King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

# MATH 101 - Exam I - Term 131

**Duration: 120 minutes**

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Name:______________________ ID Number:______________________

Section Number:______________________ Serial Number:______________________

Class Time: ______________________ Instructor's Name:______________________

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**Instructions:**

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 6 pages of problems (Total of 8 Problems)

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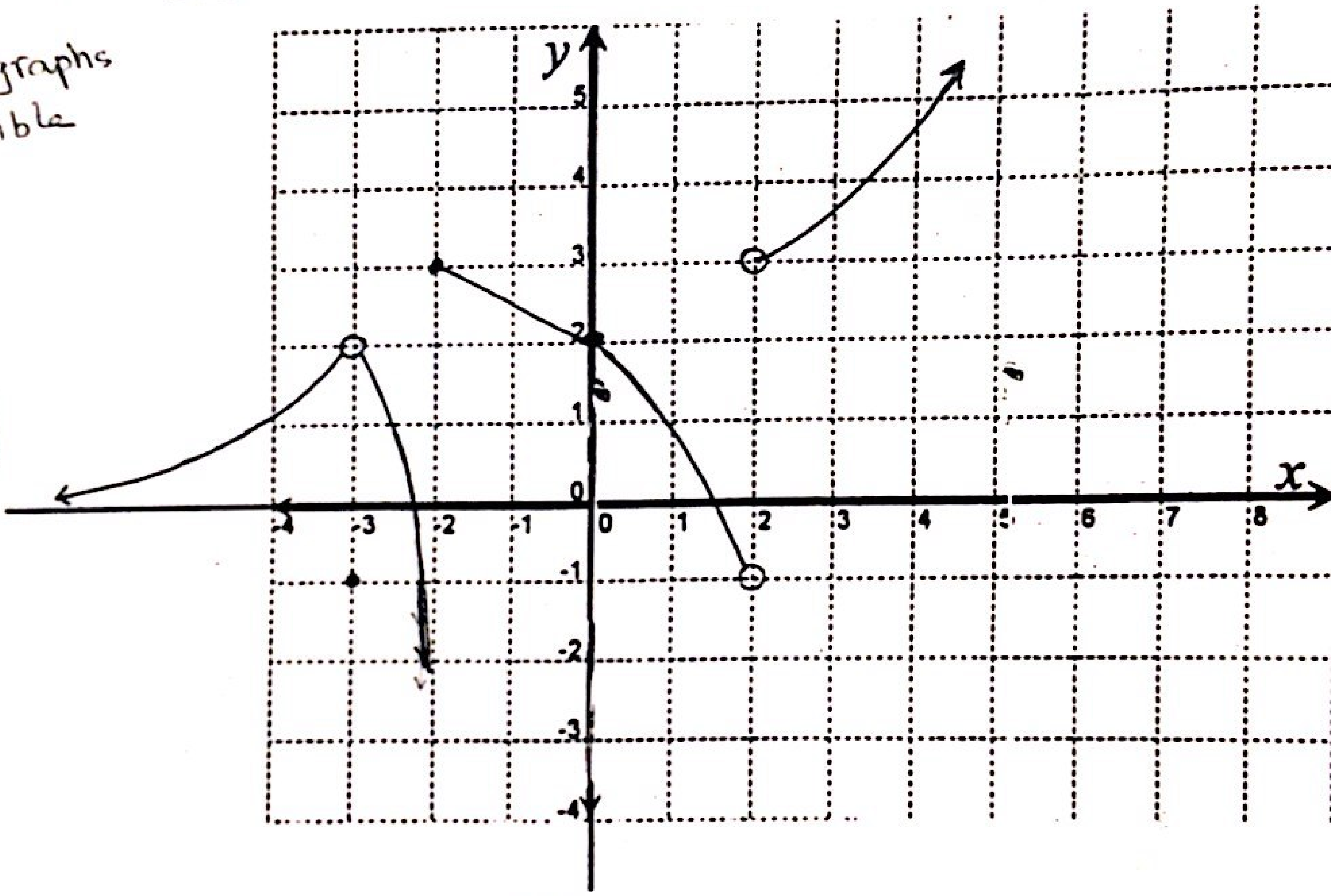
| | Points | Maximum Points |
|---|---|---|
| page 1 | | 20 |
| page 2 | | 20 |
| page 3 | | 15 |
| page 4 | | 17 |
| page 5 | | 18 |
| page 6 | | 10 |
| Total | | 100 |

1. (15 points) Sketch the graph of a function $f$ that satisfies the following conditions:

| | |
|---|---|
| 2 (i) $\lim_{x\to-\infty} f(x) = 0$ | 2 (v) $\lim_{x\to-2^+} f(x) = 3$ |
| 1 (ii) $f(-3) = -1$ | 2 (vi) $\lim_{x\to0} f(x) = 2$ |
| 2 (iii) $\lim_{x\to-3} f(x) = 2$ | 2 (vii) $f$ has a jump discontinuity at $x = 2$ |
| 2 (iv) $\lim_{x\to-2^-} f(x) = -\infty$ | 2 (viii) $\lim_{x\to\infty} f(x) = \infty$ |

Other graphs are possible



2. (5 points) Where is $f(x) = \dfrac{x+2}{x^2+x-2}$ continuous ?

Since $f$ is a rational function, It is Continuous everywhere except at the zeros of the denominator:

$x^2+x-2=0 \Rightarrow (x+2)(x-1)=0$ (2)

$\Rightarrow x=-2, 1$

So $f$ is Continuous on

$(-\infty,-2)\cup(-2,1)\cup(1,\infty)$ (3)

3. **(10 points)** Use the graph of $f(x) = \dfrac{1}{x}$ to find a number $\delta > 0$ such that for all $x$,

$$0 < |x-2| < \delta \Rightarrow \left|f(x) - \frac{1}{2}\right| < \frac{1}{8}$$

- $\varepsilon = \frac{1}{8}$
- From the graph

$f(x_1) = \frac{5}{8} \Rightarrow \frac{1}{x_1} = \frac{5}{8} \Rightarrow \boxed{x_1 = \frac{8}{5}}$ 2 (1)

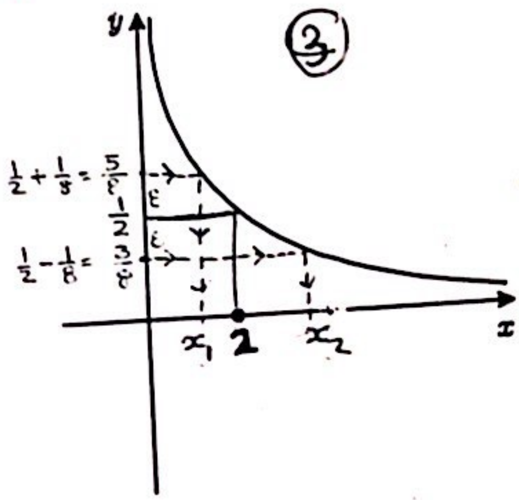
$f(x_2) = \frac{3}{8} \Rightarrow \frac{1}{x_2} = \frac{3}{8} \Rightarrow \boxed{x_2 = \frac{8}{3}}$ 2 (1)

- $\delta$ = minimum of $\{2 - x_1, x_2 - 2\}$ (2)

= minimum of $\{\frac{2}{5}, \frac{2}{3}\}$ (2)

$= \boxed{\frac{2}{5}}$ 5 (1)

(or any smaller positive number)




4. **(10 points)** For what values of $a$ and $b$ is

$$g(x) = \begin{cases} ax - 2b & x \le 0 \\ x^2 + 3a - b & 0 < x \le 2 \\ 3x - 5 & x > 2 \end{cases}$$

continuous at every $x$?

- $g$ is Continuous on $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$ as It is a polynomial on each of these intervals
- For $g$ to be continuous at every $x$, we have to check continuity at $x=0$, $x=2$

For this to happen we must have

- 2 $\boxed{\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x)}$ $\left(= g(0)\right)$ (2)

$\lim_{x\to 0^-}(ax - 2b) = \lim_{x\to 0^+}(x^2 + 3a - b) \Rightarrow -2b = 3a - b$ 2

$\Rightarrow 3a + b = 0$ ~~~ (1) (2)

- 2 $\boxed{\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x)}$ $\left(= g(2)\right)$ (2)

$\lim_{x\to 2^-}(x^2 + 3a - b) = \lim_{x\to 2^+}(3x - 5) \Rightarrow 4 + 3a - b = 6 - 5$ 2

$\Rightarrow 3a - b = -3$ ~~~ (2) (2)

Solving (1) & (2), we get $\boxed{a = -\frac{1}{2} \ \& \ b = \frac{3}{2}}$ 2 (2)

5. Find the limit if it exists. Justify your work.

a) (6 points) $\boxed{\lim_{x\to 2} \frac{\sqrt{x+7}-3}{x^3-4x} \cdot \frac{\sqrt{x+7}+3}{\sqrt{x+7}+3}}$ 2 (2)

$= \lim_{x\to 2} \frac{(x+7)-9}{(x^3-4x)(\sqrt{x+7}+3)}$ (1)

$= \lim_{x\to 2} \frac{x-2}{x(x-2)(x+2)(\sqrt{x+7}+3)}$ (1)

$= \lim_{x\to 2} \frac{1}{x(x+2)(\sqrt{x+7}+3)}$ (1)

$= \frac{1}{2\cdot 4(3+3)} \boxed{= \frac{1}{48}}$ 4 (1)

b) (5 points) $\lim_{x\to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x^3+8}$

$= \lim_{x\to -2} \frac{\frac{2+x}{2x}}{x^3+8}$ (1)

$= \lim_{x\to -2} \frac{\cancel{(2+x)}}{2x} \cdot \frac{1}{\cancel{(x+2)}(x^2-2x+4)}$ (3) (2)

$= \lim_{x\to -2} \frac{1}{2x(x^2-2x+4)}$ (1)

$= \frac{1}{-4(4+4+4)} = \boxed{-\frac{1}{48}}$ (1) (2)

c) (4 points) $\lim_{x\to 1^+} \frac{x}{1-x}$ $= \left(\lim_{x\to 1^+} x\right)\left(\lim_{x\to 1^+} \frac{1}{1-x}\right) = (1)\cdot(-\infty) = -\infty$ (2)

As $x \to 1^+$, $x \to 1$ &

$1-x \to 0^-$ (through negative values) (2) (1)

then

$\lim_{x\to 1^+} \frac{x}{1-x} = -\infty$ (3)

d) **(6 points)** $\lim_{x\to\infty}(\sqrt{3x+5}-\sqrt{x+5})$

① $= \lim_{x\to\infty} (\sqrt{3x+5}-\sqrt{x+5}) \cdot \boxed{\frac{\sqrt{3x+5}+\sqrt{x+5}}{\sqrt{3x+5}+\sqrt{x+5}}}$ ②

① $= \lim_{x\to\infty} \frac{2x}{\sqrt{3x+5}+\sqrt{x+5}}$

① $= \lim_{x\to\infty} \frac{2x}{\sqrt{x}\left[\sqrt{3+\frac{5}{x}}+\sqrt{1+\frac{5}{x}}\right]}$

① $= \lim_{x\to\infty} \boxed{\frac{2\sqrt{x}}{\sqrt{3+\frac{5}{x}}+\sqrt{1+\frac{5}{x}}}}$ ② $= \frac{\infty}{\sqrt{3}+1}$ ② $= \boxed{\infty}$ ②

e) **(5 points)** $\lim_{x\to\infty}\frac{2+\sin x}{x}$.

$-1 \le \sin x \le 1$ for all $x$

① $\Rightarrow 1 \le 2+\sin x \le 3$

① $\Rightarrow \frac{1}{x} \le \frac{2+\sin x}{x} \le \frac{3}{x}$ $(x>0$ as $x\to\infty)$

① as $\lim_{x\to\infty}\frac{1}{x} = 0 = \lim_{x\to\infty}\frac{3}{x}$, then

② $\lim_{x\to\infty}\frac{2+\sin x}{x} \boxed{=0}$, by the Squeeze Theorem (Sandwich Theorem)

$= \lim_{x\to\infty}\frac{2}{x}$ ① $+ \lim_{x\to\infty}\frac{\sin x}{x}$ ②

$= 0 + 0$

$= \boxed{0}$ ②

by Sandwitch

f) **(6 points)** $\lim_{x\to 0^-}\frac{|x\cos(2x)-\frac{x}{2}|}{\sin(3x)}$

$= \lim_{x\to 0^-}\frac{\boxed{|x|}\,|\cos(2x)-\frac{1}{2}|}{\sin(3x)}$ 2 ②

$= \lim_{x\to 0^-}\frac{-x\,|\cos(2x)-\frac{1}{2}|}{\sin(3x)}$ ①

$= \lim_{x\to 0^-} - \frac{|\cos(2x)-\frac{1}{2}|}{\boxed{3\cdot\frac{\sin(3x)}{3x}}}$ 2 ①

$= -\frac{1/2}{3\cdot 1}$ ①

$= \boxed{-\frac{1}{6}}$ 2 ①

• There are no vertical asymptotes since $|x-1|^3+9 \neq 0$ for all real numbers $x$.

②



6. **(10 points)** Find the horizontal and vertical asymptotes of the curve $y = \dfrac{x^3 - 3x + 1}{|x-1|^3 + 9}$. Justify your answer using limits.

• For H.A., we have to find $\lim_{x \to \pm\infty} f(x)$ :

* $\lim_{x\to\infty} \dfrac{x^3-3x+1}{|x-1|^3+9} = \lim_{x\to\infty} \dfrac{x^3-3x+1}{(x-1)^3+9}$ ①: $x \to \infty \Rightarrow x-1>0$ & so $|x-1| = x-1$

$= \lim_{x\to\infty} \dfrac{1 - \frac{3}{x^2} + \frac{1}{x^3}}{(1-\frac{1}{x})^3 + \frac{9}{x^3}}$ ①

$= \dfrac{1-0+0}{1+0} = 1$ ①

4

* $\lim_{x\to-\infty} \dfrac{x^3-3x+1}{|x-1|^3+9} = \lim_{x\to-\infty} \dfrac{x^3-3x+1}{(1-x)^3+9}$ ①: as $x \to -\infty \Rightarrow x-1<0$ & so $|x-1| = 1-x$

$= \lim_{x\to-\infty} \dfrac{1 - \frac{3}{x^2} + \frac{1}{x^3}}{(\frac{1}{x}-1)^3 + \frac{9}{x^3}}$ ①

$= \dfrac{1-0+0}{-1+0} = -1$ ①

4

So the H.A. are $y=1$ and $y=-1$ ②

2

7. **(8 points)** Use the Intermediate Value Theorem to show that the equation $x^2 - \cos(\pi x) = 4$ has a solution.

• Let $f(x) = x^2 - \cos(\pi x) - 4$ ②

• $f(0) = -5$ ; $f(1) = -2$ ; $f(2) = -1$ ; $f(3) = 6$

Since $f$ is continuous on $(-\infty, \infty)$ & hence on $[2,3]$ ②

4

& $f(2) = -1 < 0$ & $f(3) = 6 > 0$ ②

then, by the Intermediate Value Theorem,

$f$ has a zero in $(2,3)$ ②

4

& so the equation $x^2 - \cos(\pi x) = 4$ has a solution in $(2,3)$.

8. **(10 points)** Use limits to find the equation of the tangent line to the graph of $f(x) = x - \frac{1}{x}$ at $x = 3$.

2

- $\boxed{\text{Slope} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}}$ 2

$= \lim_{h \to 0} \frac{1}{h} \left[ (3+h) - \frac{1}{3+h} - \frac{8}{3} \right]$

$= \lim_{h \to 0} \frac{1}{h} \left[ \frac{8+6h+h^2}{3+h} - \frac{8}{3} \right]$ 1

$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{24+18h+3h^2-24-8h}{3(3+h)}$ 1

$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{3h^2+10h}{3(3+h)}$ 1

$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{h(3h+10)}{3(3+h)}$

$= \lim_{h \to 0} \frac{3h+10}{3(3+h)} =$ 1

$= \boxed{\frac{10}{9}}$ 5 1

- point of tangency is $\boxed{\left(3, \frac{8}{3}\right)}$ 1 1

- The equation of the tangent line is

$\boxed{y - \frac{8}{3} = \frac{10}{9}(x-3)}$ 2 2

or $y = \frac{10}{9}x - \frac{2}{3}$

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