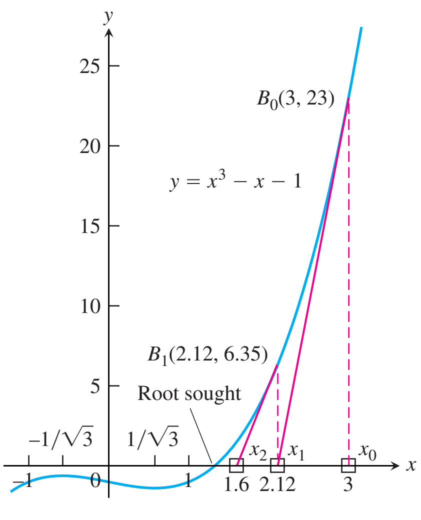



**FIGURE 4.48** Any starting value $x_0$ to the right of $x = 1/\sqrt{3}$ will lead to the root.

In Figure 4.48 we have indicated that the process in Example 2 might have started at the point $B_0(3, 23)$ on the curve, with $x_0 = 3$. Point $B_0$ is quite far from the $x$-axis, but the tangent at $B_0$ crosses the $x$-axis at about (2.12, 0), so $x_1$ is still an improvement over $x_0$. If we use Equation (1) repeatedly as before, with $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$, we obtain the nine-place solution $x_7 = x_6 = 1.3247\,17957$ in seven steps.

## Convergence of the Approximations

In Chapter 10 we define precisely the idea of *convergence* for the approximations $x_n$ in Newton's method. Intuitively, we mean that as the number $n$ of approximations increases without bound, the values $x_n$ get arbitrarily close to the desired root $r$. (This notion is similar to the idea of the limit of a function $g(t)$ as $t$ approaches infinity, as defined in Section 2.6.)

In practice, Newton's method usually gives convergence with impressive speed, but this is not guaranteed. One way to test convergence is to begin by graphing the function to estimate a good starting value for $x_0$. You can test that you are getting closer to a zero of the function by evaluating $|f(x_n)|$, and check that the approximations are converging by evaluating $|x_n - x_{n+1}|$.

*Newton's method does not always converge*. For instance, if

$$f(x) = \begin{cases} -\sqrt{r - x}, & x < r \\ \sqrt{x - r}, & x \geq r, \end{cases}$$

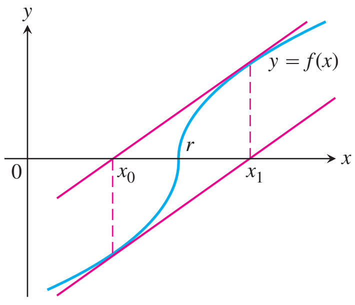



**FIGURE 4.49** Newton's method fails to converge. You go from $x_0$ to $x_1$ and back to $x_0$, never getting any closer to $r$.

the graph will be like the one in Figure 4.49. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

*If Newton's method does converge, it converges to a root*. Be careful, however. There are situations in which the method appears to converge but there is no root there. Fortunately, such situations are rare.

*When Newton's method converges to a root, it may not be the root you have in mind*. Figure 4.50 shows two ways this can happen.

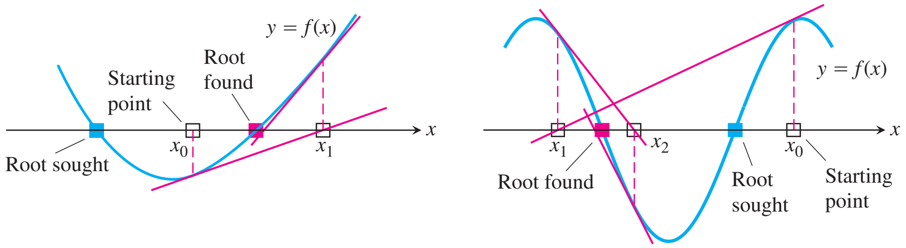



**FIGURE 4.50** If you start too far away, Newton's method may miss the root you want.

# Exercises 4.7

### Root Finding

1. Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find $x_2$.

2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find $x_2$.

3. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find $x_2$.

4. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find $x_2$.

5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find $x_2$.

6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find $x_2$.

7. **Guessing a root** Suppose that your first guess is lucky, in the sense that $x_0$ is a root of $f(x) = 0$. Assuming that $f'(x_0)$ is defined and not 0, what happens to $x_1$ and later approximations?

8. **Estimating pi** You plan to estimate $\pi/2$ to five decimal places by using Newton's method to solve the equation $\cos x = 0$. Does it matter what your starting value is? Give reasons for your answer.

### Theory and Examples

9. **Oscillation** Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_1 = -h$ if $x_0 = h$ and to $x_1 = h$ if $x_0 = -h$. Draw a picture that shows what is going on.

**10. Approximations that get worse and worse** Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$ and calculate $x_1, x_2, x_3$, and $x_4$. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

**11.** Explain why the following four statements ask for the same information:

**i)** Find the roots of $f(x) = x^3 - 3x - 1$.

**ii)** Find the $x$-coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.

**iii)** Find the $x$-coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = 1$.

**iv)** Find the values of $x$ where the derivative of $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$ equals zero.

**12. Locating a planet** To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5 \sin x$. Graphing the function $f(x) = x - 1 - 0.5 \sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's method to improve this estimate. That is, start with $x_0 = 1.5$ and find $x_1$. (The value of the root is 1.49870 to five decimal places.) Remember to use radians.

T **13. Intersecting curves** The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.

T **14. Real solutions of a quartic** Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

T **15. a.** How many solutions does the equation $\sin 3x = 0.99 - x^2$ have?

**b.** Use Newton's method to find them.

**16. Intersection of curves**

**a.** Does $\cos 3x$ ever equal $x$? Give reasons for your answer.

**b.** Use Newton's method to find where.

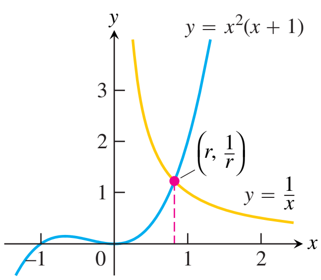
**17.** Find the four real zeros of the function $f(x) = 2x^4 - 4x^2 + 1$.

T **18. Estimating pi** Estimate $\pi$ to as many decimal places as your calculator will display by using Newton's method to solve the equation $\tan x = 0$ with $x_0 = 3$.

**19. Intersection of curves** At what value(s) of $x$ does $\cos x = 2x$?

**20. Intersection of curves** At what value(s) of $x$ does $\cos x = -x$?

**21.** The graphs of $y = x^2(x + 1)$ and $y = 1/x$ $(x > 0)$ intersect at one point $x = r$. Use Newton's method to estimate the value of $r$ to four decimal places.




**22.** The graphs of $y = \sqrt{x}$ and $y = 3 - x^2$ intersect at one point $x = r$. Use Newton's method to estimate the value of $r$ to four decimal places.

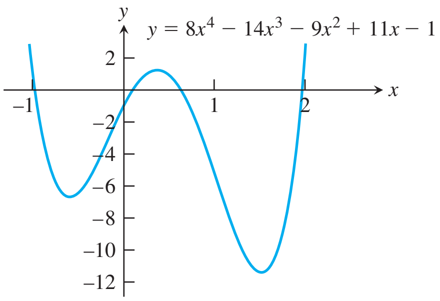
**23. Intersection of curves** At what value(s) of $x$ does $e^{-x^2} = x^2 - x + 1$?

**24. Intersection of curves** At what value(s) of $x$ does $\ln(1 - x^2) = x - 1$?

**25.** Use the Intermediate Value Theorem from Section 2.5 to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then find the root to five decimal places.

**26. Factoring a quartic** Find the approximate values of $r_1$ through $r_4$ in the factorization

$$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$




T **27. Converging to different zeros** Use Newton's method to find the zeros of $f(x) = 4x^4 - 4x^2$ using the given starting values.

**a.** $x_0 = -2$ and $x_0 = -0.8$, lying in $\left(-\infty, -\sqrt{2}/2\right)$

**b.** $x_0 = -0.5$ and $x_0 = 0.25$, lying in $\left(-\sqrt{21}/7, \sqrt{21}/7\right)$

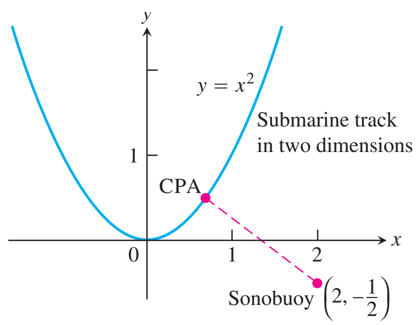
**c.** $x_0 = 0.8$ and $x_0 = 2$, lying in $\left(\sqrt{2}/2, \infty\right)$

**d.** $x_0 = -\sqrt{21}/7$ and $x_0 = \sqrt{21}/7$

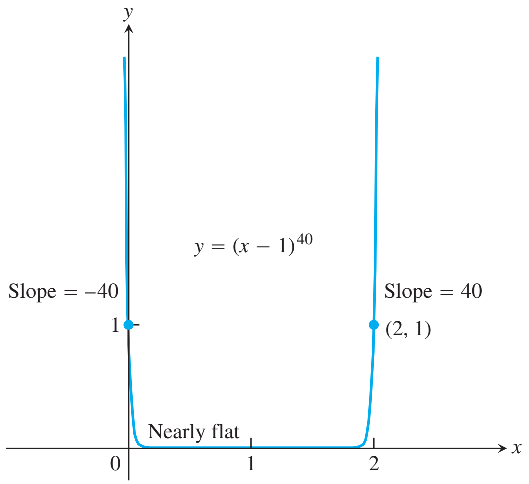
**28. The sonobuoy problem** In submarine location problems, it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on the parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.

**a.** Show that the value of $x$ that minimizes the distance between the submarine and the buoy is a solution of the equation $x = 1/(x^2 + 1)$.

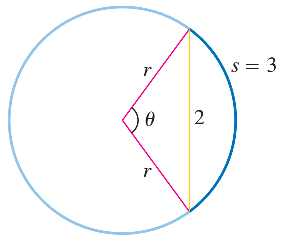
**b.** Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.




T **29. Curves that are nearly flat at the root** Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on $f(x) = (x - 1)^{40}$ with a starting value of $x_0 = 2$ to see how close your machine comes to the root $x = 1$. See the accompanying graph.




**30.** The accompanying figure shows a circle of radius r with a chord of length 2 and an arc $s$ of length 3. Use Newton's method to solve for $r$ and $\theta$ (radians) to four decimal places. Assume $0 < \theta < \pi$.




# 4.8 Antiderivatives

We have studied how to find the derivative of a function. However, many problems require that we recover a function from its known derivative (from its known rate of change). For instance, we may know the velocity function of an object falling from an initial height and need to know its height at any time. More generally, we want to find a function $F$ from its derivative $f$. If such a function $F$ exists, it is called an *antiderivative* of $f$. We will see in the next chapter that antiderivatives are the link connecting the two major elements of calculus: derivatives and definite integrals.

## Finding Antiderivatives

**DEFINITION** A function $F$ is an **antiderivative** of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as $F$ to represent an antiderivative of a function $f$, $G$ to represent an antiderivative of $g$, and so forth.

**EXAMPLE 1** Find an antiderivative for each of the following functions.

**(a)** $f(x) = 2x$ **(b)** $g(x) = \cos x$ **(c)** $h(x) = \dfrac{1}{x} + 2e^{2x}$

**Solution** We need to think backward here: What function do we know has a derivative equal to the given function?

**(a)** $F(x) = x^2$ **(b)** $G(x) = \sin x$ **(c)** $H(x) = \ln|x| + e^{2x}$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$ and the derivative of $H(x) = \ln|x| + e^{2x}$ is $(1/x) + 2e^{2x}$. ■

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant $C$. Are there others?

Corollary 2 of the Mean Value Theorem in Section 4.2 gives the answer: Any two antiderivatives of a function differ by a constant. So the functions $x^2 + C$, where $C$ is an **arbitrary constant**, form *all* the antiderivatives of $f(x) = 2x$. More generally, we have the following result.

**THEOREM 8** If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is

$$F(x) + C$$

where $C$ is an arbitrary constant.

Thus the most general antiderivative of $f$ on $I$ is a *family* of functions $F(x) + C$ whose graphs are vertical translations of one another. We can select a particular antiderivative from this family by assigning a specific value to $C$. Here is an example showing how such an assignment might be made.

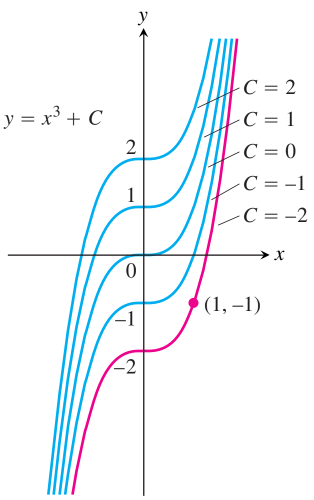



**FIGURE 4.51** The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

**EXAMPLE 2** Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

**Solution** Since the derivative of $x^3$ is $3x^2$, the general antiderivative

$$F(x) = x^3 + C$$

gives all the antiderivatives of $f(x)$. The condition $F(1) = -1$ determines a specific value for $C$. Substituting $x = 1$ into $F(x) = x^3 + C$ gives

$$F(1) = (1)^3 + C = 1 + C.$$

Since $F(1) = -1$, solving $1 + C = -1$ for $C$ gives $C = -2$. So

$$F(x) = x^3 - 2$$

is the antiderivative satisfying $F(1) = -1$. Notice that this assignment for $C$ selects the particular curve from the family of curves $y = x^3 + C$ that passes through the point $(1, -1)$ in the plane (Figure 4.51). ■

By working backward from assorted differentiation rules, we can derive formulas and rules for antiderivatives. In each case there is an arbitrary constant $C$ in the general expression representing all antiderivatives of a given function. Table 4.2 gives antiderivative formulas for a number of important functions.

The rules in Table 4.2 are easily verified by differentiating the general antiderivative formula to obtain the function to its left. For example, the derivative of $(\tan kx)/k + C$ is $\sec^2 kx$, whatever the value of the constants $C$ or $k \neq 0$, and this establishes Formula 4 for the most general antiderivative of $\sec^2 kx$.

**EXAMPLE 3** Find the general antiderivative of each of the following functions.

**(a)** $f(x) = x^5$ **(b)** $g(x) = \dfrac{1}{\sqrt{x}}$ **(c)** $h(x) = \sin 2x$

**(d)** $i(x) = \cos \dfrac{x}{2}$ **(e)** $j(x) = e^{-3x}$ **(f)** $k(x) = 2^x$

**TABLE 4.2** Antiderivative formulas, $k$ a nonzero constant

| | Function | General antiderivative | | Function | General antiderivative |
|---|---|---|---|---|---|
| **1.** | $x^n$ | $\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$ | **8.** | $e^{kx}$ | $\frac{1}{k}e^{kx} + C$ |
| **2.** | $\sin kx$ | $-\frac{1}{k}\cos kx + C$ | **9.** | $\frac{1}{x}$ | $\ln\lvert x\rvert + C, \quad x \neq 0$ |
| **3.** | $\cos kx$ | $\frac{1}{k}\sin kx + C$ | **10.** | $\frac{1}{\sqrt{1 - k^2x^2}}$ | $\frac{1}{k}\sin^{-1} kx + C$ |
| **4.** | $\sec^2 kx$ | $\frac{1}{k}\tan kx + C$ | **11.** | $\frac{1}{1 + k^2x^2}$ | $\frac{1}{k}\tan^{-1} kx + C$ |
| **5.** | $\csc^2 kx$ | $-\frac{1}{k}\cot kx + C$ | **12.** | $\frac{1}{x\sqrt{k^2x^2 - 1}}$ | $\sec^{-1} kx + C, \quad kx > 1$ |
| **6.** | $\sec kx \tan kx$ | $\frac{1}{k}\sec kx + C$ | **13.** | $a^{kx}$ | $\left(\frac{1}{k \ln a}\right)a^{kx} + C, \quad a > 0, \ a \neq 1$ |
| **7.** | $\csc kx \cot kx$ | $-\frac{1}{k}\csc kx + C$ | | | |

**Solution** In each case, we can use one of the formulas listed in Table 4.2.

**(a)** $F(x) = \frac{x^6}{6} + C$ — Formula 1 with $n = 5$

**(b)** $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$

Formula 1 with $n = -1/2$

**(c)** $H(x) = \frac{-\cos 2x}{2} + C$ — Formula 2 with $k = 2$

**(d)** $I(x) = \frac{\sin(x/2)}{1/2} + C = 2\sin\frac{x}{2} + C$ — Formula 3 with $k = 1/2$

**(e)** $J(x) = -\frac{1}{3}e^{-3x} + C$ — Formula 8 with $k = -3$

**(f)** $K(x) = \left(\frac{1}{\ln 2}\right)2^x + C$ — Formula 13 with $a = 2, k = 1$ ∎

Other derivative rules also lead to corresponding antiderivative rules. We can add and subtract antiderivatives and multiply them by constants.

**TABLE 4.3** Antiderivative linearity rules

| | | Function | General antiderivative |
|---|---|---|---|
| **1.** | *Constant Multiple Rule*: | $kf(x)$ | $kF(x) + C, \quad k$ a constant |
| **2.** | *Negative Rule*: | $-f(x)$ | $-F(x) + C$ |
| **3.** | *Sum or Difference Rule*: | $f(x) \pm g(x)$ | $F(x) \pm G(x) + C$ |

The formulas in Table 4.3 are easily proved by differentiating the antiderivatives and verifying that the result agrees with the original function. Formula 2 is the special case $k = -1$ in Formula 1.

**EXAMPLE 4** Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x.$$

**Solution** We have that $f(x) = 3g(x) + h(x)$ for the functions $g$ and $h$ in Example 3. Since $G(x) = 2\sqrt{x}$ is an antiderivative of $g(x)$ from Example 3b, it follows from the Constant Multiple Rule for antiderivatives that $3G(x) = 3 \cdot 2\sqrt{x} = 6\sqrt{x}$ is an antiderivative of $3g(x) = 3/\sqrt{x}$. Likewise, from Example 3c we know that $H(x) = (-1/2)\cos 2x$ is an antiderivative of $h(x) = \sin 2x$. From the Sum Rule for antiderivatives, we then get that

$$\begin{aligned} F(x) &= 3G(x) + H(x) + C \\ &= 6\sqrt{x} - \frac{1}{2}\cos 2x + C \end{aligned}$$

is the general antiderivative formula for $f(x)$, where $C$ is an arbitrary constant. ■

## Initial Value Problems and Differential Equations

Antiderivatives play several important roles in mathematics and its applications. Methods and techniques for finding them are a major part of calculus, and we take up that study in Chapter 8. Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x).$$

This is called a **differential equation**, since it is an equation involving an unknown function $y$ that is being differentiated. To solve it, we need a function $y(x)$ that satisfies the equation. This function is found by taking the antiderivative of $f(x)$. We fix the arbitrary constant arising in the antidifferentiation process by specifying an initial condition

$$y(x_0) = y_0.$$

This condition means the function $y(x)$ has the value $y_0$ when $x = x_0$. The combination of a differential equation and an initial condition is called an **initial value problem**. Such problems play important roles in all branches of science.

The most general antiderivative $F(x) + C$ (such as $x^3 + C$ in Example 2) of the function $f(x)$ gives the **general solution** $y = F(x) + C$ of the differential equation $dy/dx = f(x)$. The general solution gives *all* the solutions of the equation (there are infinitely many, one for each value of $C$). We **solve** the differential equation by finding its general solution. We then solve the initial value problem by finding the **particular solution** that satisfies the initial condition $y(x_0) = y_0$. In Example 2, the function $y = x^3 - 2$ is the particular solution of the differential equation $dy/dx = 3x^2$ satisfying the initial condition $y(1) = -1$.

## Antiderivatives and Motion

We have seen that the derivative of the position function of an object gives its velocity, and the derivative of its velocity function gives its acceleration. If we know an object's acceleration, then by finding an antiderivative we can recover the velocity, and from an antiderivative of the velocity we can recover its position function. This procedure was used as an application of Corollary 2 in Section 4.2. Now that we have a terminology and conceptual framework in terms of antiderivatives, we revisit the problem from the point of view of differential equations.

**EXAMPLE 5** A hot-air balloon ascending at the rate of 12 ft/sec is at a height 80 ft above the ground when a package is dropped. How long does it take the package to reach the ground?

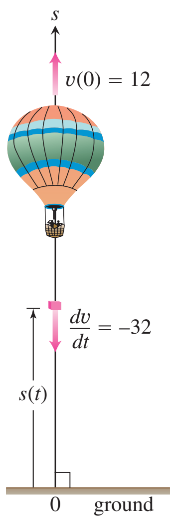



**FIGURE 4.52** A package dropped from a rising hot-air balloon (Example 5).

**Solution** Let $v(t)$ denote the velocity of the package at time $t$, and let $s(t)$ denote its height above the ground. The acceleration of gravity near the surface of the earth is 32 ft/sec$^2$. Assuming no other forces act on the dropped package, we have

$$\frac{dv}{dt} = -32. \qquad \text{Negative because gravity acts in the direction of decreasing } s$$

This leads to the following initial value problem (Figure 4.52):

$$\begin{aligned} &\textit{Differential equation:} && \frac{dv}{dt} = -32 \\ &\textit{Initial condition:} && v(0) = 12. && \text{Balloon initially rising} \end{aligned}$$

This is our mathematical model for the package's motion. We solve the initial value problem to obtain the velocity of the package.

1. *Solve the differential equation*: The general formula for an antiderivative of $-32$ is

$$v = -32t + C.$$

Having found the general solution of the differential equation, we use the initial condition to find the particular solution that solves our problem.

2. *Evaluate C*:

$$\begin{aligned} 12 &= -32(0) + C && \text{Initial condition } v(0) = 12 \\ C &= 12. \end{aligned}$$

The solution of the initial value problem is

$$v = -32t + 12.$$

Since velocity is the derivative of height, and the height of the package is 80 ft at time $t = 0$ when it is dropped, we now have a second initial value problem.

$$\begin{aligned} &\textit{Differential equation:} && \frac{ds}{dt} = -32t + 12 && \text{Set } v = ds/dt \text{ in the previous equation.} \\ &\textit{Initial condition:} && s(0) = 80 \end{aligned}$$

We solve this initial value problem to find the height as a function of $t$.

1. *Solve the differential equation*: Finding the general antiderivative of $-32t + 12$ gives

$$s = -16t^2 + 12t + C.$$

2. *Evaluate C*:

$$\begin{aligned} 80 &= -16(0)^2 + 12(0) + C && \text{Initial condition } s(0) = 80 \\ C &= 80. \end{aligned}$$

The package's height above ground at time $t$ is

$$s = -16t^2 + 12t + 80.$$

*Use the solution*: To find how long it takes the package to reach the ground, we set $s$ equal to 0 and solve for $t$:

$$\begin{aligned} -16t^2 + 12t + 80 &= 0 \\ -4t^2 + 3t + 20 &= 0 \\ t &= \frac{-3 \pm \sqrt{329}}{-8} && \text{Quadratic formula} \\ t \approx -1.89, &\qquad t \approx 2.64. \end{aligned}$$

The package hits the ground about 2.64 sec after it is dropped from the balloon. (The negative root has no physical meaning.) ■

## Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function $f$.

**DEFINITION** The collection of all antiderivatives of $f$ is called the **indefinite integral** of $f$ with respect to $x$, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function $f$ is the **integrand** of the integral, and $x$ is the **variable of integration**.

After the integral sign in the notation we just defined, the integrand function is always followed by a differential to indicate the variable of integration. We will have more to say about why this is important in Chapter 5. Using this notation, we restate the solutions of Example 1, as follows:

$$\int 2x\,dx = x^2 + C,$$

$$\int \cos x\,dx = \sin x + C,$$

$$\int \left(\frac{1}{x} + 2e^{2x}\right) dx = \ln|x| + e^{2x} + C.$$

This notation is related to the main application of antiderivatives, which will be explored in Chapter 5. Antiderivatives play a key role in computing limits of certain infinite sums, an unexpected and wonderfully useful role that is described in a central result of Chapter 5, called the Fundamental Theorem of Calculus.

**EXAMPLE 6** Evaluate

$$\int (x^2 - 2x + 5)\,dx.$$

**Solution** If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\,dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\ &= \int x^2\,dx - 2\int x\,dx + 5\int 1\,dx \\ &= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3) \\ &= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.\end{aligned}$$

This formula is more complicated than it needs to be. If we combine $C_1$, $-2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the possible antiderivatives there are. For this reason, we recommend that you go right to the final form even if you elect to integrate term-by-term. Write

$$\int (x^2 - 2x + 5)\, dx = \int x^2\, dx - \int 2x\, dx + \int 5\, dx$$
$$= \frac{x^3}{3} - x^2 + 5x + C.$$

Find the simplest antiderivative you can for each part and add the arbitrary constant of integration at the end. ■

## Exercises 4.8

### Finding Antiderivatives

In Exercises 1–24, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a. $2x$ b. $x^2$ c. $x^2 - 2x + 1$
2. a. $6x$ b. $x^7$ c. $x^7 - 6x + 8$
3. a. $-3x^{-4}$ b. $x^{-4}$ c. $x^{-4} + 2x + 3$
4. a. $2x^{-3}$ b. $\frac{x^{-3}}{2} + x^2$ c. $-x^{-3} + x - 1$
5. a. $\frac{1}{x^2}$ b. $\frac{5}{x^2}$ c. $2 - \frac{5}{x^2}$
6. a. $-\frac{2}{x^3}$ b. $\frac{1}{2x^3}$ c. $x^3 - \frac{1}{x^3}$
7. a. $\frac{3}{2}\sqrt{x}$ b. $\frac{1}{2\sqrt{x}}$ c. $\sqrt{x} + \frac{1}{\sqrt{x}}$
8. a. $\frac{4}{3}\sqrt[3]{x}$ b. $\frac{1}{3\sqrt[3]{x}}$ c. $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
9. a. $\frac{2}{3}x^{-1/3}$ b. $\frac{1}{3}x^{-2/3}$ c. $-\frac{1}{3}x^{-4/3}$
10. a. $\frac{1}{2}x^{-1/2}$ b. $-\frac{1}{2}x^{-3/2}$ c. $-\frac{3}{2}x^{-5/2}$
11. a. $\frac{1}{x}$ b. $\frac{7}{x}$ c. $1 - \frac{5}{x}$
12. a. $\frac{1}{3x}$ b. $\frac{2}{5x}$ c. $1 + \frac{4}{3x} - \frac{1}{x^2}$
13. a. $-\pi \sin \pi x$ b. $3 \sin x$ c. $\sin \pi x - 3 \sin 3x$
14. a. $\pi \cos \pi x$ b. $\frac{\pi}{2} \cos \frac{\pi x}{2}$ c. $\cos \frac{\pi x}{2} + \pi \cos x$
15. a. $\sec^2 x$ b. $\frac{2}{3} \sec^2 \frac{x}{3}$ c. $-\sec^2 \frac{3x}{2}$
16. a. $\csc^2 x$ b. $-\frac{3}{2} \csc^2 \frac{3x}{2}$ c. $1 - 8 \csc^2 2x$
17. a. $\csc x \cot x$ b. $-\csc 5x \cot 5x$ c. $-\pi \csc \frac{\pi x}{2} \cot \frac{\pi x}{2}$
18. a. $\sec x \tan x$ b. $4 \sec 3x \tan 3x$ c. $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$
19. a. $e^{3x}$ b. $e^{-x}$ c. $e^{x/2}$
20. a. $e^{-2x}$ b. $e^{4x/3}$ c. $e^{-x/5}$
21. a. $3^x$ b. $2^{-x}$ c. $\left(\frac{5}{3}\right)^x$
22. a. $x^{\sqrt{3}}$ b. $x^{\pi}$ c. $x^{\sqrt{2}-1}$
23. a. $\frac{2}{\sqrt{1 - x^2}}$ b. $\frac{1}{2(x^2 + 1)}$ c. $\frac{1}{1 + 4x^2}$
24. a. $x - \left(\frac{1}{2}\right)^x$ b. $x^2 + 2^x$ c. $\pi^x - x^{-1}$

### Finding Indefinite Integrals

In Exercises 25–70, find the most general antiderivative or indefinite integral. Check your answers by differentiation.

25. $\int (x + 1)\, dx$
26. $\int (5 - 6x)\, dx$
27. $\int \left(3t^2 + \frac{t}{2}\right) dt$
28. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$
29. $\int (2x^3 - 5x + 7)\, dx$
30. $\int (1 - x^2 - 3x^5)\, dx$
31. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$
32. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$
33. $\int x^{-1/3}\, dx$
34. $\int x^{-5/4}\, dx$
35. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$
36. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$
37. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$
38. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$
39. $\int 2x(1 - x^{-3})\, dx$
40. $\int x^{-3}(x + 1)\, dx$
41. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\, dt$
42. $\int \frac{4 + \sqrt{t}}{t^3}\, dt$

43. $\int (-2\cos t)\,dt$

44. $\int (-5\sin t)\,dt$

45. $\int 7\sin\frac{\theta}{3}\,d\theta$

46. $\int 3\cos 5\theta\,d\theta$

47. $\int (-3\csc^2 x)\,dx$

48. $\int \left(-\frac{\sec^2 x}{3}\right)dx$

49. $\int \frac{\csc\theta\cot\theta}{2}\,d\theta$

50. $\int \frac{2}{5}\sec\theta\tan\theta\,d\theta$

51. $\int (e^{3x} + 5e^{-x})\,dx$

52. $\int (2e^x - 3e^{-2x})\,dx$

53. $\int (e^{-x} + 4^x)\,dx$

54. $\int (1.3)^x\,dx$

55. $\int (4\sec x\tan x - 2\sec^2 x)\,dx$

56. $\int \frac{1}{2}(\csc^2 x - \csc x\cot x)\,dx$

57. $\int (\sin 2x - \csc^2 x)\,dx$

58. $\int (2\cos 2x - 3\sin 3x)\,dx$

59. $\int \frac{1 + \cos 4t}{2}\,dt$

60. $\int \frac{1 - \cos 6t}{2}\,dt$

61. $\int \left(\frac{1}{x} - \frac{5}{x^2 + 1}\right)dx$

62. $\int \left(\frac{2}{\sqrt{1 - y^2}} - \frac{1}{y^{1/4}}\right)dy$

63. $\int 3x^{\sqrt{3}}\,dx$

64. $\int x^{\sqrt{2}-1}\,dx$

65. $\int (1 + \tan^2\theta)\,d\theta$

(*Hint:* $1 + \tan^2\theta = \sec^2\theta$)

66. $\int (2 + \tan^2\theta)\,d\theta$

67. $\int \cot^2 x\,dx$

(*Hint:* $1 + \cot^2 x = \csc^2 x$)

68. $\int (1 - \cot^2 x)\,dx$

69. $\int \cos\theta\,(\tan\theta + \sec\theta)\,d\theta$

70. $\int \frac{\csc\theta}{\csc\theta - \sin\theta}\,d\theta$

## Checking Antiderivative Formulas

Verify the formulas in Exercises 71–82 by differentiation.

71. $\int (7x - 2)^3\,dx = \frac{(7x - 2)^4}{28} + C$

72. $\int (3x + 5)^{-2}\,dx = -\frac{(3x + 5)^{-1}}{3} + C$

73. $\int \sec^2(5x - 1)\,dx = \frac{1}{5}\tan(5x - 1) + C$

74. $\int \csc^2\left(\frac{x - 1}{3}\right)dx = -3\cot\left(\frac{x - 1}{3}\right) + C$

75. $\int \frac{1}{(x + 1)^2}\,dx = -\frac{1}{x + 1} + C$

76. $\int \frac{1}{(x + 1)^2}\,dx = \frac{x}{x + 1} + C$

77. $\int \frac{1}{x + 1}\,dx = \ln(x + 1) + C, \quad x > -1$

78. $\int xe^x\,dx = xe^x - e^x + C$

79. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

80. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

81. $\int \frac{\tan^{-1} x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1} x}{x} + C$

82. $\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1 - x^2}\,\sin^{-1} x + C$

83. Right, or wrong? Say which for each formula and give a brief reason for each answer.

**a.** $\int x\sin x\,dx = \frac{x^2}{2}\sin x + C$

**b.** $\int x\sin x\,dx = -x\cos x + C$

**c.** $\int x\sin x\,dx = -x\cos x + \sin x + C$

84. Right, or wrong? Say which for each formula and give a brief reason for each answer.

**a.** $\int \tan\theta\sec^2\theta\,d\theta = \frac{\sec^3\theta}{3} + C$

**b.** $\int \tan\theta\sec^2\theta\,d\theta = \frac{1}{2}\tan^2\theta + C$

**c.** $\int \tan\theta\sec^2\theta\,d\theta = \frac{1}{2}\sec^2\theta + C$

85. Right, or wrong? Say which for each formula and give a brief reason for each answer.

**a.** $\int (2x + 1)^2\,dx = \frac{(2x + 1)^3}{3} + C$

**b.** $\int 3(2x + 1)^2\,dx = (2x + 1)^3 + C$

**c.** $\int 6(2x + 1)^2\,dx = (2x + 1)^3 + C$

86. Right, or wrong? Say which for each formula and give a brief reason for each answer.

**a.** $\int \sqrt{2x + 1}\,dx = \sqrt{x^2 + x + C}$

**b.** $\int \sqrt{2x + 1}\,dx = \sqrt{x^2 + x} + C$

**c.** $\int \sqrt{2x + 1}\,dx = \frac{1}{3}\left(\sqrt{2x + 1}\right)^3 + C$

87. Right, or wrong? Give a brief reason why.

$$\int \frac{-15(x + 3)^2}{(x - 2)^4}\,dx = \left(\frac{x + 3}{x - 2}\right)^3 + C$$

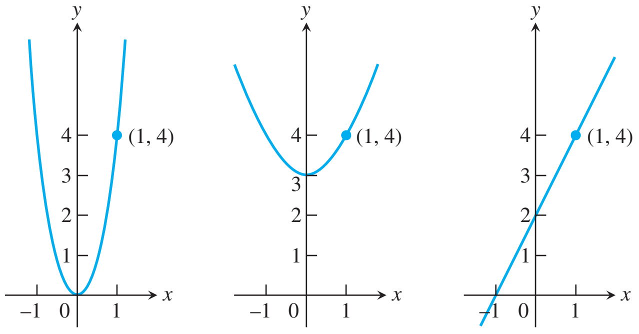
88. Right, or wrong? Give a brief reason why.

$$\int \frac{x\cos(x^2) - \sin(x^2)}{x^2}\,dx = \frac{\sin(x^2)}{x} + C$$

## Initial Value Problems

**89.** Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$

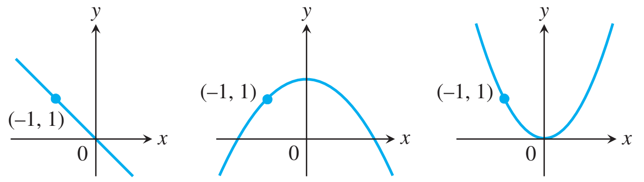


(a) (b) (c)

Give reasons for your answer.

**90.** Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1?$$



(a) (b) (c)

Give reasons for your answer.

Solve the initial value problems in Exercises 91–112.

**91.** $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

**92.** $\frac{dy}{dx} = 10 - x, \quad y(0) = -1$

**93.** $\frac{dy}{dx} = \frac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1$

**94.** $\frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$

**95.** $\frac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5$

**96.** $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad y(4) = 0$

**97.** $\frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

**98.** $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

**99.** $\frac{dr}{d\theta} = -\pi \sin \pi\theta, \quad r(0) = 0$

**100.** $\frac{dr}{d\theta} = \cos \pi\theta, \quad r(0) = 1$

**101.** $\frac{dv}{dt} = \frac{1}{2}\sec t \tan t, \quad v(0) = 1$

**102.** $\frac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\frac{\pi}{2}\right) = -7$

**103.** $\frac{dv}{dt} = \frac{3}{t\sqrt{t^2 - 1}}, \quad t > 1, v(2) = 0$

**104.** $\frac{dv}{dt} = \frac{8}{1 + t^2} + \sec^2 t, \quad v(0) = 1$

**105.** $\frac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1$

**106.** $\frac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0$

**107.** $\frac{d^2r}{dt^2} = \frac{2}{t^3}; \quad \left.\frac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$

**108.** $\frac{d^2s}{dt^2} = \frac{3t}{8}; \quad \left.\frac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$

**109.** $\frac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5$

**110.** $\frac{d^3\theta}{dt^3} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\frac{1}{2}, \quad \theta(0) = \sqrt{2}$

**111.** $y^{(4)} = -\sin t + \cos t;$

$y'''(0) = 7, \quad y''(0) = y'(0) = -1, \quad y(0) = 0$

**112.** $y^{(4)} = -\cos x + 8\sin 2x;$

$y'''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3$

**113.** Find the curve $y = f(x)$ in the $xy$-plane that passes through the point (9, 4) and whose slope at each point is $3\sqrt{x}$.

**114. a.** Find a curve $y = f(x)$ with the following properties:

**i)** $\frac{d^2y}{dx^2} = 6x$

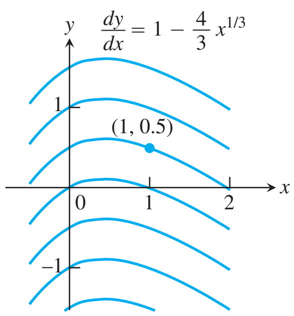
**ii)** Its graph passes through the point (0, 1), and has a horizontal tangent there.

**b.** How many curves like this are there? How do you know?

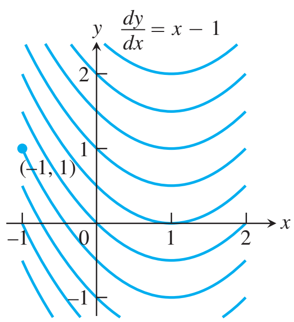
## Solution (Integral) Curves

Exercises 115–118 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

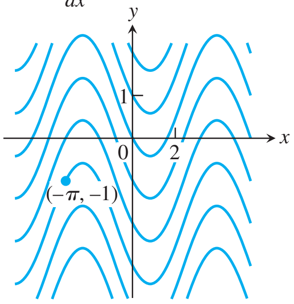
**115.**



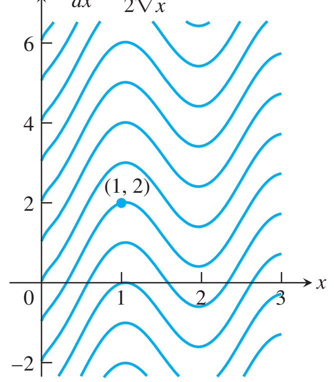

**116.**



**117.** $\dfrac{dy}{dx} = \sin x - \cos x$




**118.** $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}} + \pi \sin \pi x$




## Applications

**119. Finding displacement from an antiderivative of velocity**

**a.** Suppose that the velocity of a body moving along the $s$-axis is

$$\frac{ds}{dt} = v = 9.8t - 3.$$

**i)** Find the body's displacement over the time interval from $t = 1$ to $t = 3$ given that $s = 5$ when $t = 0$.

**ii)** Find the body's displacement from $t = 1$ to $t = 3$ given that $s = -2$ when $t = 0$.

**iii)** Now find the body's displacement from $t = 1$ to $t = 3$ given that $s = s_0$ when $t = 0$.

**b.** Suppose that the position $s$ of a body moving along a coordinate line is a differentiable function of time $t$. Is it true that once you know an antiderivative of the velocity function $ds/dt$ you can find the body's displacement from $t = a$ to $t = b$ even if you do not know the body's exact position at either of those times? Give reasons for your answer.

**120. Liftoff from Earth** A rocket lifts off the surface of Earth with a constant acceleration of $20 \text{ m/sec}^2$. How fast will the rocket be going 1 min later?

**121. Stopping a car in time** You are driving along a highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 ft? To find out, carry out the following steps.

**1.** Solve the initial value problem

Differential equation: $\dfrac{d^2s}{dt^2} = -k \quad (k \text{ constant})$

Initial conditions: $\dfrac{ds}{dt} = 88$ and $s = 0$ when $t = 0$.

Measuring time and distance from when the brakes are applied

**2.** Find the value of $t$ that makes $ds/dt = 0$. (The answer will involve $k$.)

**3.** Find the value of $k$ that makes $s = 242$ for the value of $t$ you found in Step 2.

**122. Stopping a motorcycle** The State of Illinois Cycle Rider Safety Program requires motorcycle riders to be able to brake from 30 mph (44 ft/sec) to 0 in 45 ft. What constant deceleration does it take to do that?

**123. Motion along a coordinate line** A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - \left(3/\sqrt{t}\right)$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find

**a.** the velocity $v = ds/dt$ in terms of $t$

**b.** the position $s$ in terms of $t$.

T **124. The hammer and the feather** When *Apollo 15* astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 4 ft above the ground. The television footage of the event shows the hammer and the feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 4 ft. How long did it take the hammer and feather to fall 4 ft on the moon? To find out, solve the following initial value problem for $s$ as a function of $t$. Then find the value of $t$ that makes $s$ equal to 0.

Differential equation: $\dfrac{d^2s}{dt^2} = -5.2 \text{ ft/sec}^2$

Initial conditions: $\dfrac{ds}{dt} = 0$ and $s = 4$ when $t = 0$

**125. Motion with constant acceleration** The standard equation for the position $s$ of a body moving with a constant acceleration $a$ along a coordinate line is

$$s = \frac{a}{2}t^2 + v_0 t + s_0, \tag{1}$$

where $v_0$ and $s_0$ are the body's velocity and position at time $t = 0$. Derive this equation by solving the initial value problem

Differential equation: $\dfrac{d^2s}{dt^2} = a$

Initial conditions: $\dfrac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$.

**126. Free fall near the surface of a planet** For free fall near the surface of a planet where the acceleration due to gravity has a constant magnitude of $g$ length-units/sec$^2$, Equation (1) in Exercise 125 takes the form

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \tag{2}$$

where $s$ is the body's height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing $s$. The velocity $v_0$ is positive if the object is rising at time $t = 0$ and negative if the object is falling.

Instead of using the result of Exercise 125, you can derive Equation (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

**127.** Suppose that

$$f(x) = \frac{d}{dx}\left(1 - \sqrt{x}\right) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2).$$

Find:

**a.** $\displaystyle\int f(x)\,dx$ **b.** $\displaystyle\int g(x)\,dx$

**c.** $\int [-f(x)]\,dx$ **d.** $\int [-g(x)]\,dx$

**e.** $\int [f(x) + g(x)]\,dx$ **f.** $\int [f(x) - g(x)]\,dx$

**128. Uniqueness of solutions** If differentiable functions $y = F(x)$ and $y = G(x)$ both solve the initial value problem

$$\frac{dy}{dx} = f(x), \qquad y(x_0) = y_0,$$

on an interval $I$, must $F(x) = G(x)$ for every $x$ in $I$? Give reasons for your answer.

### COMPUTER EXPLORATIONS

Use a CAS to solve the initial value problems in Exercises 129–132. Plot the solution curves.

**129.** $y' = \cos^2 x + \sin x, \quad y(\pi) = 1$

**130.** $y' = \frac{1}{x} + x, \quad y(1) = -1$

**131.** $y' = \frac{1}{\sqrt{4 - x^2}}, \quad y(0) = 2$

**132.** $y'' = \frac{2}{x} + \sqrt{x}, \quad y(1) = 0, \quad y'(1) = 0$

## Chapter 4 Questions to Guide Your Review

1. What can be said about the extreme values of a function that is continuous on a closed interval?
2. What does it mean for a function to have a local extreme value on its domain? An absolute extreme value? How are local and absolute extreme values related, if at all? Give examples.
3. How do you find the absolute extrema of a continuous function on a closed interval? Give examples.
4. What are the hypotheses and conclusion of Rolle's Theorem? Are the hypotheses really necessary? Explain.
5. What are the hypotheses and conclusion of the Mean Value Theorem? What physical interpretations might the theorem have?
6. State the Mean Value Theorem's three corollaries.
7. How can you sometimes identify a function $f(x)$ by knowing $f'$ and knowing the value of $f$ at a point $x = x_0$? Give an example.
8. What is the First Derivative Test for Local Extreme Values? Give examples of how it is applied.
9. How do you test a twice-differentiable function to determine where its graph is concave up or concave down? Give examples.
10. What is an inflection point? Give an example. What physical significance do inflection points sometimes have?
11. What is the Second Derivative Test for Local Extreme Values? Give examples of how it is applied.
12. What do the derivatives of a function tell you about the shape of its graph?
13. List the steps you would take to graph a polynomial function. Illustrate with an example.
14. What is a cusp? Give examples.
15. List the steps you would take to graph a rational function. Illustrate with an example.
16. Outline a general strategy for solving max-min problems. Give examples.
17. Describe l'Hôpital's Rule. How do you know when to use the rule and when to stop? Give an example.
18. How can you sometimes handle limits that lead to indeterminate forms $\infty/\infty$, $\infty \cdot 0$, and $\infty - \infty$? Give examples.
19. How can you sometimes handle limits that lead to indeterminate forms $1^\infty$, $0^0$, and $\infty^\infty$? Give examples.
20. Describe Newton's method for solving equations. Give an example. What is the theory behind the method? What are some of the things to watch out for when you use the method?
21. Can a function have more than one antiderivative? If so, how are the antiderivatives related? Explain.
22. What is an indefinite integral? How do you evaluate one? What general formulas do you know for finding indefinite integrals?
23. How can you sometimes solve a differential equation of the form $dy/dx = f(x)$?
24. What is an initial value problem? How do you solve one? Give an example.
25. If you know the acceleration of a body moving along a coordinate line as a function of time, what more do you need to know to find the body's position function? Give an example.

## Chapter 4 Practice Exercises

### Extreme Values

1. Does $f(x) = x^3 + 2x + \tan x$ have any local maximum or minimum values? Give reasons for your answer.
2. Does $g(x) = \csc x + 2\cot x$ have any local maximum values? Give reasons for your answer.
3. Does $f(x) = (7 + x)(11 - 3x)^{1/3}$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of $f$.

**4.** Find values of $a$ and $b$ such that the function

$$f(x) = \frac{ax + b}{x^2 - 1}$$

has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum, or a local minimum? Give reasons for your answer.

**5.** Does $g(x) = e^x - x$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of $g$.

**6.** Does $f(x) = 2e^x/(1 + x^2)$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of $f$.

In Exercises 7 and 8, find the absolute maximum and absolute minimum values of $f$ over the interval.

**7.** $f(x) = x - 2\ln x, \quad 1 \le x \le 3$

**8.** $f(x) = (4/x) + \ln x^2, \quad 1 \le x \le 4$

**9.** The greatest integer function $f(x) = \lfloor x \rfloor$, defined for all values of $x$, assumes a local maximum value of 0 at each point of [0, 1). Could any of these local maximum values also be local minimum values of $f$? Give reasons for your answer.

**10.** **a.** Give an example of a differentiable function $f$ whose first derivative is zero at some point $c$ even though $f$ has neither a local maximum nor a local minimum at $c$.

**b.** How is this consistent with Theorem 2 in Section 4.1? Give reasons for your answer.

**11.** The function $y = 1/x$ does not take on either a maximum or a minimum on the interval $0 < x < 1$ even though the function is continuous on this interval. Does this contradict the Extreme Value Theorem for continuous functions? Why?

**12.** What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \le x < 1$? Notice that the interval is not closed. Is this consistent with the Extreme Value Theorem for continuous functions? Why?

T **13.** A graph that is large enough to show a function's global behavior may fail to reveal important local features. The graph of $f(x) = (x^8/8) - (x^6/2) - x^5 + 5x^3$ is a case in point.

**a.** Graph $f$ over the interval $-2.5 \le x \le 2.5$. Where does the graph appear to have local extreme values or points of inflection?

**b.** Now factor $f'(x)$ and show that $f$ has a local maximum at $x = \sqrt[3]{5} \approx 1.70998$ and local minima at $x = \pm\sqrt{3} \approx \pm 1.73205$.

**c.** Zoom in on the graph to find a viewing window that shows the presence of the extreme values at $x = \sqrt[3]{5}$ and $x = \sqrt{3}$.

The moral here is that without calculus the existence of two of the three extreme values would probably have gone unnoticed. On any normal graph of the function, the values would lie close enough together to fall within the dimensions of a single pixel on the screen.

(*Source*: *Uses of Technology in the Mathematics Curriculum*, by Benny Evans and Jerry Johnson, Oklahoma State University, published in 1990 under National Science Foundation Grant USE-8950044.)

T **14.** (*Continuation of Exercise 13.*)

**a.** Graph $f(x) = (x^8/8) - (2/5)x^5 - 5x - (5/x^2) + 11$ over the interval $-2 \le x \le 2$. Where does the graph appear to have local extreme values or points of inflection?

**b.** Show that $f$ has a local maximum value at $x = \sqrt[7]{5} \approx 1.2585$ and a local minimum value at $x = \sqrt[3]{2} \approx 1.2599$.

**c.** Zoom in to find a viewing window that shows the presence of the extreme values at $x = \sqrt[7]{5}$ and $x = \sqrt[3]{2}$.

## The Mean Value Theorem

**15.** **a.** Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.

**b.** How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer.

**16.** **a.** Show that $y = \tan\theta$ increases on every interval in its domain.

**b.** If the conclusion in part (a) is really correct, how do you explain the fact that $\tan\pi = 0$ is less than $\tan(\pi/4) = 1$?

**17.** **a.** Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on [0, 1].

T **b.** Find the solution to as many decimal places as you can.

**18.** **a.** Show that $f(x) = x/(x + 1)$ increases on every interval in its domain.

**b.** Show that $f(x) = x^3 + 2x$ has no local maximum or minimum values.

**19.** **Water in a reservoir** As a result of a heavy rain, the volume of water in a reservoir increased by 1400 acre-ft in 24 hours. Show that at some instant during that period the reservoir's volume was increasing at a rate in excess of 225,000 gal/min. (An acre-foot is 43,560 ft$^3$, the volume that would cover 1 acre to the depth of 1 ft. A cubic foot holds 7.48 gal.)

**20.** The formula $F(x) = 3x + C$ gives a different function for each value of $C$. All of these functions, however, have the same derivative with respect to $x$, namely $F'(x) = 3$. Are these the only differentiable functions whose derivative is 3? Could there be any others? Give reasons for your answers.

**21.** Show that

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{d}{dx}\left(-\frac{1}{x+1}\right)$$

even though

$$\frac{x}{x+1} \ne -\frac{1}{x+1}.$$

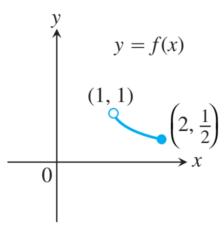
Doesn't this contradict Corollary 2 of the Mean Value Theorem? Give reasons for your answer.

**22.** Calculate the first derivatives of $f(x) = x^2/(x^2 + 1)$ and $g(x) = -1/(x^2 + 1)$. What can you conclude about the graphs of these functions?

## Analyzing Graphs

In Exercises 23 and 24, use the graph to answer the questions.

**23.** Identify any global extreme values of $f$ and the values of $x$ at which they occur.

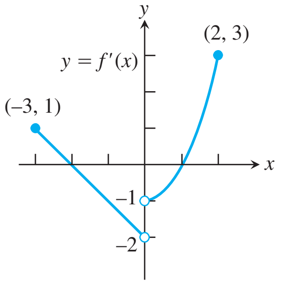


**24.** Estimate the intervals on which the function $y = f(x)$ is

**a.** increasing.

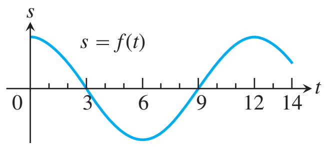
**b.** decreasing.

**c.** Use the given graph of $f'$ to indicate where any local extreme values of the function occur, and whether each extreme is a relative maximum or minimum.

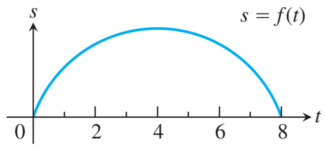



Each of the graphs in Exercises 25 and 26 is the graph of the position function $s = f(t)$ of an object moving on a coordinate line ($t$ represents time). At approximately what times (if any) is each object's **(a)** velocity equal to zero? **(b)** acceleration equal to zero? During approximately what time intervals does the object move **(c)** forward? **(d)** backward?

**25.**




**26.**




## Graphs and Graphing

Graph the curves in Exercises 27–42.

**27.** $y = x^2 - (x^3/6)$

**28.** $y = x^3 - 3x^2 + 3$

**29.** $y = -x^3 + 6x^2 - 9x + 3$

**30.** $y = (1/8)(x^3 + 3x^2 - 9x - 27)$

**31.** $y = x^3(8 - x)$

**32.** $y = x^2(2x^2 - 9)$

**33.** $y = x - 3x^{2/3}$

**34.** $y = x^{1/3}(x - 4)$

**35.** $y = x\sqrt{3 - x}$

**36.** $y = x\sqrt{4 - x^2}$

**37.** $y = (x - 3)^2 e^x$

**38.** $y = xe^{-x^2}$

**39.** $y = \ln(x^2 - 4x + 3)$

**40.** $y = \ln(\sin x)$

**41.** $y = \sin^{-1}\left(\frac{1}{x}\right)$

**42.** $y = \tan^{-1}\left(\frac{1}{x}\right)$

Each of Exercises 43–48 gives the first derivative of a function $y = f(x)$. **(a)** At what points, if any, does the graph of $f$ have a local maximum, local minimum, or inflection point? **(b)** Sketch the general shape of the graph.

**43.** $y' = 16 - x^2$

**44.** $y' = x^2 - x - 6$

**45.** $y' = 6x(x + 1)(x - 2)$

**46.** $y' = x^2(6 - 4x)$

**47.** $y' = x^4 - 2x^2$

**48.** $y' = 4x^2 - x^4$

In Exercises 49–52, graph each function. Then use the function's first derivative to explain what you see.

**49.** $y = x^{2/3} + (x - 1)^{1/3}$

**50.** $y = x^{2/3} + (x - 1)^{2/3}$

**51.** $y = x^{1/3} + (x - 1)^{1/3}$

**52.** $y = x^{2/3} - (x - 1)^{1/3}$

Sketch the graphs of the rational functions in Exercises 53–60.

**53.** $y = \dfrac{x + 1}{x - 3}$

**54.** $y = \dfrac{2x}{x + 5}$

**55.** $y = \dfrac{x^2 + 1}{x}$

**56.** $y = \dfrac{x^2 - x + 1}{x}$

**57.** $y = \dfrac{x^3 + 2}{2x}$

**58.** $y = \dfrac{x^4 - 1}{x^2}$

**59.** $y = \dfrac{x^2 - 4}{x^2 - 3}$

**60.** $y = \dfrac{x^2}{x^2 - 4}$

## Using L'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 61–72.

**61.** $\lim_{x \to 1} \dfrac{x^2 + 3x - 4}{x - 1}$

**62.** $\lim_{x \to 1} \dfrac{x^a - 1}{x^b - 1}$

**63.** $\lim_{x \to \pi} \dfrac{\tan x}{x}$

**64.** $\lim_{x \to 0} \dfrac{\tan x}{x + \sin x}$

**65.** $\lim_{x \to 0} \dfrac{\sin^2 x}{\tan(x^2)}$

**66.** $\lim_{x \to 0} \dfrac{\sin mx}{\sin nx}$

**67.** $\lim_{x \to \pi/2^-} \sec 7x \cos 3x$

**68.** $\lim_{x \to 0^+} \sqrt{x} \sec x$

**69.** $\lim_{x \to 0} (\csc x - \cot x)$

**70.** $\lim_{x \to 0} \left(\dfrac{1}{x^4} - \dfrac{1}{x^2}\right)$

**71.** $\lim_{x \to \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}\right)$

**72.** $\lim_{x \to \infty} \left(\dfrac{x^3}{x^2 - 1} - \dfrac{x^3}{x^2 + 1}\right)$

Find the limits in Exercises 73–84.

**73.** $\lim_{x \to 0} \dfrac{10^x - 1}{x}$

**74.** $\lim_{\theta \to 0} \dfrac{3^\theta - 1}{\theta}$

**75.** $\lim_{x \to 0} \dfrac{2^{\sin x} - 1}{e^x - 1}$

**76.** $\lim_{x \to 0} \dfrac{2^{-\sin x} - 1}{e^x - 1}$

**77.** $\lim_{x \to 0} \dfrac{5 - 5\cos x}{e^x - x - 1}$

**78.** $\lim_{x \to 0} \dfrac{4 - 4e^x}{xe^x}$

**79.** $\lim_{t \to 0^+} \dfrac{t - \ln(1 + 2t)}{t^2}$

**80.** $\lim_{x \to 4} \dfrac{\sin^2(\pi x)}{e^{x-4} + 3 - x}$

**81.** $\lim_{t \to 0^+} \left(\dfrac{e^t}{t} - \dfrac{1}{t}\right)$

**82.** $\lim_{y \to 0^+} e^{-1/y} \ln y$

**83.** $\lim_{x \to \infty} \left(1 + \dfrac{b}{x}\right)^{kx}$

**84.** $\lim_{x \to \infty} \left(1 + \dfrac{2}{x} + \dfrac{7}{x^2}\right)$

## Optimization

**85.** The sum of two nonnegative numbers is 36. Find the numbers if

**a.** the difference of their square roots is to be as large as possible.

**b.** the sum of their square roots is to be as large as possible.

**86.** The sum of two nonnegative numbers is 20. Find the numbers

**a.** if the product of one number and the square root of the other is to be as large as possible.

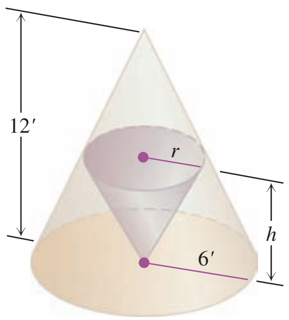
**b.** if one number plus the square root of the other is to be as large as possible.

**87.** An isosceles triangle has its vertex at the origin and its base parallel to the $x$-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

**88.** A customer has asked you to design an open-top rectangular stainless steel vat. It is to have a square base and a volume of 32 $\text{ft}^3$, to be welded from quarter-inch plate, and to weigh no more than necessary. What dimensions do you recommend?

**89.** Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$.

**90.** The figure here shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of $r$ and $h$ will give the smaller cone the largest possible volume?




**91. Manufacturing tires** Your company can manufacture $x$ hundred grade A tires and $y$ hundred grade B tires a day, where $0 \le x \le 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

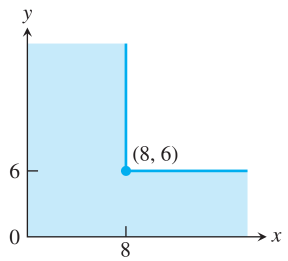
**92. Particle motion** The positions of two particles on the $s$-axis are $s_1 = \cos t$ and $s_2 = \cos(t + \pi/4)$.

**a.** What is the farthest apart the particles ever get?

**b.** When do the particles collide?

T **93. Open-top box** An open-top rectangular box is constructed from a 10-in.-by-16-in. piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume. Support your answers graphically.

**94. The ladder problem** What is the approximate length (in feet) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest foot.




## Newton's Method

**95.** Let $f(x) = 3x - x^3$. Show that the equation $f(x) = -4$ has a solution in the interval [2, 3] and use Newton's method to find it.

**96.** Let $f(x) = x^4 - x^3$. Show that the equation $f(x) = 75$ has a solution in the interval [3, 4] and use Newton's method to find it.

## Finding Indefinite Integrals

Find the indefinite integrals (most general antiderivatives) in Exercises 97–120. Check your answers by differentiation.

**97.** $\int (x^3 + 5x - 7)\,dx$

**98.** $\int \left(8t^3 - \frac{t^2}{2} + t\right) dt$

**99.** $\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt$

**100.** $\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt$

**101.** $\int \frac{dr}{(r + 5)^2}$

**102.** $\int \frac{6\,dr}{\left(r - \sqrt{2}\right)^3}$

**103.** $\int 3\theta\sqrt{\theta^2 + 1}\,d\theta$

**104.** $\int \frac{\theta}{\sqrt{7 + \theta^2}}\,d\theta$

**105.** $\int x^3(1 + x^4)^{-1/4}\,dx$

**106.** $\int (2 - x)^{3/5}\,dx$

**107.** $\int \sec^2 \frac{s}{10}\,ds$

**108.** $\int \csc^2 \pi s\,ds$

**109.** $\int \csc \sqrt{2}\theta \cot \sqrt{2}\theta\,d\theta$

**110.** $\int \sec \frac{\theta}{3} \tan \frac{\theta}{3}\,d\theta$

**111.** $\int \sin^2 \frac{x}{4}\,dx \quad \left(\textit{Hint}: \sin^2\theta = \frac{1 - \cos 2\theta}{2}\right)$

**112.** $\int \cos^2 \frac{x}{2}\,dx$

**113.** $\int \left(\frac{3}{x} - x\right) dx$

**114.** $\int \left(\frac{5}{x^2} + \frac{2}{x^2 + 1}\right) dx$

**115.** $\int \left(\frac{1}{2}e^t - e^{-t}\right) dt$

**116.** $\int (5^s + s^5)\,ds$

**117.** $\int \theta^{1-\pi}\,d\theta$

**118.** $\int 2^{\pi + r}\,dr$

**119.** $\int \frac{3}{2x\sqrt{x^2 - 1}}\,dx$

**120.** $\int \frac{d\theta}{\sqrt{16 - \theta^2}}$

## Initial Value Problems

Solve the initial value problems in Exercises 121–124.

**121.** $\frac{dy}{dx} = \frac{x^2 + 1}{x^2}, \quad y(1) = -1$

**122.** $\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^2, \quad y(1) = 1$

**123.** $\frac{d^2r}{dt^2} = 15\sqrt{t} + \frac{3}{\sqrt{t}}; \quad r'(1) = 8, \quad r(1) = 0$

**124.** $\frac{d^3r}{dt^3} = -\cos t; \quad r''(0) = r'(0) = 0, \quad r(0) = -1$

## Applications and Examples

**125.** Can the integrations in (a) and (b) both be correct? Explain.

**a.** $\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$

**b.** $\int \frac{dx}{\sqrt{1 - x^2}} = -\int -\frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1} x + C$

**126.** Can the integrations in (a) and (b) both be correct? Explain.

**a.** $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = -\int -\frac{dx}{\sqrt{1-x^2}} = -\cos^{-1}x + C$

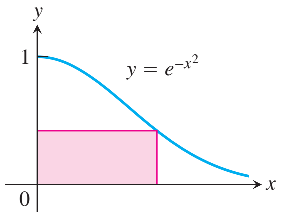
**b.** $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{-du}{\sqrt{1-(-u)^2}}$ $\quad x = -u$, $dx = -du$

$\displaystyle = \int \frac{-du}{\sqrt{1-u^2}}$

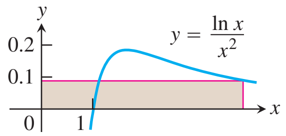
$= \cos^{-1} u + C$

$= \cos^{-1}(-x) + C$ $\quad u = -x$

**127.** The rectangle shown here has one side on the positive $y$-axis, one side on the positive $x$-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What dimensions give the rectangle its largest area, and what is that area?




**128.** The rectangle shown here has one side on the positive $y$-axis, one side on the positive $x$-axis, and its upper right-hand vertex on the curve $y = (\ln x)/x^2$. What dimensions give the rectangle its largest area, and what is that area?




In Exercises 129 and 130, find the absolute maximum and minimum values of each function on the given interval.

**129.** $y = x\ln 2x - x, \quad \left[\frac{1}{2e}, \frac{e}{2}\right]$

**130.** $y = 10x(2 - \ln x), \quad (0, e^2]$

In Exercises 131 and 132, find the absolute maxima and minima of the functions and say where they are assumed.

**131.** $f(x) = e^{x/\sqrt{x^4+1}}$

**132.** $g(x) = e^{\sqrt{3-2x-x^2}}$

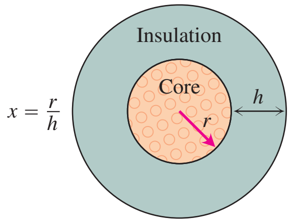
**T 133.** Graph the following functions and use what you see to locate and estimate the extreme values, identify the coordinates of the inflection points, and identify the intervals on which the graphs are concave up and concave down. Then confirm your estimates by working with the functions' derivatives.

**a.** $y = (\ln x)/\sqrt{x}$ **b.** $y = e^{-x^2}$ **c.** $y = (1 + x)e^{-x}$

**T 134.** Graph $f(x) = x\ln x$. Does the function appear to have an absolute minimum value? Confirm your answer with calculus.

**T 135.** Graph $f(x) = (\sin x)^{\sin x}$ over $[0, 3\pi]$. Explain what you see.

**136.** A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If $x$ denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $v = x^2 \ln(1/x)$. If the radius of the core is 1 cm, what insulation thickness $h$ will allow the greatest transmission speed?




# Chapter 4 Additional and Advanced Exercises

## Functions and Derivatives

**1.** What can you say about a function whose maximum and minimum values on an interval are equal? Give reasons for your answer.

**2.** Is it true that a discontinuous function cannot have both an absolute maximum and an absolute minimum value on a closed interval? Give reasons for your answer.

**3.** Can you conclude anything about the extreme values of a continuous function on an open interval? On a half-open interval? Give reasons for your answer.

**4. Local extrema** Use the sign pattern for the derivative

$$\frac{df}{dx} = 6(x-1)(x-2)^2(x-3)^3(x-4)^4$$

to identify the points where $f$ has local maximum and minimum values.

**5. Local extrema**

**a.** Suppose that the first derivative of $y = f(x)$ is

$$y' = 6(x+1)(x-2)^2.$$

At what points, if any, does the graph of $f$ have a local maximum, local minimum, or point of inflection?

**b.** Suppose that the first derivative of $y = f(x)$ is

$$y' = 6x(x+1)(x-2).$$

At what points, if any, does the graph of $f$ have a local maximum, local minimum, or point of inflection?

**6.** If $f'(x) \le 2$ for all $x$, what is the most the values of $f$ can increase on $[0, 6]$? Give reasons for your answer.

**7. Bounding a function** Suppose that $f$ is continuous on $[a, b]$ and that $c$ is an interior point of the interval. Show that if $f'(x) \le 0$ on $[a, c)$ and $f'(x) \ge 0$ on $(c, b]$, then $f(x)$ is never less than $f(c)$ on $[a, b]$.

8. **An inequality**

   a. Show that $-1/2 \le x/(1 + x^2) \le 1/2$ for every value of $x$.

   b. Suppose that $f$ is a function whose derivative is $f'(x) = x/(1 + x^2)$. Use the result in part (a) to show that

$$\left|f(b) - f(a)\right| \le \frac{1}{2}\left|b - a\right|$$

   for any $a$ and $b$.

9. The derivative of $f(x) = x^2$ is zero at $x = 0$, but $f$ is not a constant function. Doesn't this contradict the corollary of the Mean Value Theorem that says that functions with zero derivatives are constant? Give reasons for your answer.

10. **Extrema and inflection points** Let $h = fg$ be the product of two differentiable functions of $x$.

   a. If $f$ and $g$ are positive, with local maxima at $x = a$, and if $f'$ and $g'$ change sign at $a$, does $h$ have a local maximum at $a$?

   b. If the graphs of $f$ and $g$ have inflection points at $x = a$, does the graph of $h$ have an inflection point at $a$?

In either case, if the answer is yes, give a proof. If the answer is no, give a counterexample.

11. **Finding a function** Use the following information to find the values of $a$, $b$, and $c$ in the formula $f(x) = (x + a)/(bx^2 + cx + 2)$.

   i) The values of $a$, $b$, and $c$ are either 0 or 1.

   ii) The graph of $f$ passes through the point $(-1, 0)$.

   iii) The line $y = 1$ is an asymptote of the graph of $f$.

12. **Horizontal tangent** For what value or values of the constant $k$ will the curve $y = x^3 + kx^2 + 3x - 4$ have exactly one horizontal tangent?

## Optimization

13. **Largest inscribed triangle** Points $A$ and $B$ lie at the ends of a diameter of a unit circle and point $C$ lies on the circumference. Is it true that the area of triangle $ABC$ is largest when the triangle is isosceles? How do you know?

14. **Proving the second derivative test** The Second Derivative Test for Local Maxima and Minima (Section 4.4) says:

   a. $f$ has a local maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$

   b. $f$ has a local minimum value at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.

   To prove statement (a), let $\epsilon = (1/2)\left|f''(c)\right|$. Then use the fact that

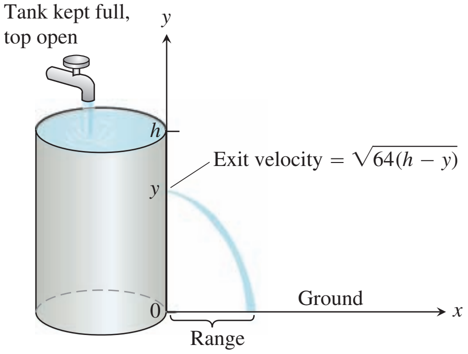
$$f''(c) = \lim_{h \to 0} \frac{f'(c + h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c + h)}{h}$$

   to conclude that for some $\delta > 0$,

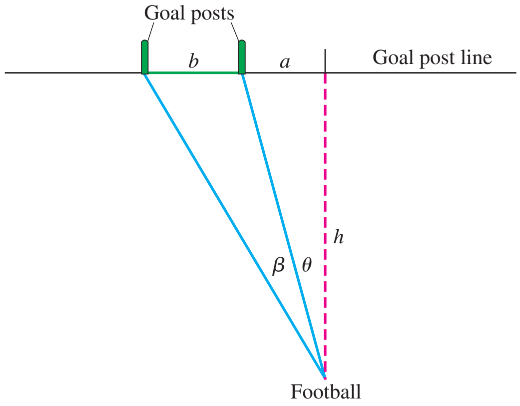
$$0 < |h| < \delta \quad \Rightarrow \quad \frac{f'(c + h)}{h} < f''(c) + \epsilon < 0.$$

   Thus, $f'(c + h)$ is positive for $-\delta < h < 0$ and negative for $0 < h < \delta$. Prove statement (b) in a similar way.

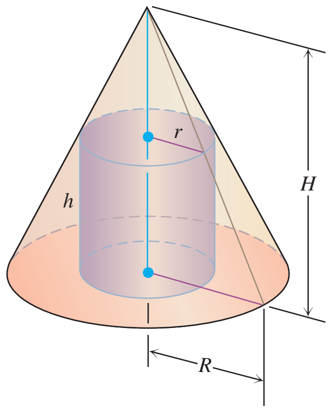
15. **Hole in a water tank** You want to bore a hole in the side of the tank shown here at a height that will make the stream of water coming out hit the ground as far from the tank as possible. If you drill the hole near the top, where the pressure is low, the water will exit slowly but spend a relatively long time in the air. If you drill the hole near the bottom, the water will exit at a higher velocity but have only a short time to fall. Where is the best place, if any, for the hole? (*Hint:* How long will it take an exiting particle of water to fall from height $y$ to the ground?)




16. **Kicking a field goal** An American football player wants to kick a field goal with the ball being on a right hash mark. Assume that the goal posts are $b$ feet apart and that the hash mark line is a distance $a > 0$ feet from the right goal post. (See the accompanying figure.) Find the distance $h$ from the goal post line that gives the kicker his largest angle $\beta$. Assume that the football field is flat.




17. **A max-min problem with a variable answer** Sometimes the solution of a max-min problem depends on the proportions of the shapes involved. As a case in point, suppose that a right circular cylinder of radius $r$ and height $h$ is inscribed in a right circular cone of radius $R$ and height $H$, as shown here. Find the value of $r$ (in terms of $R$ and $H$) that maximizes the total surface area of the cylinder (including top and bottom). As you will see, the solution depends on whether $H \le 2R$ or $H > 2R$.



**18. Minimizing a parameter** Find the smallest value of the positive constant $m$ that will make $mx - 1 + (1/x)$ greater than or equal to zero for all positive values of $x$.

## Limits

**19.** Evaluate the following limits.

**a.** $\lim_{x\to 0} \frac{2\sin 5x}{3x}$ **b.** $\lim_{x\to 0} \sin 5x \cot 3x$

**c.** $\lim_{x\to 0} x\csc^2\sqrt{2x}$ **d.** $\lim_{x\to\pi/2} (\sec x - \tan x)$

**e.** $\lim_{x\to 0} \frac{x - \sin x}{x - \tan x}$ **f.** $\lim_{x\to 0} \frac{\sin x^2}{x\sin x}$

**g.** $\lim_{x\to 0} \frac{\sec x - 1}{x^2}$ **h.** $\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4}$

**20.** L'Hôpital's Rule does not help with the following limits. Find them some other way.

**a.** $\lim_{x\to\infty} \frac{\sqrt{x+5}}{\sqrt{x}+5}$ **b.** $\lim_{x\to\infty} \frac{2x}{x + 7\sqrt{x}}$

## Theory and Examples

**21.** Suppose that it costs a company $y = a + bx$ dollars to produce $x$ units per week. It can sell $x$ units per week at a price of $P = c - ex$ dollars per unit. Each of $a$, $b$, $c$, and $e$ represents a positive constant. **(a)** What production level maximizes the profit? **(b)** What is the corresponding price? **(c)** What is the weekly profit at this level of production? **(d)** At what price should each item be sold to maximize profits if the government imposes a tax of $t$ dollars per item sold? Comment on the difference between this price and the price before the tax.

**22. Estimating reciprocals without division** You can estimate the value of the reciprocal of a number $a$ without ever dividing by $a$ if you apply Newton's method to the function $f(x) = (1/x) - a$. For example, if $a = 3$, the function involved is $f(x) = (1/x) - 3$.

**a.** Graph $y = (1/x) - 3$. Where does the graph cross the $x$-axis?

**b.** Show that the recursion formula in this case is

$$x_{n+1} = x_n(2 - 3x_n),$$

so there is no need for division.

**23.** To find $x = \sqrt[q]{a}$, we apply Newton's method to $f(x) = x^q - a$. Here we assume that $a$ is a positive real number and $q$ is a positive integer. Show that $x_1$ is a "weighted average" of $x_0$ and $a/x_0^{q-1}$, and find the coefficients $m_0$, $m_1$ such that

$$x_1 = m_0x_0 + m_1\left(\frac{a}{x_0^{q-1}}\right), \qquad \begin{array}{l} m_0 > 0, m_1 > 0, \\ m_0 + m_1 = 1. \end{array}$$

What conclusion would you reach if $x_0$ and $a/x_0^{q-1}$ were equal? What would be the value of $x_1$ in that case?

**24.** The family of straight lines $y = ax + b$ ($a$, $b$ arbitrary constants) can be characterized by the relation $y'' = 0$. Find a similar relation satisfied by the family of all circles

$$(x - h)^2 + (y - h)^2 = r^2,$$

where $h$ and $r$ are arbitrary constants. (*Hint:* Eliminate $h$ and $r$ from the set of three equations including the given one and two obtained by successive differentiation.)

**25. Free fall in the fourteenth century** In the middle of the fourteenth century, Albert of Saxony (1316–1390) proposed a model of free fall that assumed that the velocity of a falling body was proportional to the distance fallen. It seemed reasonable to think that a body that had fallen 20 ft might be moving twice as fast as a body that had fallen 10 ft. And besides, none of the instruments in use at the time were accurate enough to prove otherwise. Today we can see just how far off Albert of Saxony's model was by solving the initial value problem implicit in his model. Solve the problem and compare your solution graphically with the equation $s = 16t^2$. You will see that it describes a motion that starts too slowly at first and then becomes too fast too soon to be realistic.

T **26. Group blood testing** During World War II it was necessary to administer blood tests to large numbers of recruits. There are two standard ways to administer a blood test to $N$ people. In method 1, each person is tested separately. In method 2, the blood samples of $x$ people are pooled and tested as one large sample. If the test is negative, this one test is enough for all $x$ people. If the test is positive, then each of the $x$ people is tested separately, requiring a total of $x + 1$ tests. Using the second method and some probability theory it can be shown that, on the average, the total number of tests $y$ will be

$$y = N\left(1 - q^x + \frac{1}{x}\right).$$

With $q = 0.99$ and $N = 1000$, find the integer value of $x$ that minimizes $y$. Also find the integer value of $x$ that maximizes $y$. (This second result is not important to the real-life situation.) The group testing method was used in World War II with a savings of 80% over the individual testing method, but not with the given value of $q$.

**27.** Assume that the brakes of an automobile produce a constant deceleration of $k$ ft/sec$^2$. **(a)** Determine what $k$ must be to bring an automobile traveling 60 mi/hr (88 ft/sec) to rest in a distance of 100 ft from the point where the brakes are applied. **(b)** With the same $k$, how far would a car traveling 30 mi/hr travel before being brought to a stop?

**28.** Let $f(x)$, $g(x)$ be two continuously differentiable functions satisfying the relationships $f'(x) = g(x)$ and $f''(x) = -f(x)$. Let $h(x) = f^2(x) + g^2(x)$. If $h(0) = 5$, find $h(10)$.

**29.** Can there be a curve satisfying the following conditions? $d^2y/dx^2$ is everywhere equal to zero and, when $x = 0$, $y = 0$ and $dy/dx = 1$. Give a reason for your answer.

**30.** Find the equation for the curve in the $xy$-plane that passes through the point $(1, -1)$ if its slope at $x$ is always $3x^2 + 2$.

**31.** A particle moves along the $x$-axis. Its acceleration is $a = -t^2$. At $t = 0$, the particle is at the origin. In the course of its motion, it reaches the point $x = b$, where $b > 0$, but no point beyond $b$. Determine its velocity at $t = 0$.

**32.** A particle moves with acceleration $a = \sqrt{t} - (1/\sqrt{t})$. Assuming that the velocity $v = 4/3$ and the position $s = -4/15$ when $t = 0$, find

**a.** the velocity $v$ in terms of $t$.

**b.** the position $s$ in terms of $t$.

**33.** Given $f(x) = ax^2 + 2bx + c$ with $a > 0$. By considering the minimum, prove that $f(x) \geq 0$ for all real $x$ if and only if $b^2 - ac \leq 0$.

**34. Schwarz's inequality**

**a.** In Exercise 33, let

$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2,$$

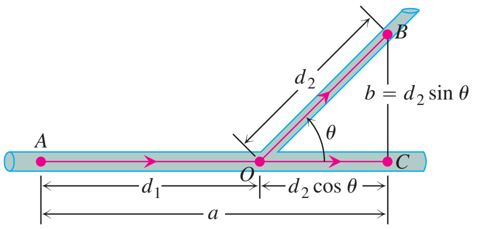
and deduce Schwarz's inequality:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le \left(a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2\right)\left(b_1{}^2 + b_2{}^2 + \cdots + b_n{}^2\right).$$

**b.** Show that equality holds in Schwarz's inequality only if there exists a real number $x$ that makes $a_ix$ equal $-b_i$ for every value of $i$ from 1 to $n$.

**35. The best branching angles for blood vessels and pipes** When a smaller pipe branches off from a larger one in a flow system, we may want it to run off at an angle that is best from some energy-saving point of view. We might require, for instance, that energy loss due to friction be minimized along the section $AOB$ shown in the accompanying figure. In this diagram, $B$ is a given point to be reached by the smaller pipe, $A$ is a point in the larger pipe upstream from $B$, and $O$ is the point where the branching occurs. A law due to Poiseuille states that the loss of energy due to friction in nonturbulent flow is proportional to the length of the path and inversely proportional to the fourth power of the radius. Thus, the loss along $AO$ is $(kd_1)/R^4$ and along $OB$ is $(kd_2)/r^4$, where $k$ is a constant, $d_1$ is the length of $AO$, $d_2$ is the length of $OB$, $R$ is the radius of the larger pipe, and $r$ is the radius of the smaller pipe. The angle $\theta$ is to be chosen to minimize the sum of these two losses:

$$L = k\frac{d_1}{R^4} + k\frac{d_2}{r^4}.$$




In our model, we assume that $AC = a$ and $BC = b$ are fixed. Thus we have the relations

$$d_1 + d_2\cos\theta = a \qquad d_2\sin\theta = b,$$

so that

$$d_2 = b\csc\theta,$$
$$d_1 = a - d_2\cos\theta = a - b\cot\theta.$$

We can express the total loss $L$ as a function of $\theta$:

$$L = k\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right).$$

**a.** Show that the critical value of $\theta$ for which $dL/d\theta$ equals zero is

$$\theta_c = \cos^{-1}\frac{r^4}{R^4}.$$

**b.** If the ratio of the pipe radii is $r/R = 5/6$, estimate to the nearest degree the optimal branching angle given in part (a).

## Chapter 4 Technology Application Projects

### Mathematica/Maple Modules:

***Motion Along a Straight Line: Position → Velocity → Acceleration***

You will observe the shape of a graph through dramatic animated visualizations of the derivative relations among the position, velocity, and acceleration. Figures in the text can be animated.

***Newton's Method: Estimate $\pi$ to How Many Places?***

Plot a function, observe a root, pick a starting point near the root, and use Newton's Iteration Procedure to approximate the root to a desired accuracy. The numbers $\pi$, $e$, and $\sqrt{2}$ are approximated.

# 5

# INTEGRATION

**OVERVIEW** A great achievement of classical geometry was obtaining formulas for the areas and volumes of triangles, spheres, and cones. In this chapter we develop a method to calculate the areas and volumes of very general shapes. This method, called *integration*, is a tool for calculating much more than areas and volumes. The *integral* is of fundamental importance in statistics, the sciences, and engineering. We use it to calculate quantities ranging from probabilities and averages to energy consumption and the forces against a dam's floodgates. We study a variety of these applications in the next chapter, but in this chapter we focus on the integral concept and its use in computing areas of various regions with curved boundaries.

## 5.1 Area and Estimating with Finite Sums

The *definite integral* is the key tool in calculus for defining and calculating quantities important to mathematics and science, such as areas, volumes, lengths of curved paths, probabilities, and the weights of various objects, just to mention a few. The idea behind the integral is that we can effectively compute such quantities by breaking them into small pieces and then summing the contributions from each piece. We then consider what happens when more and more, smaller and smaller pieces are taken in the summation process. Finally, if the number of terms contributing to the sum approaches infinity and we take the limit of these sums in the way described in Section 5.3, the result is a definite integral. We prove in Section 5.4 that integrals are connected to antiderivatives, a connection that is one of the most important relationships in calculus.

The basis for formulating definite integrals is the construction of appropriate finite sums. Although we need to define precisely what we mean by the area of a general region in the plane, or the average value of a function over a closed interval, we have intuitive ideas of what these notions mean. So in this section we begin our approach to integration by *approximating* these quantities with finite sums. We also consider what happens when we take more and more terms in the summation process. In subsequent sections we look at taking the limit of these sums as the number of terms goes to infinity, which then leads to precise definitions of the quantities being approximated here.

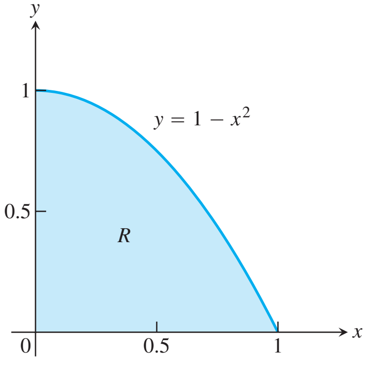



**FIGURE 5.1** The area of the region $R$ cannot be found by a simple formula.

### Area

Suppose we want to find the area of the shaded region $R$ that lies above the $x$-axis, below the graph of $y = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$ (Figure 5.1). Unfortunately, there is no simple geometric formula for calculating the areas of general shapes having curved boundaries like the region $R$. How, then, can we find the area of $R$?

While we do not yet have a method for determining the exact area of $R$, we can approximate it in a simple way. Figure 5.2a shows two rectangles that together contain the region $R$. Each rectangle has width $1/2$ and they have heights 1 and $3/4$, moving from left to right. The height of each rectangle is the maximum value of the function $f$,

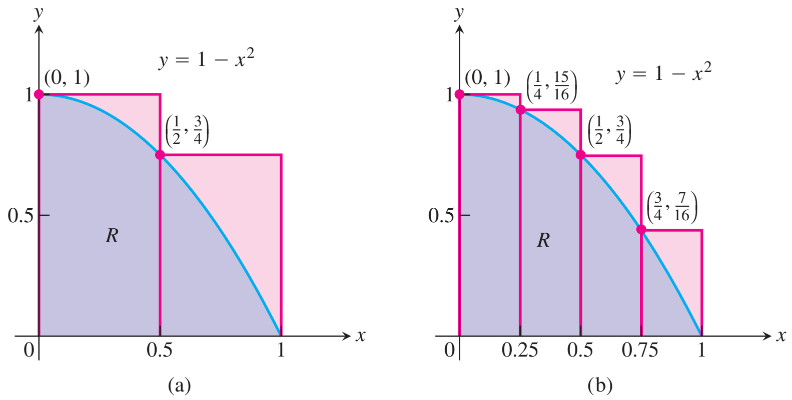



**FIGURE 5.2** (a) We get an upper estimate of the area of $R$ by using two rectangles containing $R$. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area by the amount shaded in light red.

obtained by evaluating $f$ at the left endpoint of the subinterval of [0, 1] forming the base of the rectangle. The total area of the two rectangles approximates the area $A$ of the region $R$,

$$A \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875.$$

This estimate is larger than the true area $A$ since the two rectangles contain $R$. We say that 0.875 is an **upper sum** because it is obtained by taking the height of each rectangle as the maximum (uppermost) value of $f(x)$ for a point $x$ in the base interval of the rectangle. In Figure 5.2b, we improve our estimate by using four thinner rectangles, each of width $1/4$, which taken together contain the region $R$. These four rectangles give the approximation

$$A \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125,$$

which is still greater than $A$ since the four rectangles contain $R$.

Suppose instead we use four rectangles contained *inside* the region $R$ to estimate the area, as in Figure 5.3a. Each rectangle has width $1/4$ as before, but the rectangles are

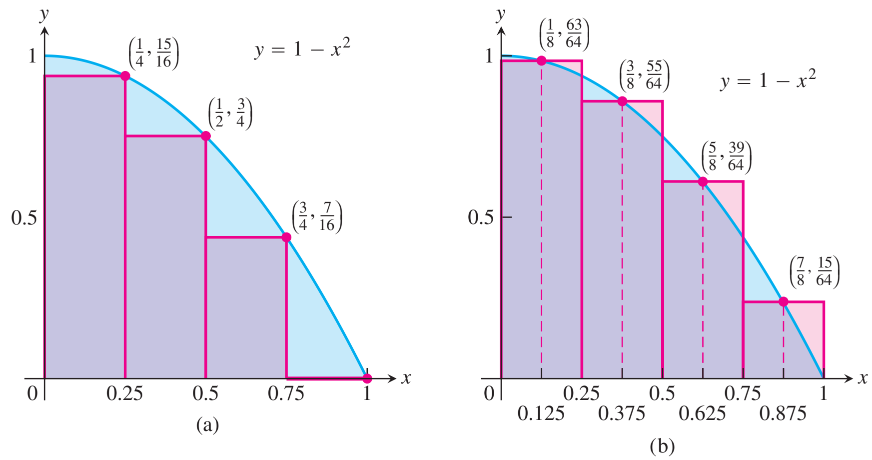



**FIGURE 5.3** (a) Rectangles contained in $R$ give an estimate for the area that undershoots the true value by the amount shaded in light blue. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases. The estimate appears closer to the true value of the area because the light red overshoot areas roughly balance the light blue undershoot areas.

shorter and lie entirely beneath the graph of $f$. The function $f(x) = 1 - x^2$ is decreasing on [0, 1], so the height of each of these rectangles is given by the value of $f$ at the right endpoint of the subinterval forming its base. The fourth rectangle has zero height and therefore contributes no area. Summing these rectangles with heights equal to the minimum value of $f(x)$ for a point $x$ in each base subinterval gives a **lower sum** approximation to the area,

$$A \approx \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{17}{32} = 0.53125.$$

This estimate is smaller than the area $A$ since the rectangles all lie inside of the region $R$. The true value of $A$ lies somewhere between these lower and upper sums:

$$0.53125 < A < 0.78125.$$

By considering both lower and upper sum approximations we get not only estimates for the area, but also a bound on the size of the possible error in these estimates since the true value of the area lies somewhere between them. Here the error cannot be greater than the difference $0.78125 - 0.53125 = 0.25$.

Yet another estimate can be obtained by using rectangles whose heights are the values of $f$ at the midpoints of their bases (Figure 5.3b). This method of estimation is called the **midpoint rule** for approximating the area. The midpoint rule gives an estimate that is between a lower sum and an upper sum, but it is not quite so clear whether it overestimates or underestimates the true area. With four rectangles of width $1/4$ as before, the midpoint rule estimates the area of $R$ to be

$$A \approx \frac{63}{64} \cdot \frac{1}{4} + \frac{55}{64} \cdot \frac{1}{4} + \frac{39}{64} \cdot \frac{1}{4} + \frac{15}{64} \cdot \frac{1}{4} = \frac{172}{64} \cdot \frac{1}{4} = 0.671875.$$

In each of our computed sums, the interval $[a, b]$ over which the function $f$ is defined was subdivided into $n$ subintervals of equal width (also called length) $\Delta x = (b - a)/n$, and $f$ was evaluated at a point in each subinterval: $c_1$ in the first subinterval, $c_2$ in the second subinterval, and so on. The finite sums then all take the form

$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \cdots + f(c_n)\,\Delta x.$$

By taking more and more rectangles, with each rectangle thinner than before, it appears that these finite sums give better and better approximations to the true area of the region $R$.

Figure 5.4a shows a lower sum approximation for the area of $R$ using 16 rectangles of equal width. The sum of their areas is 0.634765625, which appears close to the true area, but is still smaller since the rectangles lie inside $R$.

Figure 5.4b shows an upper sum approximation using 16 rectangles of equal width. The sum of their areas is 0.697265625, which is somewhat larger than the true area because the rectangles taken together contain $R$. The midpoint rule for 16 rectangles gives a total area approximation of 0.6669921875, but it is not immediately clear whether this estimate is larger or smaller than the true area.

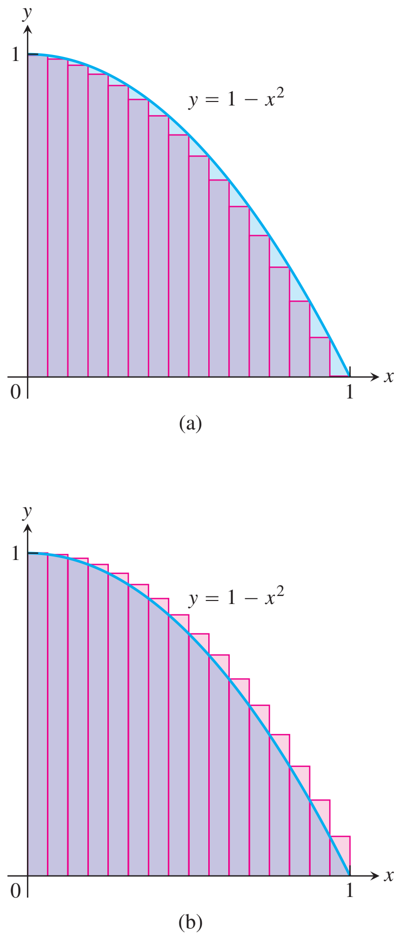



**FIGURE 5.4** (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

**EXAMPLE 1** Table 5.1 shows the values of upper and lower sum approximations to the area of $R$ using up to 1000 rectangles. In Section 5.2 we will see how to get an exact value of the areas of regions such as $R$ by taking a limit as the base width of each rectangle goes to zero and the number of rectangles goes to infinity. With the techniques developed there, we will be able to show that the area of $R$ is exactly $2/3$. ■

## Distance Traveled

Suppose we know the velocity function $v(t)$ of a car moving down a highway, without changing direction, and want to know how far it traveled between times $t = a$ and $t = b$. If we already know an antiderivative $F(t)$ of $v(t)$ we can find the car's position function $s(t)$ by setting

**TABLE 5.1** Finite approximations for the area of $R$

| Number of subintervals | Lower sum | Midpoint rule | Upper sum |
|---|---|---|---|
| 2 | .375 | .6875 | .875 |
| 4 | .53125 | .671875 | .78125 |
| 16 | .634765625 | .6669921875 | .697265625 |
| 50 | .6566 | .6667 | .6766 |
| 100 | .66165 | .666675 | .67165 |
| 1000 | .6661665 | .66666675 | .6671665 |

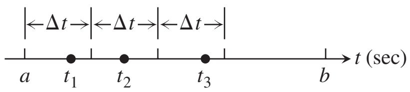
$s(t) = F(t) + C$. The distance traveled can then be found by calculating the change in position, $s(b) - s(a) = F(b) - F(a)$. If the velocity function is known only by the readings at various times of a speedometer on the car, then we have no formula from which to obtain an antiderivative function for velocity. So what do we do in this situation?

When we don't know an antiderivative for the velocity function $v(t)$, we can apply the same principle of approximating the distance traveled with finite sums in a way similar to our estimates for area discussed before. We subdivide the interval $[a, b]$ into short time intervals on each of which the velocity is considered to be fairly constant. Then we approximate the distance traveled on each time subinterval with the usual distance formula

$$\text{distance} = \text{velocity} \times \text{time}$$

and add the results across $[a, b]$.

Suppose the subdivided interval looks like




with the subintervals all of equal length $\Delta t$. Pick a number $t_1$ in the first interval. If $\Delta t$ is so small that the velocity barely changes over a short time interval of duration $\Delta t$, then the distance traveled in the first time interval is about $v(t_1)\,\Delta t$. If $t_2$ is a number in the second interval, the distance traveled in the second time interval is about $v(t_2)\,\Delta t$. The sum of the distances traveled over all the time intervals is

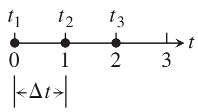
$$D \approx v(t_1)\,\Delta t + v(t_2)\,\Delta t + \cdots + v(t_n)\,\Delta t,$$

where $n$ is the total number of subintervals.

**EXAMPLE 2** The velocity function of a projectile fired straight into the air is $f(t) = 160 - 9.8t$ m/sec. Use the summation technique just described to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact value of 435.9 m?

**Solution** We explore the results for different numbers of intervals and different choices of evaluation points. Notice that $f(t)$ is decreasing, so choosing left endpoints gives an upper sum estimate; choosing right endpoints gives a lower sum estimate.

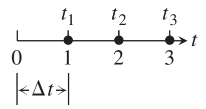
**(a)** *Three subintervals of length* 1, *with $f$ evaluated at left endpoints giving an upper sum*:



With $f$ evaluated at $t = 0, 1$, and 2, we have

$$\begin{aligned} D &\approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + f(t_3)\,\Delta t \\ &= [160 - 9.8(0)](1) + [160 - 9.8(1)](1) + [160 - 9.8(2)](1) \\ &= 450.6. \end{aligned}$$

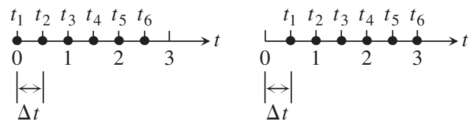
**(b)** *Three subintervals of length* 1, *with $f$ evaluated at right endpoints giving a lower sum*:




With $f$ evaluated at $t = 1, 2$, and 3, we have

$$\begin{aligned} D &\approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + f(t_3)\,\Delta t \\ &= [160 - 9.8(1)](1) + [160 - 9.8(2)](1) + [160 - 9.8(3)](1) \\ &= 421.2. \end{aligned}$$

**(c)** *With six subintervals of length* 1/2, *we get*




These estimates give an upper sum using left endpoints: $D \approx 443.25$; and a lower sum using right endpoints: $D \approx 428.55$. These six-interval estimates are somewhat closer than the three-interval estimates. The results improve as the subintervals get shorter.

As we can see in Table 5.2, the left-endpoint upper sums approach the true value 435.9 from above, whereas the right-endpoint lower sums approach it from below. The true value lies between these upper and lower sums. The magnitude of the error in the closest entries is 0.23, a small percentage of the true value.

$$\begin{aligned} \text{Error magnitude} &= |\text{true value} - \text{calculated value}| \\ &= |435.9 - 435.67| = 0.23. \end{aligned}$$

$$\text{Error percentage} = \frac{0.23}{435.9} \approx 0.05\%.$$

It would be reasonable to conclude from the table's last entries that the projectile rose about 436 m during its first 3 sec of flight. ■

**TABLE 5.2** Travel-distance estimates

| Number of subintervals | Length of each subinterval | Upper sum | Lower sum |
|---|---|---|---|
| 3 | 1 | 450.6 | 421.2 |
| 6 | 1/2 | 443.25 | 428.55 |
| 12 | 1/4 | 439.58 | 432.23 |
| 24 | 1/8 | 437.74 | 434.06 |
| 48 | 1/16 | 436.82 | 434.98 |
| 96 | 1/32 | 436.36 | 435.44 |
| 192 | 1/64 | 436.13 | 435.67 |

## Displacement Versus Distance Traveled

If an object with position function $s(t)$ moves along a coordinate line without changing direction, we can calculate the total distance it travels from $t = a$ to $t = b$ by summing the distance traveled over small intervals, as in Example 2. If the object reverses direction one or more times during the trip, then we need to use the object's *speed* $|v(t)|$, which is the absolute value of its velocity function, $v(t)$, to find the total distance traveled. Using the velocity itself, as in Example 2, gives instead an estimate to the object's **displacement**, $s(b) - s(a)$, the difference between its initial and final positions.

To see why using the velocity function in the summation process gives an estimate to the displacement, partition the time interval $[a, b]$ into small enough equal subintervals $\Delta t$ so that the object's velocity does not change very much from time $t_{k-1}$ to $t_k$. Then $v(t_k)$ gives a good approximation of the velocity throughout the interval. Accordingly, the change in the object's position coordinate during the time interval is about

$$v(t_k)\,\Delta t.$$

The change is positive if $v(t_k)$ is positive and negative if $v(t_k)$ is negative.

In either case, the distance traveled by the object during the subinterval is about

$$|v(t_k)|\,\Delta t.$$

The **total distance traveled** is approximately the sum

$$|v(t_1)|\,\Delta t + |v(t_2)|\,\Delta t + \cdots + |v(t_n)|\,\Delta t.$$

We revisit these ideas in Section 5.4.

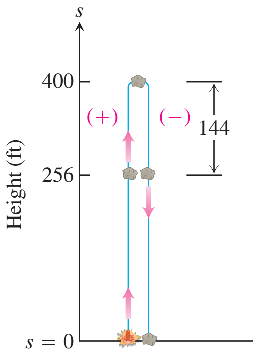



**FIGURE 5.5** The rock in Example 3. The height 256 ft is reached at $t = 2$ and $t = 8$ sec. The rock falls 144 ft from its maximum height when $t = 8$.

**EXAMPLE 3** In Example 4 in Section 3.4, we analyzed the motion of a heavy rock blown straight up by a dynamite blast. In that example, we found the velocity of the rock at any time during its motion to be $v(t) = 160 - 32t$ ft/sec. The rock was 256 ft above the ground 2 sec after the explosion, continued upwards to reach a maximum height of 400 ft at 5 sec after the explosion, and then fell back down to reach the height of 256 ft again at $t = 8$ sec after the explosion. (See Figure 5.5.)

If we follow a procedure like that presented in Example 2, and use the velocity function $v(t)$ in the summation process over the time interval [0, 8], we will obtain an estimate to 256 ft, the rock's *height* above the ground at $t = 8$. The positive upward motion (which yields a positive distance change of 144 ft from the height of 256 ft to the maximum height) is cancelled by the negative downward motion (giving a negative change of 144 ft from the maximum height down to 256 ft again), so the displacement or height above the ground is being estimated from the velocity function.

On the other hand, if the absolute value $|v(t)|$ is used in the summation process, we will obtain an estimate to the *total distance* the rock has traveled: the maximum height reached of 400 ft plus the additional distance of 144 ft it has fallen back down from that maximum when it again reaches the height of 256 ft at $t = 8$ sec. That is, using the absolute value of the velocity function in the summation process over the time interval [0, 8], we obtain an estimate to 544 ft, the total distance up and down that the rock has traveled in 8 sec. There is no cancellation of distance changes due to sign changes in the velocity function, so we estimate distance traveled rather than displacement when we use the absolute value of the velocity function (that is, the speed of the rock).

As an illustration of our discussion, we subdivide the interval [0, 8] into sixteen subintervals of length $\Delta t = 1/2$ and take the right endpoint of each subinterval in our calculations. Table 5.3 shows the values of the velocity function at these endpoints.

**TABLE 5.3** Velocity Function

| $t$ | $v(t)$ | $t$ | $v(t)$ |
|---|---|---|---|
| 0 | 160 | 4.5 | 16 |
| 0.5 | 144 | 5.0 | 0 |
| 1.0 | 128 | 5.5 | −16 |
| 1.5 | 112 | 6.0 | −32 |
| 2.0 | 96 | 6.5 | −48 |
| 2.5 | 80 | 7.0 | −64 |
| 3.0 | 64 | 7.5 | −80 |
| 3.5 | 48 | 8.0 | −96 |
| 4.0 | 32 | | |

Using $v(t)$ in the summation process, we estimate the displacement at $t = 8$:

$$(144 + 128 + 112 + 96 + 80 + 64 + 48 + 32 + 16$$
$$+ 0 - 16 - 32 - 48 - 64 - 80 - 96) \cdot \frac{1}{2} = 192$$

$$\text{Error magnitude} = 256 - 192 = 64$$

Using $|v(t)|$ in the summation process, we estimate the total distance traveled over the time interval [0, 8]:

$$(144 + 128 + 112 + 96 + 80 + 64 + 48 + 32 + 16 + 0 + 16 + 32 + 48 + 64 + 80 + 96) \cdot \frac{1}{2} = 528$$

$$\text{Error magnitude} = 544 - 528 = 16$$

If we take more and more subintervals of [0, 8] in our calculations, the estimates to 256 ft and 544 ft improve, approaching their true values. ■

## Average Value of a Nonnegative Continuous Function

The average value of a collection of $n$ numbers $x_1, x_2, \ldots, x_n$ is obtained by adding them together and dividing by $n$. But what is the average value of a continuous function $f$ on an interval $[a, b]$? Such a function can assume infinitely many values. For example, the temperature at a certain location in a town is a continuous function that goes up and down each day. What does it mean to say that the average temperature in the town over the course of a day is 73 degrees?

When a function is constant, this question is easy to answer. A function with constant value $c$ on an interval $[a, b]$ has average value $c$. When $c$ is positive, its graph over $[a, b]$ gives a rectangle of height $c$. The average value of the function can then be interpreted geometrically as the area of this rectangle divided by its width $b - a$ (Figure 5.6a).

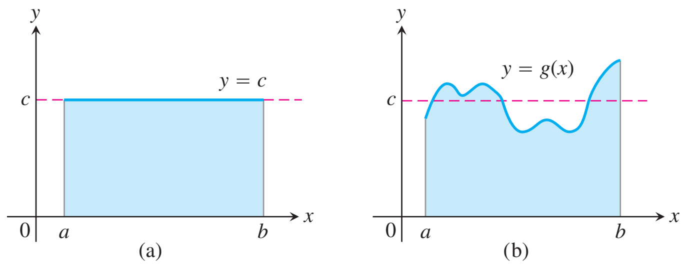



**FIGURE 5.6** (a) The average value of $f(x) = c$ on $[a, b]$ is the area of the rectangle divided by $b - a$. (b) The average value of $g(x)$ on $[a, b]$ is the area beneath its graph divided by $b - a$.

What if we want to find the average value of a nonconstant function, such as the function $g$ in Figure 5.6b? We can think of this graph as a snapshot of the height of some water that is sloshing around in a tank between enclosing walls at $x = a$ and $x = b$. As the water moves, its height over each point changes, but its average height remains the same. To get the average height of the water, we let it settle down until it is level and its height is constant. The resulting height $c$ equals the area under the graph of $g$ divided by $b - a$. We are led to *define* the average value of a nonnegative function on an interval $[a, b]$ to be the area under its graph divided by $b - a$. For this definition to be valid, we need a precise understanding of what is meant by the area under a graph. This will be obtained in Section 5.3, but for now we look at an example.

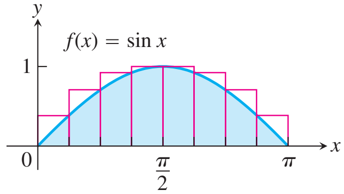



**FIGURE 5.7** Approximating the area under $f(x) = \sin x$ between 0 and $\pi$ to compute the average value of $\sin x$ over $[0, \pi]$, using eight rectangles (Example 4).

**EXAMPLE 4** Estimate the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

**Solution** Looking at the graph of $\sin x$ between 0 and $\pi$ in Figure 5.7, we can see that its average height is somewhere between 0 and 1. To find the average we need to calculate the area $A$ under the graph and then divide this area by the length of the interval, $\pi - 0 = \pi$.

We do not have a simple way to determine the area, so we approximate it with finite sums. To get an upper sum approximation, we add the areas of eight rectangles of equal width $\pi/8$ that together contain the region beneath the graph of $y = \sin x$ and above the $x$-axis on $[0, \pi]$. We choose the heights of the rectangles to be the largest value of $\sin x$ on each subinterval. Over a particular subinterval, this largest value may occur at the left endpoint, the right endpoint, or somewhere between them. We evaluate $\sin x$ at this point to get the height of the rectangle for an upper sum. The sum of the rectangle areas then estimates the total area (Figure 5.7):

$$A \approx \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right)\cdot\frac{\pi}{8}$$

$$\approx (.38 + .71 + .92 + 1 + 1 + .92 + .71 + .38)\cdot\frac{\pi}{8} = (6.02)\cdot\frac{\pi}{8} \approx 2.365.$$

To estimate the average value of $\sin x$ we divide the estimated area by $\pi$ and obtain the approximation $2.365/\pi \approx 0.753$.

Since we used an upper sum to approximate the area, this estimate is greater than the actual average value of $\sin x$ over $[0, \pi]$. If we use more and more rectangles, with each rectangle getting thinner and thinner, we get closer and closer to the true average value. Using the techniques covered in Section 5.3, we will show that the true average value is $2/\pi \approx 0.64$.

As before, we could just as well have used rectangles lying under the graph of $y = \sin x$ and calculated a lower sum approximation, or we could have used the midpoint rule. In Section 5.3 we will see that in each case, the approximations are close to the true area if all the rectangles are sufficiently thin. ■

## Summary

The area under the graph of a positive function, the distance traveled by a moving object that doesn't change direction, and the average value of a nonnegative function over an interval can all be approximated by finite sums. First we subdivide the interval into subintervals, treating the appropriate function $f$ as if it were constant over each particular subinterval. Then we multiply the width of each subinterval by the value of $f$ at some point within it, and add these products together. If the interval $[a, b]$ is subdivided into $n$ subintervals of equal widths $\Delta x = (b - a)/n$, and if $f(c_k)$ is the value of $f$ at the chosen point $c_k$ in the $k$th subinterval, this process gives a finite sum of the form

$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \cdots + f(c_n)\,\Delta x.$$

The choices for the $c_k$ could maximize or minimize the value of $f$ in the $k$th subinterval, or give some value in between. The true value lies somewhere between the approximations given by upper sums and lower sums. The finite sum approximations we looked at improved as we took more subintervals of thinner width.

# Exercises 5.1

### Area

In Exercises 1–4, use finite approximations to estimate the area under the graph of the function using

**a.** a lower sum with two rectangles of equal width.

**b.** a lower sum with four rectangles of equal width.

**c.** an upper sum with two rectangles of equal width.

**d.** an upper sum with four rectangles of equal width.

**1.** $f(x) = x^2$ between $x = 0$ and $x = 1$.

**2.** $f(x) = x^3$ between $x = 0$ and $x = 1$.

**3.** $f(x) = 1/x$ between $x = 1$ and $x = 5$.

**4.** $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Using rectangles whose height is given by the value of the function at the midpoint of the rectangle's base (*the midpoint rule*), estimate the area under the graphs of the following functions, using first two and then four rectangles.

**5.** $f(x) = x^2$ between $x = 0$ and $x = 1$.

**6.** $f(x) = x^3$ between $x = 0$ and $x = 1$.

**7.** $f(x) = 1/x$ between $x = 1$ and $x = 5$.

**8.** $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

## Distance

**9. Distance traveled** The accompanying table shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine using 10 subintervals of length 1 with

**a.** left-endpoint values.

**b.** right-endpoint values.

| Time (sec) | Velocity (in./sec) | Time (sec) | Velocity (in./sec) |
|---|---|---|---|
| 0 | 0 | 6 | 11 |
| 1 | 12 | 7 | 6 |
| 2 | 22 | 8 | 2 |
| 3 | 10 | 9 | 6 |
| 4 | 5 | 10 | 0 |
| 5 | 13 | | |

**10. Distance traveled upstream** You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with

**a.** left-endpoint values.

**b.** right-endpoint values.

| Time (min) | Velocity (m/sec) | Time (min) | Velocity (m/sec) |
|---|---|---|---|
| 0 | 1 | 35 | 1.2 |
| 5 | 1.2 | 40 | 1.0 |
| 10 | 1.7 | 45 | 1.8 |
| 15 | 2.0 | 50 | 1.5 |
| 20 | 1.8 | 55 | 1.2 |
| 25 | 1.6 | 60 | 0 |
| 30 | 1.4 | | |

**11. Length of a road** You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the accompanying table. Estimate the length of the road using

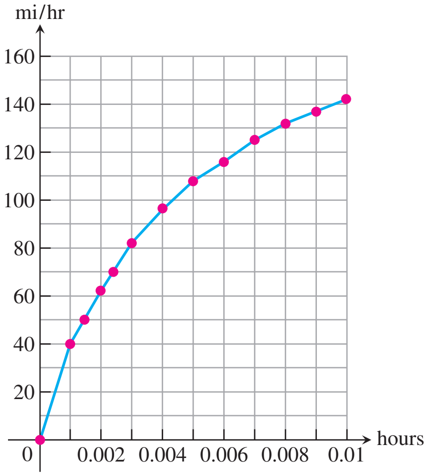
**a.** left-endpoint values.

**b.** right-endpoint values.

| Time (sec) | Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec) | Time (sec) | Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec) |
|---|---|---|---|
| 0 | 0 | 70 | 15 |
| 10 | 44 | 80 | 22 |
| 20 | 15 | 90 | 35 |
| 30 | 35 | 100 | 44 |
| 40 | 30 | 110 | 30 |
| 50 | 44 | 120 | 35 |
| 60 | 35 | | |

**12. Distance from velocity data** The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour).

| Time (h) | Velocity (mi/h) | Time (h) | Velocity (mi/h) |
|---|---|---|---|
| 0.0 | 0 | 0.006 | 116 |
| 0.001 | 40 | 0.007 | 125 |
| 0.002 | 62 | 0.008 | 132 |
| 0.003 | 82 | 0.009 | 137 |
| 0.004 | 96 | 0.010 | 142 |
| 0.005 | 108 | | |




**a.** Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

**b.** Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

**13. Free fall with air resistance** An object is dropped straight down from a helicopter. The object falls faster and faster but its acceleration (rate of change of its velocity) decreases over time because of air resistance. The acceleration is measured in ft/sec$^2$ and recorded every second after the drop for 5 sec, as shown:

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $a$ | 32.00 | 19.41 | 11.77 | 7.14 | 4.33 | 2.63 |

**a.** Find an upper estimate for the speed when $t = 5$.

**b.** Find a lower estimate for the speed when $t = 5$.

**c.** Find an upper estimate for the distance fallen when $t = 3$.

**14. Distance traveled by a projectile** An object is shot straight upward from sea level with an initial velocity of 400 ft/sec.

**a.** Assuming that gravity is the only force acting on the object, give an upper estimate for its velocity after 5 sec have elapsed. Use $g = 32$ ft/sec$^2$ for the gravitational acceleration.

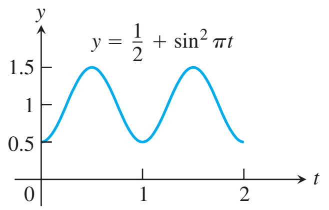
**b.** Find a lower estimate for the height attained after 5 sec.

## Average Value of a Function

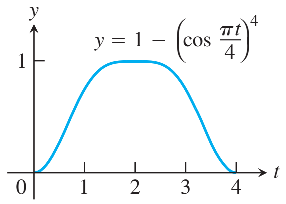
In Exercises 15–18, use a finite sum to estimate the average value of $f$ on the given interval by partitioning the interval into four subintervals of equal length and evaluating $f$ at the subinterval midpoints.

**15.** $f(x) = x^3$ on $[0, 2]$  **16.** $f(x) = 1/x$ on $[1, 9]$

**17.** $f(t) = (1/2) + \sin^2 \pi t$ on $[0, 2]$




**18.** $f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4$ on $[0, 4]$




## Examples of Estimations

**19. Water pollution** Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

| **Time (h)** | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| **Leakage (gal/h)** | 50 | 70 | 97 | 136 | 190 |

| **Time (h)** | 5 | 6 | 7 | 8 |
|---|---|---|---|---|
| **Leakage (gal/h)** | 265 | 369 | 516 | 720 |

**a.** Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.

**b.** Repeat part (a) for the quantity of oil that has escaped after 8 hours.

**c.** The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil, approximately how many more hours will elapse in the worst case before all the oil has spilled? In the best case?

**20. Air pollution** A power plant generates electricity by burning oil. Pollutants produced as a result of the burning process are removed by scrubbers in the smokestacks. Over time, the scrubbers become less efficient and eventually they must be replaced when the amount of pollution released exceeds government standards. Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere, recorded as follows.

| **Month** | **Jan** | **Feb** | **Mar** | **Apr** | **May** | **Jun** |
|---|---|---|---|---|---|---|
| Pollutant release rate (tons/day) | 0.20 | 0.25 | 0.27 | 0.34 | 0.45 | 0.52 |

| **Month** | **Jul** | **Aug** | **Sep** | **Oct** | **Nov** | **Dec** |
|---|---|---|---|---|---|---|
| Pollutant release rate (tons/day) | 0.63 | 0.70 | 0.81 | 0.85 | 0.89 | 0.95 |

**a.** Assuming a 30-day month and that new scrubbers allow only 0.05 ton/day to be released, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate?

**b.** In the best case, approximately when will a total of 125 tons of pollutants have been released into the atmosphere?

**21.** Inscribe a regular $n$-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of $n$:

**a.** 4 (square)  **b.** 8 (octagon)  **c.** 16

**d.** Compare the areas in parts (a), (b), and (c) with the area of the circle.

**22.** (*Continuation of Exercise 21.*)

**a.** Inscribe a regular $n$-sided polygon inside a circle of radius 1 and compute the area of one of the $n$ congruent triangles formed by drawing radii to the vertices of the polygon.

**b.** Compute the limit of the area of the inscribed polygon as $n \to \infty$.

**c.** Repeat the computations in parts (a) and (b) for a circle of radius $r$.

## COMPUTER EXPLORATIONS

In Exercises 23–26, use a CAS to perform the following steps.

**a.** Plot the functions over the given interval.

**b.** Subdivide the interval into $n = 100, 200$, and $1000$ subintervals of equal length and evaluate the function at the midpoint of each subinterval.

**c.** Compute the average value of the function values generated in part (b).

**d.** Solve the equation $f(x) = $ (average value) for $x$ using the average value calculated in part (c) for the $n = 1000$ partitioning.

**23.** $f(x) = \sin x$ on $[0, \pi]$  **24.** $f(x) = \sin^2 x$ on $[0, \pi]$

**25.** $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$  **26.** $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

# 5.2 Sigma Notation and Limits of Finite Sums

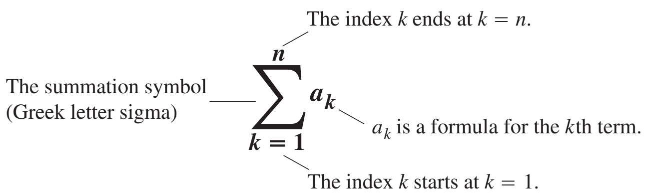
In estimating with finite sums in Section 5.1, we encountered sums with many terms (up to 1000 in Table 5.1, for instance). In this section we introduce a more convenient notation for sums with a large number of terms. After describing the notation and stating several of its properties, we look at what happens to a finite sum approximation as the number of terms approaches infinity.

## Finite Sums and Sigma Notation

**Sigma notation** enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

The Greek letter $\Sigma$ (capital sigma, corresponding to our letter S), stands for "sum." The **index of summation** $k$ tells us where the sum begins (at the number below the $\Sigma$ symbol) and where it ends (at the number above $\Sigma$). Any letter can be used to denote the index, but the letters $i$, $j$, and $k$ are customary.




Thus we can write

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2,$$

and

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i).$$

The lower limit of summation does not have to be 1; it can be any integer.

**EXAMPLE 1**

| A sum in sigma notation | The sum written out, one term for each value of $k$ | The value of the sum |
|---|---|---|
| $\sum_{k=1}^{5} k$ | $1 + 2 + 3 + 4 + 5$ | $15$ |
| $\sum_{k=1}^{3} (-1)^k k$ | $(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$ | $-1 + 2 - 3 = -2$ |
| $\sum_{k=1}^{2} \frac{k}{k+1}$ | $\frac{1}{1+1} + \frac{2}{2+1}$ | $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$ |
| $\sum_{k=4}^{5} \frac{k^2}{k-1}$ | $\frac{4^2}{4-1} + \frac{5^2}{5-1}$ | $\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$ |

**EXAMPLE 2** Express the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

**Solution** The formula generating the terms changes with the lower limit of summation, but the terms generated remain the same. It is often simplest to start with $k = 0$ or $k = 1$, but we can start with any integer.

$$\textit{Starting with } k = 0: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=0}^{4} (2k + 1)$$

$$\textit{Starting with } k = 1: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=1}^{5} (2k - 1)$$

$$\textit{Starting with } k = 2: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=2}^{6} (2k - 3)$$

$$\textit{Starting with } k = -3: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=-3}^{1} (2k + 7)$$

■

When we have a sum such as

$$\sum_{k=1}^{3} (k + k^2)$$

we can rearrange its terms,

$$\begin{aligned}\sum_{k=1}^{3} (k + k^2) &= (1 + 1^2) + (2 + 2^2) + (3 + 3^2) \\ &= (1 + 2 + 3) + (1^2 + 2^2 + 3^2) \qquad \text{Regroup terms.} \\ &= \sum_{k=1}^{3} k + \sum_{k=1}^{3} k^2.\end{aligned}$$

This illustrates a general rule for finite sums:

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

Four such rules are given below. A proof that they are valid can be obtained using mathematical induction (see Appendix 2).

### Algebra Rules for Finite Sums

1. *Sum Rule:* $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

2. *Difference Rule:* $$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

3. *Constant Multiple Rule:* $$\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k \qquad \text{(Any number } c\text{)}$$

4. *Constant Value Rule:* $$\sum_{k=1}^{n} c = n \cdot c \qquad (c \text{ is any constant value.})$$

**EXAMPLE 3** We demonstrate the use of the algebra rules.

**(a)** $$\sum_{k=1}^{n} (3k - k^2) = 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$$ Difference Rule and Constant Multiple Rule

**(b)** $$\sum_{k=1}^{n} (-a_k) = \sum_{k=1}^{n} (-1) \cdot a_k = -1 \cdot \sum_{k=1}^{n} a_k = -\sum_{k=1}^{n} a_k$$ Constant Multiple Rule

**(c)** $\sum_{k=1}^{3}(k+4) = \sum_{k=1}^{3}k + \sum_{k=1}^{3}4$ — Sum Rule

$= (1+2+3) + (3\cdot 4)$ — Constant Value Rule

$= 6 + 12 = 18$

**(d)** $\sum_{k=1}^{n}\frac{1}{n} = n\cdot\frac{1}{n} = 1$ — Constant Value Rule ($1/n$ is constant) ■

Historical Biography

Carl Friedrich Gauss
(1777–1855)

Over the years people have discovered a variety of formulas for the values of finite sums. The most famous of these are the formula for the sum of the first $n$ integers (Gauss is said to have discovered it at age 8) and the formulas for the sums of the squares and cubes of the first $n$ integers.

**EXAMPLE 4** Show that the sum of the first $n$ integers is

$$\sum_{k=1}^{n}k = \frac{n(n+1)}{2}.$$

**Solution** The formula tells us that the sum of the first 4 integers is

$$\frac{(4)(5)}{2} = 10.$$

Addition verifies this prediction:

$$1 + 2 + 3 + 4 = 10.$$

To prove the formula in general, we write out the terms in the sum twice, once forward and once backward.

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \end{array}$$

If we add the two terms in the first column we get $1 + n = n + 1$. Similarly, if we add the two terms in the second column we get $2 + (n-1) = n + 1$. The two terms in any column sum to $n + 1$. When we add the $n$ columns together we get $n$ terms, each equal to $n + 1$, for a total of $n(n+1)$. Since this is twice the desired quantity, the sum of the first $n$ integers is $(n)(n+1)/2$. ■

Formulas for the sums of the squares and cubes of the first $n$ integers are proved using mathematical induction (see Appendix 2). We state them here.

The first $n$ squares: $\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$

The first $n$ cubes: $\sum_{k=1}^{n}k^3 = \left(\frac{n(n+1)}{2}\right)^2$

## Limits of Finite Sums

The finite sum approximations we considered in Section 5.1 became more accurate as the number of terms increased and the subinterval widths (lengths) narrowed. The next example shows how to calculate a limiting value as the widths of the subintervals go to zero and their number grows to infinity.

**EXAMPLE 5** Find the limiting value of lower sum approximations to the area of the region $R$ below the graph of $y = 1 - x^2$ and above the interval [0, 1] on the $x$-axis using equal-width rectangles whose widths approach zero and whose number approaches infinity. (See Figure 5.4a.)

**Solution** We compute a lower sum approximation using $n$ rectangles of equal width $\Delta x = (1 - 0)/n$, and then we see what happens as $n \to \infty$. We start by subdividing $[0, 1]$ into $n$ equal width subintervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, \frac{n}{n}\right].$$

Each subinterval has width $1/n$. The function $1 - x^2$ is decreasing on $[0, 1]$, and its smallest value in a subinterval occurs at the subinterval's right endpoint. So a lower sum is constructed with rectangles whose height over the subinterval $[(k-1)/n, k/n]$ is $f(k/n) = 1 - (k/n)^2$, giving the sum

$$\left[f\left(\frac{1}{n}\right)\right]\left(\frac{1}{n}\right) + \left[f\left(\frac{2}{n}\right)\right]\left(\frac{1}{n}\right) + \cdots + \left[f\left(\frac{k}{n}\right)\right]\left(\frac{1}{n}\right) + \cdots + \left[f\left(\frac{n}{n}\right)\right]\left(\frac{1}{n}\right).$$

We write this in sigma notation and simplify,

$$\begin{aligned}
\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right) &= \sum_{k=1}^{n}\left(1 - \left(\frac{k}{n}\right)^2\right)\left(\frac{1}{n}\right) \\
&= \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3} && \text{Difference Rule} \\
&= n \cdot \frac{1}{n} - \frac{1}{n^3}\sum_{k=1}^{n} k^2 && \text{Constant Value and Constant Multiple Rules} \\
&= 1 - \left(\frac{1}{n^3}\right)\frac{(n)(n+1)(2n+1)}{6} && \text{Sum of the First } n \text{ Squares} \\
&= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}. && \text{Numerator expanded}
\end{aligned}$$

We have obtained an expression for the lower sum that holds for any $n$. Taking the limit of this expression as $n \to \infty$, we see that the lower sums converge as the number of subintervals increases and the subinterval widths approach zero:

$$\lim_{n\to\infty}\left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right) = 1 - \frac{2}{6} = \frac{2}{3}.$$

The lower sum approximations converge to $2/3$. A similar calculation shows that the upper sum approximations also converge to $2/3$. Any finite sum approximation $\sum_{k=1}^{n} f(c_k)(1/n)$ also converges to the same value, $2/3$. This is because it is possible to show that any finite sum approximation is trapped between the lower and upper sum approximations. For this reason we are led to *define* the area of the region $R$ as this limiting value. In Section 5.3 we study the limits of such finite approximations in a general setting. ■

## Riemann Sums

HISTORICAL BIOGRAPHY

Georg Friedrich Bernhard Riemann (1826–1866)

The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann. We now introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral studied in the next section.

We begin with an arbitrary bounded function $f$ defined on a closed interval $[a, b]$. Like the function pictured in Figure 5.8, $f$ may have negative as well as positive values. We subdivide the interval $[a, b]$ into subintervals, not necessarily of equal widths (or lengths), and form sums in the same way as for the finite approximations in Section 5.1. To do so, we choose $n - 1$ points $\{x_1, x_2, x_3, \ldots, x_{n-1}\}$ between $a$ and $b$ and satisfying

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

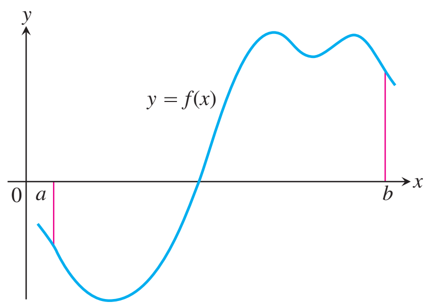



**FIGURE 5.8** A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

To make the notation consistent, we denote $a$ by $x_0$ and $b$ by $x_n$, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The set

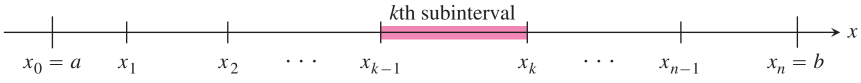
$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

is called a **partition** of $[a, b]$.

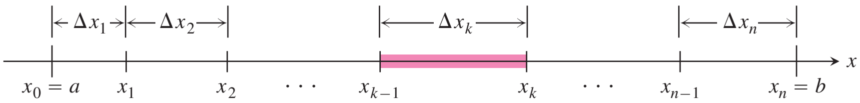
The partition $P$ divides $[a, b]$ into $n$ closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the ***k*th subinterval of** $P$ is $[x_{k-1}, x_k]$, for $k$ an integer between 1 and $n$.




The width of the first subinterval $[x_0, x_1]$ is denoted $\Delta x_1$, the width of the second $[x_1, x_2]$ is denoted $\Delta x_2$, and the width of the $k$th subinterval is $\Delta x_k = x_k - x_{k-1}$. If all $n$ subintervals have equal width, then the common width $\Delta x$ is equal to $(b - a)/n$.




In each subinterval we select some point. The point chosen in the $k$th subinterval $[x_{k-1}, x_k]$ is called $c_k$. Then on each subinterval we stand a vertical rectangle that stretches from the $x$-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the $x$-axis, depending on whether $f(c_k)$ is positive or negative, or on the $x$-axis if $f(c_k) = 0$ (Figure 5.9).

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width $\Delta x_k$. When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width $\Delta x_k$ that drops from the $x$-axis to the negative number $f(c_k)$.

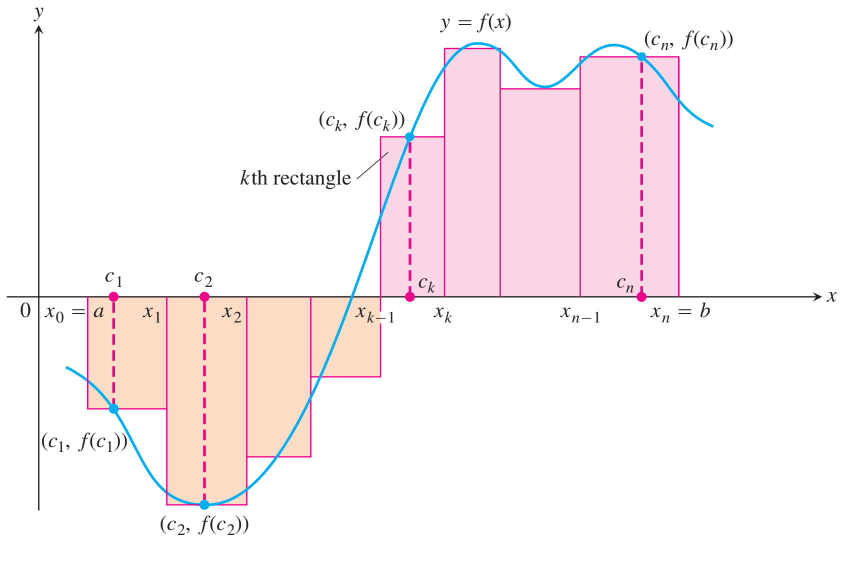



**FIGURE 5.9** The rectangles approximate the region between the graph of the function $y = f(x)$ and the $x$-axis. Figure 5.8 has been enlarged to enhance the partition of $[a, b]$ and selection of points $c_k$ that produce the rectangles.

Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k\,.$$

The sum $S_P$ is called a **Riemann sum for $f$ on the interval $[a, b]$**. There are many such sums, depending on the partition $P$ we choose, and the choices of the points $c_k$ in the subintervals. For instance, we could choose $n$ subintervals all having equal width $\Delta x = (b - a)/n$ to partition $[a, b]$, and then choose the point $c_k$ to be the right-hand endpoint of each subinterval when forming the Riemann sum (as we did in Example 5). This choice leads to the Riemann sum formula

$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\cdot\left(\frac{b-a}{n}\right).$$

Similar formulas can be obtained if instead we choose $c_k$ to be the left-hand endpoint, or the midpoint, of each subinterval.

In the cases in which the subintervals all have equal width $\Delta x = (b - a)/n$, we can make them thinner by simply increasing their number $n$. When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval. We define the **norm** of a partition $P$, written $\|P\|$, to be the largest of all the subinterval widths. If $\|P\|$ is a small number, then all of the subintervals in the partition $P$ have a small width. Let's look at an example of these ideas.

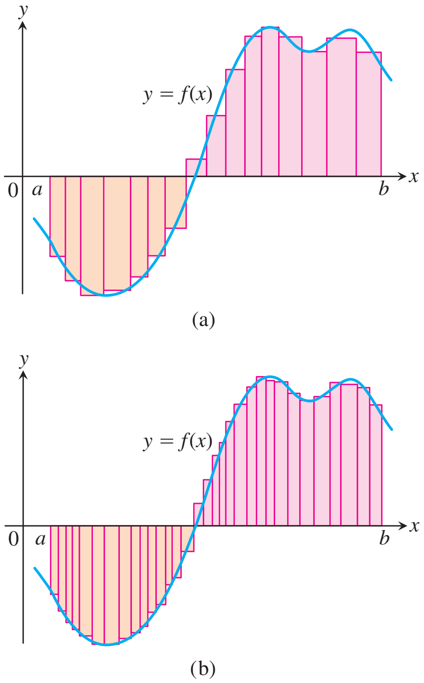
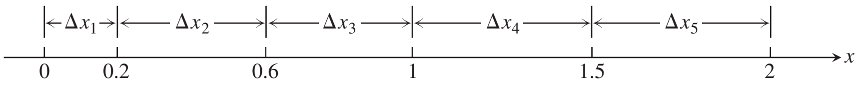



**FIGURE 5.10** The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of $f$ and the $x$-axis with increasing accuracy.

**EXAMPLE 6** The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of $P$: $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$:




The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ■

Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a continuous function $f$ and the $x$-axis. Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy, as suggested by Figure 5.10. We will see in the next section that if the function $f$ is continuous over the closed interval $[a, b]$, then no matter how we choose the partition $P$ and the points $c_k$ in its subintervals to construct a Riemann sum, a single limiting value is approached as the subinterval widths, controlled by the norm of the partition, approach zero.

## Exercises 5.2

### Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1}$

2. $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

3. $\displaystyle\sum_{k=1}^{4} \cos k\pi$

4. $\displaystyle\sum_{k=1}^{5} \sin k\pi$

5. $\displaystyle\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$

6. $\displaystyle\sum_{k=1}^{4} (-1)^{k} \cos k\pi$

7. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?

   **a.** $\displaystyle\sum_{k=1}^{6} 2^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} 2^{k}$ **c.** $\displaystyle\sum_{k=-1}^{4} 2^{k+1}$

8. Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?

   **a.** $\displaystyle\sum_{k=1}^{6} (-2)^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} (-1)^{k} 2^{k}$ **c.** $\displaystyle\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

**9.** Which formula is not equivalent to the other two?

**a.** $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ **b.** $\sum_{k=0}^{2} \frac{(-1)^k}{k+1}$ **c.** $\sum_{k=-1}^{1} \frac{(-1)^k}{k+2}$

**10.** Which formula is not equivalent to the other two?

**a.** $\sum_{k=1}^{4} (k-1)^2$ **b.** $\sum_{k=-1}^{3} (k+1)^2$ **c.** $\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice of the lower limit of summation.

**11.** $1 + 2 + 3 + 4 + 5 + 6$ **12.** $1 + 4 + 9 + 16$

**13.** $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ **14.** $2 + 4 + 6 + 8 + 10$

**15.** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$ **16.** $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$

### Values of Finite Sums

**17.** Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

**a.** $\sum_{k=1}^{n} 3a_k$ **b.** $\sum_{k=1}^{n} \frac{b_k}{6}$ **c.** $\sum_{k=1}^{n} (a_k + b_k)$

**d.** $\sum_{k=1}^{n} (a_k - b_k)$ **e.** $\sum_{k=1}^{n} (b_k - 2a_k)$

**18.** Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

**a.** $\sum_{k=1}^{n} 8a_k$ **b.** $\sum_{k=1}^{n} 250b_k$

**c.** $\sum_{k=1}^{n} (a_k + 1)$ **d.** $\sum_{k=1}^{n} (b_k - 1)$

Evaluate the sums in Exercises 19–32.

**19. a.** $\sum_{k=1}^{10} k$ **b.** $\sum_{k=1}^{10} k^2$ **c.** $\sum_{k=1}^{10} k^3$

**20. a.** $\sum_{k=1}^{13} k$ **b.** $\sum_{k=1}^{13} k^2$ **c.** $\sum_{k=1}^{13} k^3$

**21.** $\sum_{k=1}^{7} (-2k)$ **22.** $\sum_{k=1}^{5} \frac{\pi k}{15}$

**23.** $\sum_{k=1}^{6} (3 - k^2)$ **24.** $\sum_{k=1}^{6} (k^2 - 5)$

**25.** $\sum_{k=1}^{5} k(3k + 5)$ **26.** $\sum_{k=1}^{7} k(2k + 1)$

**27.** $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$ **28.** $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

**29. a.** $\sum_{k=1}^{7} 3$ **b.** $\sum_{k=1}^{500} 7$ **c.** $\sum_{k=3}^{264} 10$

**30. a.** $\sum_{k=9}^{36} k$ **b.** $\sum_{k=3}^{17} k^2$ **c.** $\sum_{k=18}^{71} k(k-1)$

**31. a.** $\sum_{k=1}^{n} 4$ **b.** $\sum_{k=1}^{n} c$ **c.** $\sum_{k=1}^{n} (k-1)$

**32. a.** $\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right)$ **b.** $\sum_{k=1}^{n} \frac{c}{n}$ **c.** $\sum_{k=1}^{n} \frac{k}{n^2}$

### Riemann Sums

In Exercises 33–36, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\Sigma_{k=1}^{4} f(c_k)\,\Delta x_k$, given that $c_k$ is the **(a)** left-hand endpoint, **(b)** right-hand endpoint, **(c)** midpoint of the $k$th subinterval. (Make a separate sketch for each set of rectangles.)

**33.** $f(x) = x^2 - 1, \quad [0, 2]$ **34.** $f(x) = -x^2, \quad [0, 1]$

**35.** $f(x) = \sin x, \quad [-\pi, \pi]$ **36.** $f(x) = \sin x + 1, \quad [-\pi, \pi]$

**37.** Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.

**38.** Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

### Limits of Riemann Sums

For the functions in Exercises 39–46, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into $n$ equal subintervals and using the right-hand endpoint for each $c_k$. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

**39.** $f(x) = 1 - x^2$ over the interval $[0, 1]$.

**40.** $f(x) = 2x$ over the interval $[0, 3]$.

**41.** $f(x) = x^2 + 1$ over the interval $[0, 3]$.

**42.** $f(x) = 3x^2$ over the interval $[0, 1]$.

**43.** $f(x) = x + x^2$ over the interval $[0, 1]$.

**44.** $f(x) = 3x + 2x^2$ over the interval $[0, 1]$.

**45.** $f(x) = 2x^3$ over the interval $[0, 1]$.

**46.** $f(x) = x^2 - x^3$ over the interval $[-1, 0]$.

# 5.3 The Definite Integral

In Section 5.2 we investigated the limit of a finite sum for a function defined over a closed interval $[a, b]$ using $n$ subintervals of equal width (or length), $(b - a)/n$. In this section we consider the limit of more general Riemann sums as the norm of the partitions of $[a, b]$ approaches zero. For general Riemann sums the subintervals of the partitions need not have equal widths. The limiting process then leads to the definition of the *definite integral* of a function over a closed interval $[a, b]$.

## Definition of the Definite Integral

The definition of the definite integral is based on the idea that for certain functions, as the norm of the partitions of $[a, b]$ approaches zero, the values of the corresponding Riemann