King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101 - Calculus I
Exam I
Term (121)

Tuesday, October 2, 2012 **Time Allowed: 2 hours**

Name: Solutions ID Number: KEY

Section Number: ______ Serial Number: ______

Class Time: ______ Instructor's Name: ______

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 10 Problems)

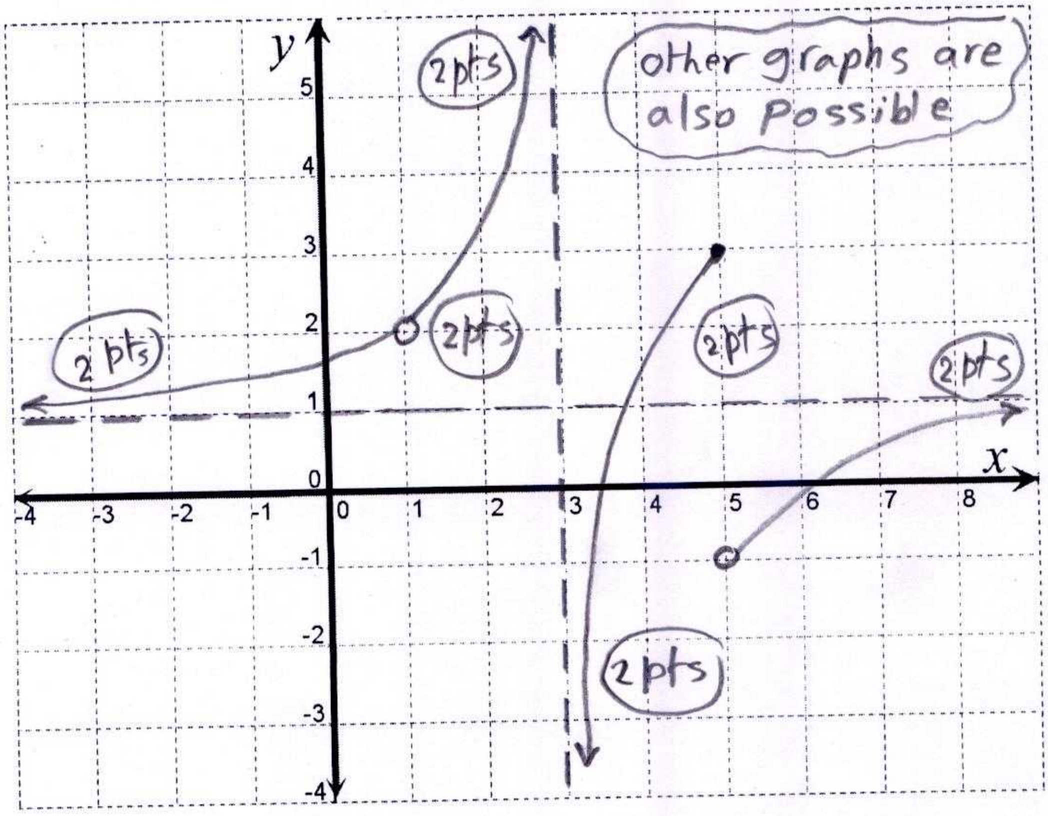
	Points	Maximum Points
Page 1		20
Page 2		13
Page 3		12
Page 4		17
Page 5		22
Page 6		10
Page 7		6
Total		100

1. (2 points) Find the average rate of change of the function $f(x) = x^3 + 1$ over the interval $[-1, 1]$.

$$\frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1 \quad \text{2 pts}$$

2. (12 points) Sketch the graph of a function f that satisfies the following conditions:

(i) $\lim_{x\to\pm\infty} f(x) = 1$, (ii) $\lim_{x\to 3^-} f(x) = \infty$, (iii) $\lim_{x\to 3+} f(x) = -\infty$,
(iv) f has a removable discontinuity at 1, (v) f has a jump discontinuity at 5.



3. (6 points) Using the Sandwich Theorem, show that if $\lim_{x\to 1} |f(x)| = 0$, then $\lim_{x\to 1} f(x) = 0$.

Since $-|f(x)| \leq f(x) \leq |f(x)|$ 2 pts

and $\lim_{x\to 1} |f(x)| = 0 \Rightarrow \lim_{x\to 1} -|f(x)| = 0$, 2pts

it follows from the Sandwich Theorem

that $\lim_{x\to 1} f(x) = 0$ 2pts

4. Evaluate the limit or show that it does not exist.

i) (4 points) $\lim\limits_{x\to 2^-} \dfrac{|x-2|(x-4)}{(x-2)}$

$$= \lim_{x\to 2^-} \frac{-(x-2)(x-4)}{(x-2)}$$ 2 pts

$$= \lim_{x\to 2^-} -(x-4) = -(-2) = 2$$ 2 pts

ii) (6 points) $\lim\limits_{\theta\to 0} \tan 3\theta \cdot \csc\theta$

$$= \lim_{\theta\to 0}\left(\frac{\tan 3\theta}{3\theta} \cdot 3 \cdot \frac{\theta}{\sin\theta}\right)$$ 3 pts

$$\left.\begin{aligned} &= 3 \lim_{\theta\to 0} \frac{\tan 3\theta}{3\theta} \cdot \lim_{\theta\to 0} \frac{\theta}{\sin\theta} \\ &= 3 \cdot 1 \cdot 1 = 3 \end{aligned}\right\}$$ 3 pts

iii) (3 points) $\lim\limits_{x\to 1} \left[\dfrac{1}{x}\right]$, where $[y]$ is the greatest integer less than or equal to y.

Since $\lim\limits_{x\to 1^-} \left[\frac{1}{x}\right] = 1$ 1 pt

and $\lim\limits_{x\to 1^+} \left[\frac{1}{x}\right] = 0$, 1 pt

it follows that $\lim\limits_{x\to 1} \left[\frac{1}{x}\right]$ DNE 1 pt

iv) (6 points) $\lim_{x\to 3} \dfrac{x^2-9}{\sqrt{x^2+7}-4}$

$$= \lim_{x\to 3} \frac{(x^2-9)(\sqrt{x^2+7}+4)}{(\sqrt{x^2+7}-4)(\sqrt{x^2+7}+4)} \quad 2 \text{ pts}$$

$$= \lim_{x\to 3} \frac{(x^2-9)(\sqrt{x^2+7}+4)}{(x^2+7)-16} = \lim_{x\to 3} \frac{(x^2-9)(\sqrt{x^2+7}+4)}{x^2-9} \quad 2 \text{pts}$$

$$= \lim_{x\to 3} \left(\sqrt{x^2+7}+4\right) = \sqrt{16}+4 = 8 \quad 2 \text{pts}$$

v) (6 points) $\lim_{x\to 0} \dfrac{x - x\cos x}{\sin^2 x}$

$$= \lim_{x\to 0} \frac{x(1-\cos x)}{\sin^2 x} \quad 1 \text{ pt}$$

$$= \lim_{x\to 0} \frac{x\,\dfrac{(1-\cos x)}{x^2}}{\dfrac{\sin^2 x}{x^2}} \quad 2 \text{ pts}$$

$$= \lim_{x\to 0} \frac{\dfrac{1-\cos x}{x}}{\left(\dfrac{\sin x}{x}\right)^2} \quad 1 \text{ pt}$$

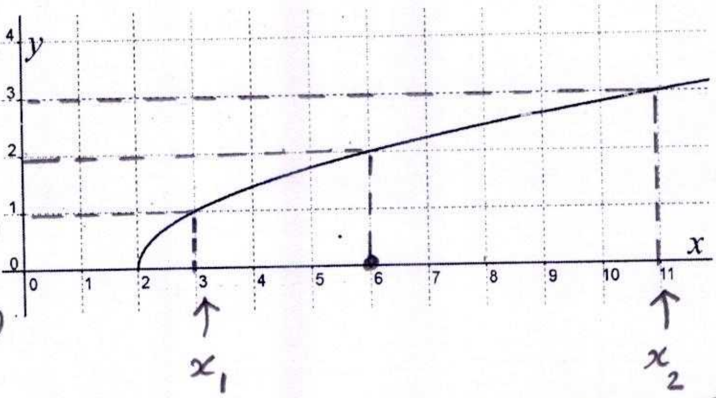
$$= \frac{0}{1} = 0 \quad 2 \text{ pts}$$

5. (9 points) Use the graph of $f(x) = \sqrt{x-2}$ to find $\delta > 0$ such that if $0 < |x-6| < \delta$, then $|f(x) - 2| < 1$

$|f(x) - 2| < 1 \Rightarrow \varepsilon = 1$ (1pt)

From the graph:

- $f(x_1) = 1 \Rightarrow x_1 = 3$ (2pts)
- $f(x_2) = 3 \Rightarrow x_2 = 11$ (2pts)



- $6 - x_1 = 6 - 3 = 3$ and $x_2 - 6 = 11 - 6 = 5$ (2pts)

$\Rightarrow \delta = 3$ (or $0 < \delta \leq 3$) (2pts)

6. (8 points) Use the Intermediate Value Theorem to prove that the equation $x^3 - 3x - 1 = 0$ has a solution.

Let $f(x) = x^3 - 3x - 1$ (2pts)

$\Rightarrow$ f is continuous everywhere because it is a polynomial. (2pts)

If $x = -1$, then $f(-1) = 1$, (1pt)

if $x = 0$, then $f(0) = -1$ (1pt)

other values of x are also possible

Then $x^3 - 3x - 1 = 0$ for some x between -1 and 0 according to the Intermediate Value Theorem (2pts)

7. (12 points) Use limits to find the values of a and b for which the function

$$g(x) = \begin{cases} ax + 5b, & x \le 0 \\ x^2 + a - 3b, & 0 < x \le 2 \\ 5x - 3, & x > 2 \end{cases}$$

is continuous at every x.

g is continuous everywhere except possibly at 0 and 2 (1pt)

g is continuous at 0 and 2 if:

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = g(0) \quad (1pt)$$

$$\Rightarrow \lim_{x\to 0^-} (ax+5b) = \lim_{x\to 0^+} (x^2+a-3b) = 5b \quad (2pts)$$

$$\Rightarrow 5b = a - 3b \Rightarrow a = 8b. \quad (A) \quad (2pts)$$

and $$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x) = g(2) \quad (1pt)$$

$$\Rightarrow 4 + a - 3b = 10 - 3 \Rightarrow a = 3b + 3 \quad (B) \quad (2pts)$$

$$(A), (B) \Rightarrow 8b = 3b + 3 \Rightarrow 5b = 3 \Rightarrow \boxed{b = \frac{3}{5}} \quad (2pts)$$

$$\Rightarrow \boxed{a = 24/5} \quad (1pt)$$

8. (10 points) Use limits to find the horizontal asymptotes of the curve $y = \dfrac{4 - 3x^3}{\sqrt{x^6 + 9}}$

$$\lim_{x\to -\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to -\infty} \frac{x^3\left(\frac{4}{x^3} - 3\right)}{|x^3|\sqrt{1+\frac{9}{x^6}}} = \lim_{x\to -\infty} \frac{-\left(\frac{4}{x^3} - 3\right)}{\sqrt{1+\frac{9}{x^6}}} \quad (3pts)$$

$$= \frac{-(-3)}{1} = 3 \quad (1pt)$$

$$\lim_{x\to \infty} \frac{4-4x^3}{\sqrt{x^6+4}} = \lim_{x\to \infty} \frac{x^3\left(\frac{4}{x^3} - 3\right)}{|x^3|\sqrt{1+\frac{9}{x^6}}} = \lim_{x\to \infty} \frac{\left(\frac{4}{x^3} - 3\right)}{\sqrt{1+9/x^6}} \quad (3pts)$$

$$= \frac{-3}{1} = -3 \quad (1pt)$$

$\Rightarrow$ $y = 3$ and $y = -3$ are the horizontal asymptotes (2pts)

9. (10 points) Use limits to find all asymptotes of the curve $y = \dfrac{x^2+1}{x-1}$.

- $\lim_{x \to 1^-} y = -\infty$ (or $\lim_{x \to 1^+} y = \infty$) 1 pt

$\Rightarrow$ $x = 1$ is a vertical asymptote 1 pt

- $$y = \frac{x^2+1}{x-1} = \frac{x^2\left(1+\frac{1}{x^2}\right)}{x\left(1-\frac{1}{x}\right)} = \frac{x\left(1+\frac{1}{x}\right)}{\left(1-\frac{1}{x}\right)}$$ 1 pt

$\Rightarrow$ $\lim_{x \to \infty} y = \infty$ and $\lim_{x \to -\infty} y = -\infty$ 2 pts

$\Rightarrow$ No horizontal asymptotes 1 pt

- $$y = x+1 + \frac{2}{x-1}$$ 1 pt $\Leftarrow$

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2+1 \\ & x^2-x \\ \hline & x+1 \\ & x-1 \\ \hline & 2 \end{array}$$

$\Rightarrow$ $$\lim_{x \to \pm\infty} (y - (x+1)) = \lim_{x \to \pm\infty} \frac{2}{x-1} = 0$$ 2 pts

$\Rightarrow$ $y = x+1$ is an oblique (slant) asymptote

1 pt

10. To each of the following statements, give an example to show that the statement is **False:**

(a) (3 points) If $\lim_{x\to 1}(f(x)\cdot g(x))$ exists, then the limit must be $f(1)\cdot g(1)$

Let $f(x) = x-1$ and $g(x) = \frac{1}{x-1}$

$$\Rightarrow \lim_{x\to 1}(f(x)\cdot g(x)) = \lim_{x\to 1} 1 = 1$$

$$\neq f(1)\cdot g(1)$$

Other examples are also possible

(b) (3 points) If f has domain $(0,\infty)$ and has no horizontal asymptotes, then $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$

$f(x) = \sin x$, $x \geq 0$

has no horizontal asymptotes

with $\lim_{x\to\infty} f(x) \neq \infty$ and $\lim_{x\to\infty} f(x) \neq -\infty$

Other examples are also possible