King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101- Calculus I
Exam I
2010-2011 (Term 102)

Saturday, March 26, 2011 **Allowed Time: 2 hours**

Name: ______________________________

ID Number: __________

Section Number:__________ **Serial Number:**__________

Instructions:

1. Write neatly and eligibly. You may lose points for messy work.
2. **Show all your work. No points for answers without justification.**
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 10 different problems (6 pages + cover page)

Page No	Grade	Maximum Points
Page 1 (# 1, 2)		17
Page 2 (# 3)		27
Page 3 (# 4, 5)		18
Page 4 (# 6,7)		16
Page 5 (# 8, 9)		13
Page 6 (# 10)		9
Total		100

1. **[10 points]** Sketch the graph of **a function** f that satisfies all of the following conditions:

1 (i) $f(0) = 3$ and $f(-2) = 3$.

1 (ii) $\lim_{x \to 0} f(x) = 1$.

1 (iii) $\lim_{x \to 1^-} f(x) = 2$.

2 (iv) $\lim_{x \to 1^+} f(x)$ does not exist.

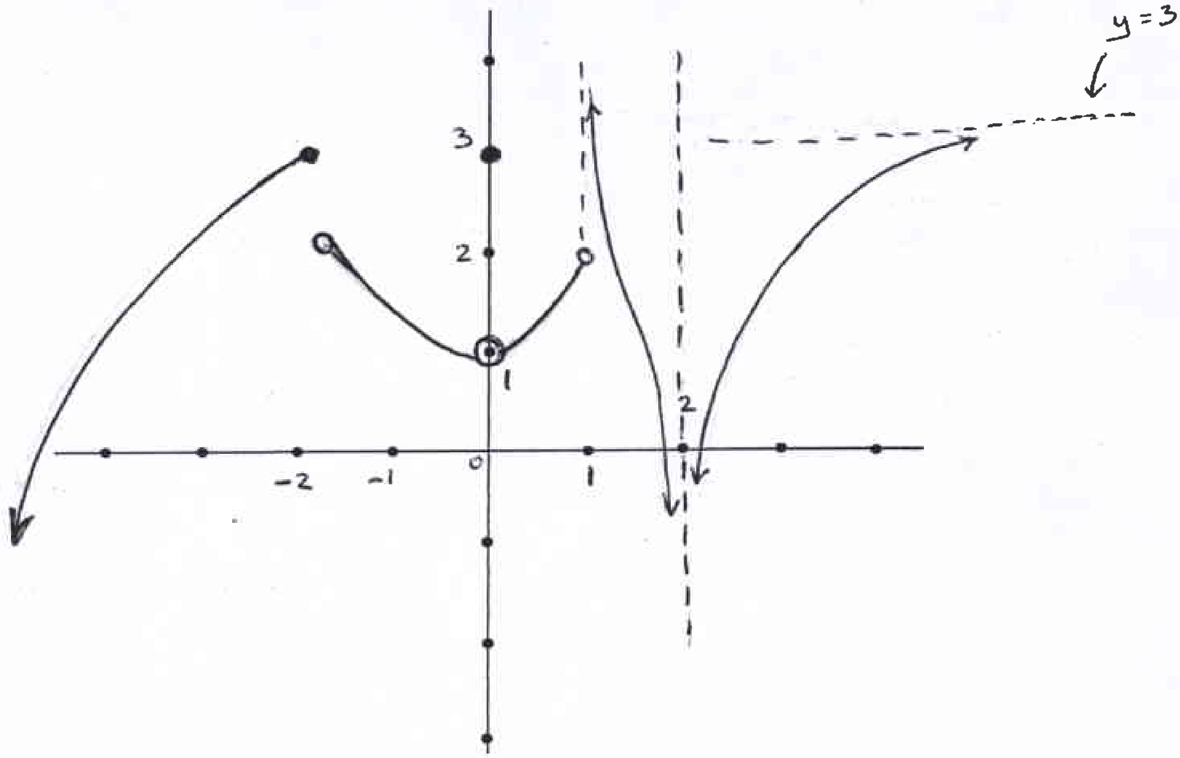
1 (v) f has a vertical asymptote at $x = 2$.

2 (vi) f has a jump discontinuity at $x = -2$.

1 (vii) $\lim_{x \to -\infty} f(x) = -\infty$.

1 (viii) $\lim_{x \to \infty} f(x) = 3$.

One possible graph



2. **[7 points]** Use the Intermediate Value Theorem to show that the equation

$$x^{10} - 100x + 5 = 0$$

has a root in the interval $(1, 2)$.

Let $f(x) = x^{10} - 100x + 5$, $[a,b] = [1,2]$ 1

Since i. f is Continuous on $[1,2]$, (a polynomial) 1

ii. $f(1) = -94 < 0$ 3

$f(2) = 829 > 0$

then, by the IVT, there is a number c in $(1,2)$ such that $f(c) = 0$; i.e,

$c^{10} - 100c + 5 = 0$ 2

Thus the given equation has a root c in $(1,2)$.

3. Find each of the following limits. Show your work.

(a) **[7 points]** $\displaystyle\lim_{x\to 7} \frac{x-7}{\sqrt{7x}-7}$.

$$= \lim_{x\to 7} \frac{x-7}{\sqrt{7x}-7} \cdot \frac{\sqrt{7x}+7}{\sqrt{7x}+7} \quad ②$$

$$= \lim_{x\to 7} \frac{(x-7)(\sqrt{7x}+7)}{7x-49} \quad ②$$

$$= \lim_{x\to 7} \frac{\sqrt{7x}+7}{7} \quad ②$$

$$= \frac{7+7}{7} = 2 \quad ①$$

(b) **[7 points]** $\displaystyle\lim_{x\to 0} \frac{(3+2x)^{-1}-3^{-1}}{2x}$.

$$= \lim_{x\to 0} \frac{\frac{1}{3+2x} - \frac{1}{3}}{2x} \quad ①$$

$$= \lim_{x\to 0} \frac{\frac{3-3-2x}{3(3+2x)}}{2x} \quad ②$$

$$= \lim_{x\to 0} \frac{-2x}{3(3+2x)} \cdot \frac{1}{2x} \quad ①$$

$$= \lim_{x\to 0} \frac{-1}{3(3+2x)} \quad ② \quad = \frac{-1}{9} \quad ①$$

(c) **[5 points]** $\displaystyle\lim_{x\to\infty} (10+x^4-3x^7)$.

$$= \lim_{x\to\infty} x^7\left(\frac{10}{x^7} + \frac{1}{x^3} - 3\right) \quad ②$$

$$= \infty\,(0+0-3) \quad ②$$

$$= -\infty \quad ①$$

(d) **[8 points]** $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{4x^2-2x+1}}{|1-x|}$.

$$\lim_{x \to -\infty} \frac{\sqrt{4x^2-2x+1}}{|1-x|}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{4x^2-2x+1}}{1-x} \quad 2 \quad (x \to -\infty \Rightarrow x<0 \Rightarrow |1-x| = 1-x)$$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2\left(4-\frac{2}{x}+\frac{1}{x^2}\right)}}{x\left(\frac{1}{x}-1\right)} \quad 2$$

$$= \lim_{x \to -\infty} \frac{|x|\sqrt{4-\frac{2}{x}+\frac{1}{x^2}}}{x\left(\frac{1}{x}-1\right)} \quad 1$$

$$= \lim_{x \to -\infty} \frac{-x\sqrt{4-\frac{2}{x}+\frac{1}{x^2}}}{x\left(\frac{1}{x}-1\right)} \quad 1 : (x \to -\infty \Rightarrow x<0 \Rightarrow |x| = -x)$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{4-\frac{2}{x}+\frac{1}{x^2}}}{\frac{1}{x}-1} \quad 1$$

$$= \frac{-\sqrt{4-0+0}}{0-1} = 2 \quad 1$$

$\underline{\text{OR}}$ $$\lim_{x \to -\infty} \frac{\sqrt{4x^2-2x+1}}{|1-x|}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{4x^2-2x+1}}{1-x} \quad 2 \quad (x \to -\infty \Rightarrow x<0 \Rightarrow |1-x| = 1-x)$$

$$= \lim_{x \to -\infty} \frac{\frac{1}{x}\sqrt{4x^2-2x+1}}{\frac{1}{x}(1-x)} \quad 2$$

$$= \lim_{x \to -\infty} \frac{-\frac{1}{\sqrt{x^2}}\sqrt{4x^2-2x+1}}{\frac{1}{x}(1-x)} \quad 2 \qquad (\text{when } x<0,\ \sqrt{x^2} = -x)$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{4-\frac{2}{x}+\frac{1}{x^2}}}{\frac{1}{x}-1} \quad 1$$

$$= \frac{-\sqrt{4-0+0}}{0-1} = 2 \quad 1$$

4. [**9 points**] Is $f(x) = [\![x]\!] + [\![-x]\!]$ continuous at $x = 0$. Use limits to justify your answer.

(1) We need to check whether $\lim_{x \to 0} f(x) = f(0)$

(1) • $f(0) = [\![0]\!] + [\![-0]\!] = 0 + 0 = 0$

(2) • $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} [\![x]\!] + [\![-x]\!] = 0 + (-1) = -1$

(2) • $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} [\![x]\!] + [\![-x]\!] = -1 + 0 = -1$

(1) • Since $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = -1$, then $\lim_{x \to 0} f(x) = -1$

(1) • Since $\lim_{x \to 0} f(x) \neq f(0)$, $[-1 \neq 0]$

(1) then f is not continuous at $x = 0$

5. [**9 points**] Let

$$f(x) = \begin{cases} a + bx & \text{if } x > 2 \\ 3 & \text{if } x = 2 \\ b - ax^2 & \text{if } x < 2 \end{cases}.$$

Find the values of a and b that make f continuous at $x = 2$.

(1) We need to find a & b so that

$$\lim_{x \to 2} f(x) = f(2)$$

i.e, $\lim_{x \to 2^+} f(x) = f(2)$ & $\lim_{x \to 2^-} f(x) = f(2)$

(3) • $\lim_{x \to 2^+} f(x) = f(2) \Rightarrow \lim_{x \to 2^+} (a + bx) = 3 \Rightarrow a + 2b = 3$

(3) • $\lim_{x \to 2^-} f(x) = f(2) \Rightarrow \lim_{x \to 2^-} (b - ax^2) = 3 \Rightarrow b - 4a = 3$

• Solving the system $\begin{cases} a + 2b = 3 \\ -4a + b = 3 \end{cases}$

(1) we get $a = -\frac{1}{3}$

(1) $b = \frac{5}{3}$

6. [**8 points**] Use the graph of $f(x) = \sqrt{x}$ to find a number $\delta > 0$ such that

$$\text{if } |x-9| < \delta, \text{ then } |\sqrt{x} - 3| < 0.2.$$

"not to a scale"

$\varepsilon = 0.2$

$f(a) = 2.8 \Rightarrow \sqrt{a} = \frac{28}{10} \Rightarrow a = \frac{784}{100} = 7.84$ (2)

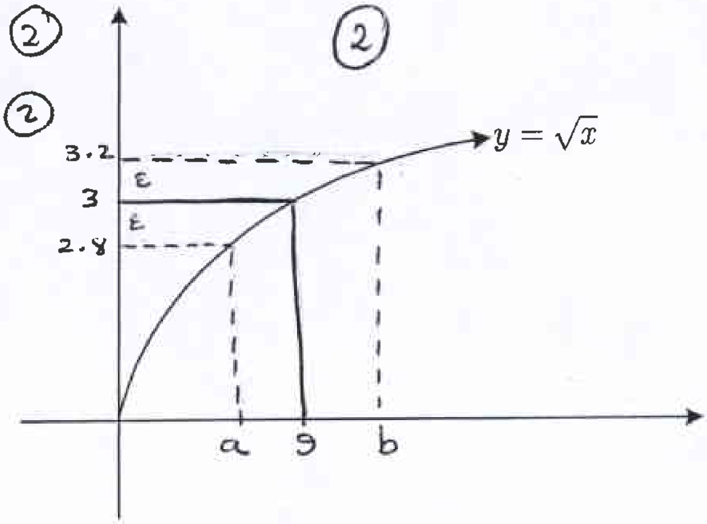
$f(b) = 3.2 \Rightarrow \sqrt{b} = \frac{32}{10} \Rightarrow b = \frac{1024}{100} = 10.24$ (2)

We may choose

$\delta = \text{minimum}\{9-a, b-9\}$ (1)

$= \text{minimum}\{1.16, 1.24\}$

$= 1.16$ (1)



7. Let $y = f(x) = x^2 - 8x + \pi$.

(a) [**2 points**] Find the average rate of change of y with respect to x over the interval $[3, 4]$.

average rate of change $= \dfrac{f(4) - f(3)}{4 - 3}$ (1)

$= \dfrac{(-16+\pi) - (-15+\pi)}{1} = -1$ (1)

(b) [**6 points**] Use limits to find the instantaneous rate of change of y with respect to x at $x = 3$.

Instantaneous rate of change at $x = 3$

$= f'(3)$ (1)

$= \lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$ (1)

$= \lim\limits_{x \to 3} \dfrac{(x^2 - 8x + \pi) - (-15 + \pi)}{x - 3} = \lim\limits_{x \to 3} \dfrac{x^2 - 8x + 15}{x - 3}$ (1)

$= \lim\limits_{x \to 3} \dfrac{(x-3)(x-5)}{x-3}$ (2)

$= \lim\limits_{x \to 3} (x - 5) = 3 - 5 = -2$ (1)

8. [**6 points**] The following limit

$$\lim_{x\to\frac{\pi}{2}} \frac{2\cos(2x)+2}{2x-\pi}$$

represents the derivative of some function f at some number a. Find f and a.

$$\lim_{x\to\frac{\pi}{2}} \frac{2\cos(2x)+2}{2x-\pi} = \lim_{x\to\frac{\pi}{2}} \frac{2\cos(2x)+2}{2(x-\frac{\pi}{2})} \quad (1)$$

a

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos(2x)+1}{x-\frac{\pi}{2}} \quad (1)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos(2x)-\cos(2\cdot\frac{\pi}{2})}{x-\frac{\pi}{2}} \quad (1)$$

$$= f'(\tfrac{\pi}{2}) \quad (1)$$

So ~~therefore~~ $f(x) = \cos(2x)$ & $a = \frac{\pi}{2}$ (1) + (1)

9. [**7 points**] Is the line $x = 1$ a vertical asymptote for the graph of $f(x) = \dfrac{-2x^2+3x-1}{3x^2-2x-1}$. Justify your answer.

- $$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{-2x^2+3x-1}{3x^2-2x-1}$$

$$= \lim_{x\to 1} \frac{(2x-1)(1-x)}{(3x+1)(x-1)} \quad (2)$$

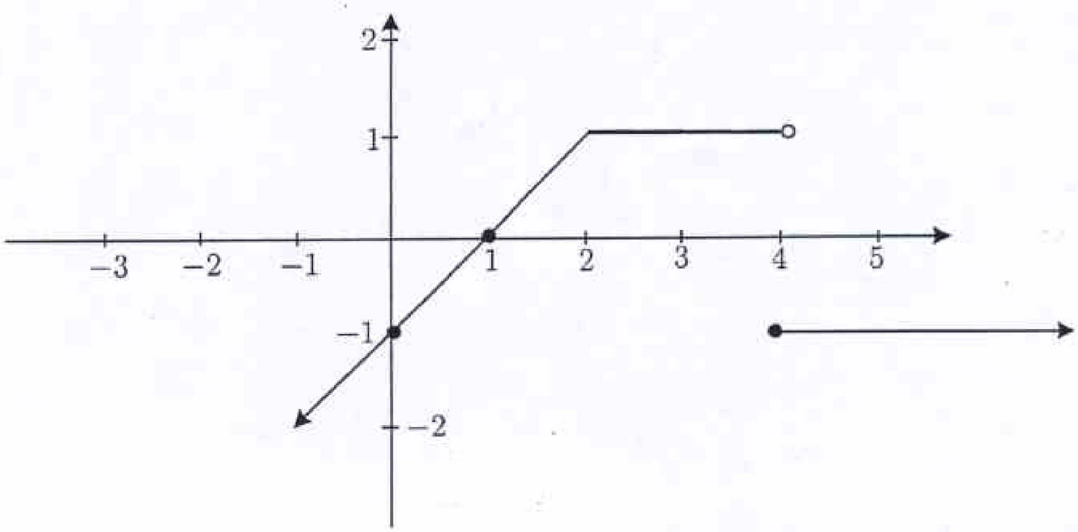
$$= \lim_{x\to 1} -\frac{2x-1}{3x+1} = -\frac{1}{4} \quad (2)$$

$$\left[= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = -\frac{1}{4}\right]$$

(2) { - Since $\lim_{x\to 1} f(x) \neq \infty$ or $-\infty$

(1) { then the line $x = 1$ is not a vertical asymptote

10. [9 **points**] For the function f whose graph is given below, find the derivative if it exists. Justify your answer.



(a) $f'(0)$ = the slope of the tangent line to the curve $y = f(x)$ at the point $(0, f(0)) = (0,-1)$ (1)

= the slope of the line segment joining $(0,-1)$ & $(1,0)$

$= \dfrac{0-(-1)}{1-0}$ (1)

$= 1$ (1)

(b) $f'(2)$

$f'(2)$ does not exist (1)

since f has a corner at $x = 2$ (1)

(c) $f'(4)$ does not exist (1)

since f is not continuous at $x = 4$ (1)

(d) $f'(5)$. = the slope of the tangent line to the curve $y = f(x)$ at the point $(5, f(5)) = (5,-1)$ (1)

= the slope of the horizontal line $y=-1$, for $x > 4$

(1) $= 0$