King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101- Calculus I
Exam I
2009-2010 (092)

Tuesday, March 30, 2010 **Allowed Time: 2 hours**

Name: Solution Key

ID Number:

Section Number:__________ **Serial Number:**__________

Instructions:

1. Write neatly and eligibly. You may lose points for messy work.
2. **Show all your work. No points for answers without justification.**
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 8 different problems (5 pages + cover page)

Problem No	Grade	Maximum Points
1		12
2		7
3(a,b,c,d)		24
3(e,f)		14
4		8
5		7
6		8
7		10
8		10
Total		100

1. [**12 points**] Sketch the graph of a function f that satisfies the following conditions:

   (i) $f(4) = 1$. 1

   (ii) f has a removable discontinuity at $x = 4$. 3

   (iii) $\lim_{x \to 2^+} f(x) = 3$. 1

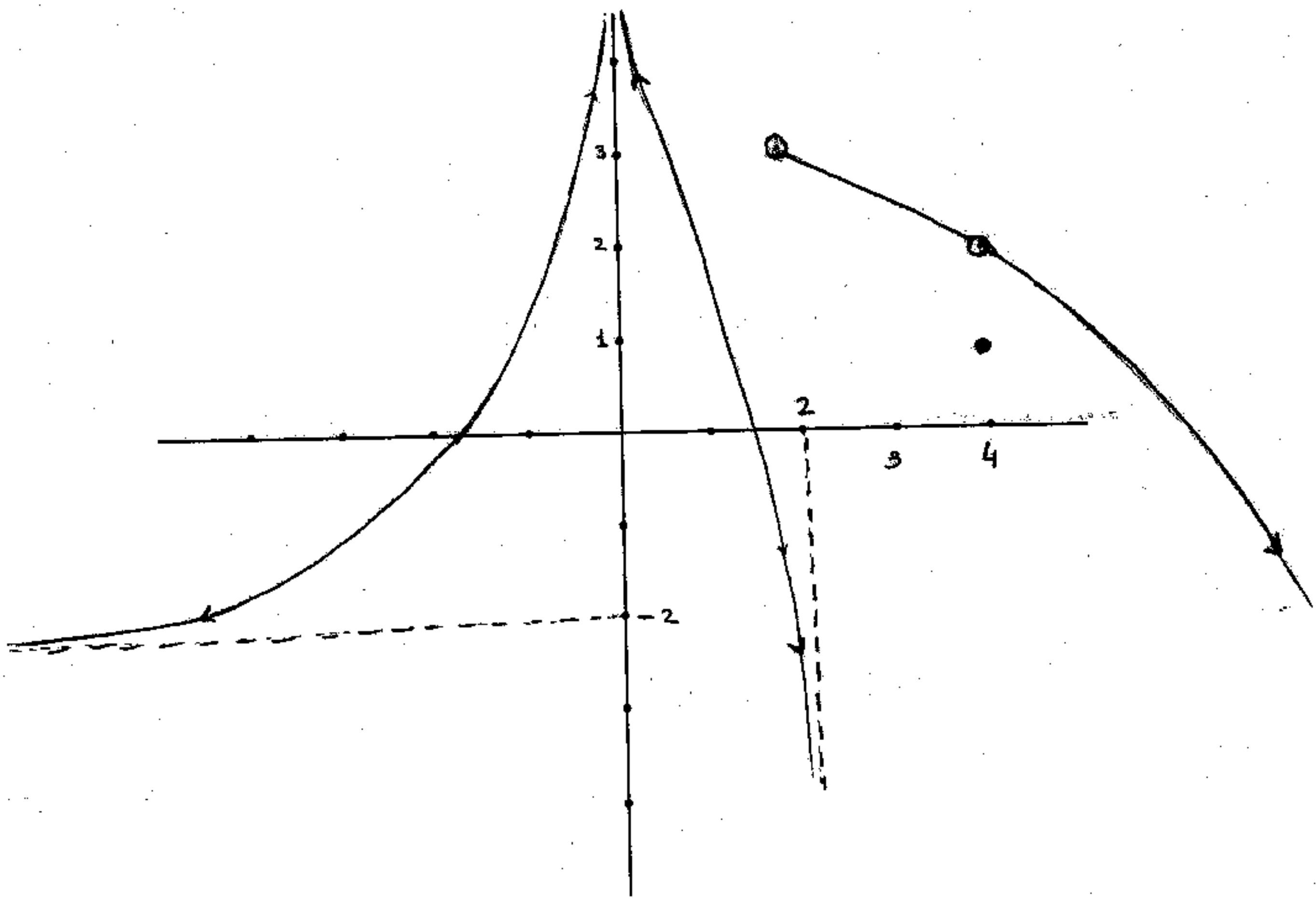
   (iv) $\lim_{x \to 2^-} f(x) = -\infty$. 2

   (v) $\lim_{x \to 0} f(x) = +\infty$. 2

   (vi) $\lim_{x \to -\infty} f(x) = -2$. 2

   (vii) $\lim_{x \to +\infty} f(x) = -\infty$. 1

one possible graph



2. [**7 points**] Where is the function $f(x) = \dfrac{\ln(1-x)}{\sqrt{1+x}}$ continuous?

1 • f is continuous in its domain

• Domain: Numerator $= \ln(1-x)$; we must have $1-x>0 \Rightarrow x<1$ 2

Denominator $= \sqrt{1+x}$; we must have $1+x>0 \Rightarrow x>-1$ 2

Domain is $(-1, 1)$

So f is continuous on $(-1,1)$ 2

3. Find the limit if it exists. Show your work.

(a) **[6 points]** $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} 6-2x & \text{if } 0 \le x \le 3 \\ x^2-10 & \text{if } 3 < x \le 5. \end{cases}$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2-10 = 9-10 = -1$ 2

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 6-2x = 6-6 = 0$ 2

Since $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$, 1

then $\lim_{x\to 3} f(x)$ does not exist. 1

(b) **[6 points]** $\lim_{x\to 1^-} \dfrac{|x^2+2x-3|}{x^2+x-2}$.

$= \lim_{x\to 1^-} \dfrac{|(x-1)(x+3)|}{(x-1)(x+2)}$ 2

$= \lim_{x\to 1^-} \dfrac{|x-1|\cdot|x+3|}{(x-1)(x+2)} = \lim_{x\to 1^-} \dfrac{-(x-1)\cdot|x+3|}{(x-1)(x+2)}$ 2

$= \lim_{x\to 1^-} -\dfrac{|x+3|}{(x+2)}$

$= -\dfrac{4}{3}$ 2

(c) **[6 points]** $\lim_{x\to 1}(x-1)^2 \sin\left(\dfrac{\pi}{x-1}\right)$.

1 $-1 \le \sin\left(\frac{\pi}{x-1}\right) \le 1$, $x \neq 1$

1 $\Rightarrow -(x-1)^2 \le (x-1)^2 \sin\left(\frac{\pi}{x-1}\right) \le (x-1)^2$, $x \neq 1$, $(x-1)^2 > 0$

1 Since $\lim_{x\to 1} -(x-1)^2 = 0 = \lim_{x\to 1} (x-1)^2$,

then $\lim_{x\to 1} (x-1)^2 \sin\left(\frac{\pi}{x-1}\right) = 0$ 2 by the Squeeze theorem. 1

(d) **[6 points]** $\lim_{x\to +\infty} (\sqrt{x} - 2x)$. $\infty - \infty$, not defined.

Factor the largest power of x;

$\lim_{x\to +\infty} (\sqrt{x} - 2x) = \lim_{x\to +\infty} x\left(\frac{1}{\sqrt{x}} - 2\right)$ 2

$= +\infty \cdot (0-2)$ 2

$= -\infty$ 2

(e) [6 points] $\lim_{x\to 0^+} \tan^{-1}\left(\frac{1}{x}\right)$.

$$x \longrightarrow 0^+ \Rightarrow \frac{1}{x} \longrightarrow +\infty \quad ③$$
$$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right) \longrightarrow \frac{\pi}{2} \quad ③$$

Thus $\lim_{x\to 0^+} \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

(f) [8 points] $\lim_{x\to 1} \frac{\sqrt{2-x^3}-1}{\sqrt{2x}-\sqrt{2}}$.

$$= \lim_{x\to 1} \frac{\sqrt{2-x^3}-1}{\sqrt{2x}-\sqrt{2}} \cdot \frac{\sqrt{2-x^3}+1}{\sqrt{2-x^3}+1} \cdot \frac{\sqrt{2x}+\sqrt{2}}{\sqrt{2x}+\sqrt{2}} \quad ②$$

$$= \lim_{x\to 1} \frac{1-x^3}{2x-2} \cdot \frac{\sqrt{2x}+\sqrt{2}}{\sqrt{2-x^3}+1} \quad ②$$

$$= \lim_{x\to 1} \frac{\cancel{(1-x)}(1+x+x^2)}{-2\cancel{(1-x)}} \cdot \frac{\sqrt{2x}+\sqrt{2}}{\sqrt{2-x^3}+1} \quad ②$$

$$= -\frac{3}{2}\sqrt{2} \quad ②$$

4. [8 points] Use the Intermediate Value Theorem to show that the equation

$$2 - e^x = \sqrt{x}$$

has a root between 0 and 1.

①+① Let $f(x) = e^x + \sqrt{x} - 2$, $[a,b] = [0,1]$

① Since • f is continuous on $[0,1]$

① • $f(0) = 1 + 0 - 2 = -1 < 0$ } opposite signs

① $f(1) = e + 1 - 2 = e - 1 > 0$

② Then, by the Intermediate Value Theorem, there is a number c in $(0,1)$ such that $f(c) = 0$;

i.e., $e^c + \sqrt{c} - 2 = 0$

or $2 - e^c = \sqrt{c}$

① Thus the given equation has a root c in $(0,1)$.

5. **[7 points]** Using the ϵ, δ definition of limit, prove that $\lim_{x \to -2} \left(\frac{x}{4} + 3 \right) = \frac{5}{2}$.

$f(x) = \frac{x}{4} + 3$, $a = -2$, $L = \frac{5}{2}$

. Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

If $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

(2) i.e, If $0 < |x+2| < \delta$, then $\left| \left(\frac{x}{4} + 3\right) - \frac{5}{2} \right| < \varepsilon$.

But $\left| \left(\frac{x}{4} + 3\right) - \frac{5}{2} \right| = \left| \frac{x}{4} + \frac{1}{2} \right| = \frac{1}{4}|x+2|$. So we want

(2) If $0 < |x+2| < \delta$, then $\frac{1}{4}|x+2| < \varepsilon$

(1) i.e, If $0 < |x+2| < \delta$, then $|x+2| < 4\varepsilon$

Thus we may choose $\boxed{\delta = 4\varepsilon}$ (2)

. Checking is optional.

6. **[8 points]** Find all values of the constant a that make f continuous on $(-\infty, +\infty)$:

$$f(x) = \begin{cases} x^2 - a & \text{if } x \leq 4 \\ a^2 x + 13 & \text{if } x > 4. \end{cases}$$

(1) . If $x < 4$, then $f(x) = x^2 - a$ is Continuous on $(-\infty, +\infty)$ for any a (f is a polynomial)

(1) . If $x > 4$, then $f(x) = a^2 x + 13$ " " " "

. It remains to check continuity at $x = 4$. For this we must have [for the limit $\lim_{x \to 4} f(x)$ to exist]

(2) $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x)$

(1) i.e, $\lim_{x \to 4^-} x^2 - a = \lim_{x \to 4^+} a^2 x + 13$

(1) $\Rightarrow$ $16 - a = 4a^2 + 13$

$\Rightarrow$ $4a^2 + a - 3 = 0 \Rightarrow (4a-3)(a+1) = 0$

(2) $\Rightarrow$ $\boxed{a = -1}$ or $\boxed{a = \frac{3}{4}}$

As in these Cases, $\lim_{x \to 4} f(x) = f(4)$, then f is Continuous on $(-\infty, +\infty)$ for these two values of a.

7. Let $f(x) = \dfrac{4 - 3x^2}{(x-2)^2}$.

(a) **[6 points]** Find the horizontal asymptotes of f. Justify your answer.

We find $\lim_{x\to+\infty} f(x)$ & $\lim_{x\to-\infty} f(x)$.

3 • $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \dfrac{4-3x^2}{x^2-4x+4}$

$= \lim_{x\to+\infty} \dfrac{x^2\left(\frac{4}{x^2}-3\right)}{x^2\left(1-\frac{4}{x}+\frac{4}{x^2}\right)} = \lim_{x\to+\infty} \dfrac{\frac{4}{x^2}-3}{1-\frac{4}{x}+\frac{4}{x^2}} = \dfrac{0-3}{1-0+0} = -3$

1 • Similarly $\lim_{x\to-\infty} f(x) = -3$

Then the line $\boxed{y=-3}$ is the only horizontal asymptote of f.
2

(b) **[4 points]** Find the vertical asymptotes of f. Justify your answer.

• Denominator $= (x-2)^2 = 0 \implies x = 2$

2 • $\lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{4-3x^2}{(x-2)^2} = -\infty$

So the line $\boxed{x=2}$ is the only Vertical asymptote for f.
2

8. **[10 points]** Find an equation of the tangent line to the curve $y = \dfrac{x-1}{x-2}$ at the point $\left(0, \dfrac{1}{2}\right)$. [You must use limits].

• Slope $= \lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0}$ 2

$= \lim_{x\to 0} \dfrac{\frac{x-1}{x-2}-\frac{1}{2}}{x}$ 1

$= \lim_{x\to 0} \dfrac{\frac{x}{2(x-2)}}{x}$ 1

$= \lim_{x\to 0} \dfrac{1}{2(x-2)}$ 1

$= -\dfrac{1}{4}$ 2

• Equation:

$y - \frac{1}{2} = -\frac{1}{4}(x-0)$ 2

$\implies y = -\frac{1}{4}x + \frac{1}{2}$ 1