King Fahd University of Petroleum and Minerals — College of Sciences — Department of Mathematical Sciences

MATH 101 Calculus I — Instructor: Dr. Faisal Fairag

EXAM (2) FORM B

Name:	ID #	Sec # (11) (28)

Question	Mark
MCQ (1-6)	7 each
7	16
8	14
9	14
10	14
Bonus	10 or zero

de cbda

Eid Mubarak. Say bismillah and start.

Circle the correct answer in questions (1 – 5):

1) $\lim_{h\to 0}\frac{\cos 5h-1}{1-\cos 7h}\cdot\frac{(\cos 5h+1)(1+\cos 7h)}{(\cos 5h+1)(1+\cos 7h)} = -\frac{25}{49}\left(\frac{\sin 5h}{5h}\right)^2\left(\frac{7h}{\sin 7h}\right)^2\frac{1+\cos 7h}{1+\cos 5h}$

Similar to 31/2.6

a) $\frac{-5}{7}$ b) $\frac{-7}{5}$ c) $\frac{-49}{25}$ d) $\frac{-25}{49}$ e) 0

2) Given that $f(x)=\sqrt{x+1}$. Find the slope of the tangent line at a=3 $f'(x) = 1/2\sqrt{x+1}$

13/3.2

a) $\frac{1}{8}$ b) $\frac{1}{5}$ c) $\frac{1}{6}$ d) $\frac{1}{7}$ e) $\frac{1}{4}$

3) If $f(x)=\frac{x+1}{x}$. Then $f''(-1)=$ $f''(x) = 2/x^3$

45/3.3

a) 0 b) 1 c) -2 d) 2 e) -1

4) If $f(x)=x\sin x-3\cos x$. Then $f''(\frac{\pi}{2})=$ $f''(x) = -x\sin x+5\cos x$

21/3.4

a) 5 b) $\frac{-\pi}{2}$ c) $\frac{\pi}{2}$ d) -5 e) π

5) If $f(x)=-2\sqrt{\cos(5x)}$. Then $f'(2\pi)=$ $f'(x) = 5\sin(5x)/\sqrt{\cos 5x}$

23/3.5

a) 2 b) 5 c) -5 d) 0 e) -2

6) Find $\lim_{x\to 0}(1-4x)^{\frac{2}{x}}=$ let $t=-4x \Rightarrow -\frac{8}{t}=\frac{2}{x}$

Similar to 57/4.2

a) e^{-8} b) e^{8} c) e^{2} d) e^{-4} e) e^{4}

EXAM (2) FORM A

Name:	ID #	Sec # (11) (28)

Question	Mark
MCQ (1-6)	7 each
7	16
8	14
9	14
10	14
Bonus	10 or Zero

da cbab

Eid Mubarak. Say bismillah and start.

Circle the correct answer in questions (1 – 5):

1) $\lim_{h\to 0}\frac{1-\cos 7h}{\cos 5h - 1} \cdot \frac{(1+\cos 7h)(\cos 5h+1)}{(1+\cos 7h)(\cos 5h+1)} = -\frac{49}{25}\left(\frac{\sin 7h}{7h}\right)^2\left(\frac{5h}{\sin 5h}\right)^2\frac{1+\cos 5h}{1+\cos 7h}$

Similar to 31/2.6

a) $\frac{-5}{7}$ b) $\frac{-7}{5}$ c) $\frac{-25}{49}$ d) $\frac{-49}{25}$ e) 0

2) Given that $f(x)=\sqrt{x+1}$. Find the slope of the tangent line at a=8 $f'(x) = \frac{1}{2\sqrt{x+1}}$

13/3.2

a) $\frac{1}{6}$ b) $\frac{1}{5}$ c) $\frac{1}{4}$ d) $\frac{1}{7}$ e) $\frac{1}{8}$

3) If $f(x)=\frac{x+1}{x}$. Then $f''(1)=$ $f''(x) = \frac{2}{x^3}$

45/3.3

a) 0 b) 1 c) 2 d) -1 e) -2

4) If $f(x) = x\sin x - 3\cos x$. Then $f''(\pi)=$ $f'(x) = 5\sin(5x)/\sqrt{\cos 5}$

21/3.4

a) 5 b) -5 c) $\frac{\pi}{2}$ d) $\frac{-\pi}{2}$ e) π

5) If $f(x) = -2\sqrt{\cos(5x)}$. Then $f'(0)=$ $f''(x) = -x\sin x + 5c$

23/3.5

a) 0 b) 5 c) -5 d) 2 e) -2

6) Find $\lim_{x\to 0}(1-4x)^{\frac{2}{x}} =$ let $t=-4x \Rightarrow \frac{-8}{t} = \frac{2}{x}$

Similar to 57/4.2

a) e^4 b) e^{-8} c) e^2 d) e^{-4} e) e^8

7) Let $f(x) = \dfrac{x^3}{x^2+1}$ and $g(x) = f^{-1}(x)$. Find $g''(\frac{1}{2})$ **(SHOW ALL YOUR WORK)**

$$f'(x) = \frac{3x^2(x^2+1) - 2x(x^3)}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2} \qquad \boxed{f'(x) = \frac{x^4+3x^2}{(x^2+1)^2}}$$

$$f''(x) = \frac{(4x^3+6x)(x^2+1)^2 - 2(x^2+1)(2x)(x^4+3x^2)}{(x^2+1)^4}$$

$$g'(x) = \left[f^{-1}(x)\right]' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(g(x))} = \left[f'(g(x))\right]^{-1} \quad (*)$$

$$g''(x) = -\left[f'(g(x))\right]^{-2} \cdot \left[f'(g(x))\right]'$$
$$= -\left[f'(g(x))\right]^{-2} \cdot f''(g(x)) \cdot g'(x)$$

$$\boxed{g''(x) = \frac{-f''(g(x)) \cdot g'(x)}{\left[f'(g(x))\right]^2}}$$

$$\Rightarrow g''(\tfrac{1}{2}) = \frac{-f''(g(\frac{1}{2})) \cdot g'(\frac{1}{2})}{\left[f'(g(\frac{1}{2}))\right]^2}$$ we need to find $g(\frac{1}{2})$, $g'(\frac{1}{2})$

let $y = g(\frac{1}{2}) \Leftrightarrow y = f^{-1}(\frac{1}{2}) \Leftrightarrow f(y) = \frac{1}{2} \Leftrightarrow \frac{y^3}{y^2+1} = \frac{1}{2}$

$$\Rightarrow 2y^3 = y^2+1 \Rightarrow 2y^3 - y^2 - 1 = 0 \Rightarrow y = 1 \Rightarrow \boxed{g(\tfrac{1}{2}) = 1}$$

From $(*)$ $$g'(\tfrac{1}{2}) = \frac{1}{f'(g(\frac{1}{2}))} = \frac{1}{f'(1)} = \frac{1}{\left[\frac{1+3}{(1+1)^2}\right]} = \frac{1}{\left[\frac{4}{4}\right]} = 1$$

$$\boxed{g'(\tfrac{1}{2}) = 1}$$

Now, $$g''(\tfrac{1}{2}) = \frac{-f''(g(\frac{1}{2})) \cdot g'(\frac{1}{2})}{\left[f'(g(\frac{1}{2}))\right]^2} = \frac{-f''(1) \cdot (1)}{\left[f'(1)\right]^2} = -\frac{f''(1)}{\left[f'(1)\right]^2}$$

$$= -\frac{\left[\frac{(10)(4) - (2)(2)(2)(4)}{16}\right]}{[1]^2} = \frac{8/16}{1} = \frac{1}{2} \Rightarrow \boxed{g''(\tfrac{1}{2}) = \tfrac{1}{2}}$$

8) Find $\frac{dy}{dx}$, $y = (x^3 - 2x)^{\ln x}$ **(SHOW ALL YOUR WORK)** (43/4.3)

$$\ln y = \ln (x^3 - 2x)^{\ln x}$$ 4

$$\ln y = (\ln x) \cdot \left[\ln (x^3 - 2x) \right]$$

diff. both sides w.r.t x

$$\frac{y'}{y} = \underbrace{\frac{1}{x} \left[\ln (x^3 - 2x) \right] + (\ln x) \cdot \frac{3x^2 - 2}{x^3 - 2x}}$$ 4

$$y' = \{ \quad \} y$$

$$y' = \{ \quad \} (x^3 - 2x)^{\ln x}$$

$$y' = \left\{ \frac{1}{x} \left[\ln (x^3 - 2x) \right] + (\ln x) \frac{3x^2 - 2}{x^3 - 2x} \right\} (x^3 - 2x)^{\ln x}$$

6

9) Find all values in the interval [0,2] at which the graph of f has a horizontal tangent line.
$f(x) = \ln(\cos e^x)$. **(SHOW ALL YOUR WORK)** (30/4.3 and 29/3.4)

If the graph of f has a horizontal tangent line at x_0 then the slope of the tangent line at x_0 is equal zero or $f'(x_0) = 0$. we need to find all $x_0 \in [0,2]$ such that $f'(x_0) = 0$.

$$f(x) = \ln(\cos e^x)$$

$$f'(x) = \frac{(\cos e^x)'}{\cos e^x} = \frac{(-\sin e^x)\cdot e^x}{\cos e^x} = -e^x \cdot \tan e^x$$

$$\boxed{f'(x) = -e^x \cdot \tan e^x}$$ 4

Now, let $f'(x) = 0 \Rightarrow -e^x \cdot \tan e^x = 0$ 3

e^x never $= 0 \Rightarrow \tan e^x = 0$

$\Rightarrow e^x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \ldots$

$\Rightarrow x = \ln 0, \ln(\mp\pi), \ln(\mp 2\pi), \ldots$

negatives and zero rejected

$\Rightarrow x = \ln(\pi), \ln(2\pi), \ln(3\pi), \ldots$

odd multiple rejected

for example $f(\ln\pi) = \ln(\cos\pi) = \ln(-1)$ = undefined

$\Rightarrow x = \ln(2\pi), \ln(4\pi), \ln(6\pi), \ldots$

Now, $\ln(2\pi) \cong 1.8379 \in [0,2]$
$\ln(4\pi) \cong 2.5310 \notin [0,2]$

Hence, $\boxed{x = \ln(2\pi)}$ 7

10) Find an equation for the line that is tangent to the curve $x = \ln(y \tan x)$ at $x = \frac{\pi}{4}$.

(SHOW ALL YOUR WORK) (Similar to 32/4.3)

$$x = \frac{\pi}{4} \Rightarrow \frac{\pi}{4} = \ln\left(y \tan\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\pi}{4} = \ln(y) \Rightarrow \boxed{y = e^{\pi/4}}$$ 3

So, $\left(\frac{\pi}{4}, e^{\pi/4}\right)$ is a point on the curve.

slope of the tangent line at the $\left(\frac{\pi}{4}, e^{\pi/4}\right)$ equals $y'\left(\frac{\pi}{4}\right)$.

$$x = \ln(y \tan x)$$

diff. w.r.t x

$$1 = \frac{(y \tan x)'}{y \tan x}$$

$$1 = \frac{y' \tan x + y \sec^2 x}{y \tan x} \Rightarrow y \tan x = y' \tan x + y \sec^2 x$$

$$\Rightarrow y' \tan x = y(\tan x - \sec^2 x) \Rightarrow \boxed{y' = \frac{y(\tan x - \sec^2 x)}{\tan x}}$$ 4

$$y'\left(\frac{\pi}{4}\right) = \frac{y\left(\frac{\pi}{4}\right) \cdot \left(\tan\frac{\pi}{4} - \sec^2\frac{\pi}{4}\right)}{\tan\frac{\pi}{4}} = \frac{e^{\pi/4} \cdot \left(1 - (\sqrt{2})^2\right)}{1} = \boxed{-e^{\pi/4}}$$ 3

Slope $= -e^{\frac{\pi}{4}}$, point: $\left(\frac{\pi}{4}, e^{\pi/4}\right)$

equation of the tangent line is

$$\boxed{y - e^{\pi/4} = -e^{\frac{\pi}{4}}\left(x - \frac{\pi}{4}\right)}$$ 4

BONUS QUESTION

Find $\frac{d^{100}}{dx^{100}}[e^x \sin x]$. **(SHOW ALL YOUR WORK)**

$$y = e^x \sin x$$

$$y^{(1)} = e^x \sin x + e^x \cos x$$

$$y^{(2)} = 2e^x \cos x$$

$$y^{(3)} = 2e^x \cos x - 2e^x \sin x$$

$$y^{(4)} = -4e^x \sin x$$

$$\boxed{y^{(4)} = -4y}$$

$$\frac{d^4}{dx^4}\left[y^{(4)}\right] = -4\frac{d^4}{dx^4}[y] \Rightarrow y^{(8)} = -4y^{(4)}$$

$$y^{(8)} = -4(-4y) = (-4)^2 y$$

$$\boxed{y^{(8)} = (-4)^2 y}$$

$$\frac{d^4}{dx^4}\left[y^{(8)}\right] = (-4)^2 y^{(4)} = (-4)^2 \cdot (-4y) = (-4)^3 y$$

$$\boxed{y^{(12)} = (-4)^3 y} \ \cdots\cdots \ y^{(16)} = (-4)^4 y \ \cdots\cdots \ y^{(20)} = (-4)^5 y$$

$$\vdots$$

$$y^{(100)} = (-4)^{25} y \Rightarrow y^{(100)} = -4^{25} y = -2^{50} y$$

$$\boxed{y^{(100)} = -2^{50} y} \text{ or } \boxed{y^{(100)} = -2^{50} \cdot e^x \sin x}$$