King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Semester (011)
MATH 101
First Major Exam

Name: KEY ID: KEY

Sec: (please circle one) 7 (9:00 - 9:50), 11 (10:00 - 10:50)

Serial #: ____________________

Problem #		Points
1		40
2		40
3		40
4		40
5		40
Total:		200

1. Find a number δ such that $| f(x) - L | < \epsilon$ if $0 < | x - a | < \delta$, when

$$\lim_{x \to 3} (5x - 2) = 13 \quad ; \quad \epsilon = 0.01$$

$$|f(x) - L| = |(5x - 2) - 13|$$

$$= |5x - 15|$$

$$= 5 \cdot |x - 3|$$

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we need $|f(x) - L| < \varepsilon = 0.01$

or $5 \cdot |x - 3| < \varepsilon$

$$\Rightarrow |x - 3| < \frac{\varepsilon}{5} = \frac{0.01}{5} = \frac{0.02}{10}$$

$$\Rightarrow |x - 3| < 0.002$$

15

pick $\delta = 0.002$

5

$$\boxed{\delta = 0.002}$$

2. Find:

(a)

$$\lim_{\alpha \to 0} \frac{\sin\alpha - \tan\alpha}{\sin^3\alpha}$$

5

Now, Consider $\frac{\sin\alpha - \tan\alpha}{\sin^3\alpha} = \frac{\sin\alpha - \frac{\sin\alpha}{\cos\alpha}}{\sin^3\alpha} =$

$= \frac{\cos\alpha \cdot \sin\alpha - \sin\alpha}{\sin^3\alpha} = \frac{-\sin\alpha\,[1 - \cos\alpha]}{\sin^3\alpha} = -\frac{(1-\cos\alpha)}{\sin^2\alpha}$

$= -\frac{(1-\cos\alpha)}{(1-\cos^2\alpha)}$ 5 $= -\frac{(1-\cancel{\cos\alpha})}{(1-\cancel{\cos\alpha})\cdot(1+\cos\alpha)}$

5

Now, $\lim_{\alpha\to 0}\frac{\sin\alpha - \tan\alpha}{\sin^3\alpha} = \lim_{\alpha\to 0}\left[-\frac{(1-\cancel{\cos\alpha})}{(1-\cancel{\cos\alpha})(1+\cos\alpha)}\right] = \lim_{\alpha\to 0}\left[\frac{-1}{1+\cos\alpha}\right]$

$= \frac{-1}{1+\cos(0)} = -\frac{1}{2}$ 5 $\boxed{-\frac{1}{2}}$

(b)

$$\lim_{t \to 0} \frac{\sin t}{t^2 + 5t}$$

$\lim_{t\to 0}\frac{\sin t}{t^2+5t} = \lim_{t\to 0}\left[\frac{\frac{\sin t}{t}}{\frac{t^2}{t} + \frac{5t}{t}}\right]$ 6

$= \lim_{t\to 0}\left[\frac{\frac{\sin t}{t}}{t+5}\right]$

8

6 $= \frac{\lim_{t\to 0}\frac{\sin t}{t}}{\lim_{t\to 0}(t+5)} = \frac{1}{(0+5)} = \frac{1}{5}$

$\boxed{\frac{1}{5}}$

3. Consider the equation

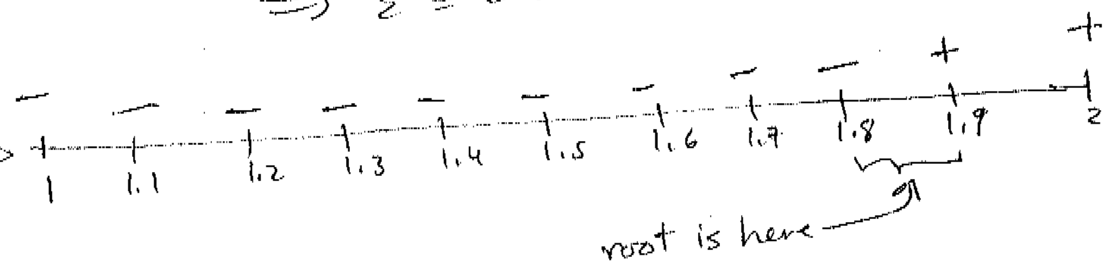
$$x^3 - 4x + 1 = 0$$

There is one real root lies inside the interval $[1, 2]$. Approximate this root with an error of at most 0.005.

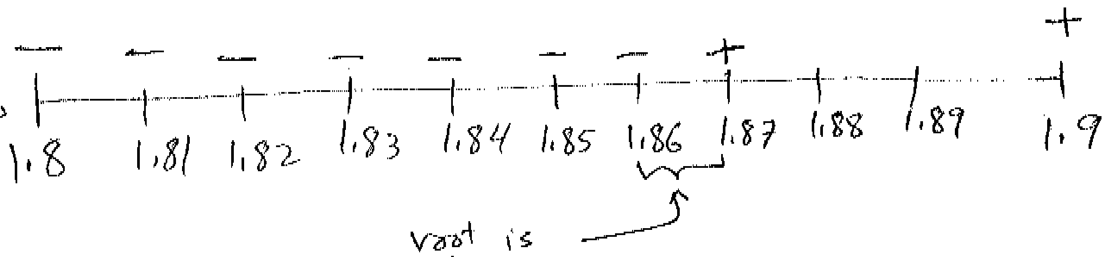
We need to find x^* an approximation for the root so that $|x^* - \text{exact root}| < 0.005 = \frac{\varepsilon}{2}$

$\Rightarrow \varepsilon = 0.005 * 2 = 0.01$ 6

12



12



and $\varepsilon = 1.87 - 1.86 = 0.01$

Consider the following interval 1.86 — 1.87

$x_1 = \text{midpoint} = \frac{1.86 + 1.87}{2} = 1.865$ 10

Now, $x^* = 1.865$ is an approximation for the root with error equals $\frac{\varepsilon}{2} = 0.005$.

$$\boxed{x^* = 1.865}$$

4. Use the defenition of the derivative to find $f'(x)$ where

$$f(x) = \sqrt{x+1}$$

10 10

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \frac{(x+h+1)-(x+1)}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)} = \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$

$$= \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$ 10

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h \to 0} \left[\frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}\right]$$

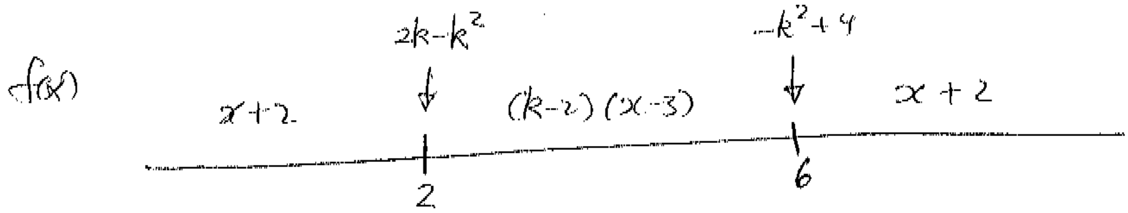
$$= \frac{1}{\sqrt{x+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$ 10

$$\boxed{f'(x) = \frac{1}{2\sqrt{x+1}}}$$

5. Let $f(x)$ be

$$f(x) = \begin{cases} x+2 & x<2 \\ 2k-k^2 & x=2 \\ (k-2)(x-3) & 2<x<6 \\ -k^2+4 & x=6 \\ x+2 & x>6 \end{cases}$$

Find all values of k so that $f(x)$ is continous on the closed interval $[2,6]$.



$f(x)$ cont. on $[2,6]$ means

*(1) $f(x)$ cont on $(2,6)$

*(2) $f(x)$ cont from left at 6

*(3) $f(x)$ cont from right at 2

(1) $f(x) = (k-2)(x-3)$ when $x \in (2,6)$

$\Longrightarrow f(x)$ cont. on $(2,6)$ because it is polynomial. 10

(2) $f(x)$ cont from the right at 2 means

$f(2) = \lim_{x \to 2^+} f(x) \Rightarrow 2k-k^2 = (k-2)(-1)$

$\Longrightarrow 2k-k^2 = -k+2 \Rightarrow k^2-3k+2=0 \Rightarrow (k-2)(k-1)=0$

$\Longrightarrow k \in \{1,2\}$ ———— (☆) 10

(3) $f(x)$ cont. from the left at 6 means

$f(6) = \lim_{x \to 6^-} f(x) \Rightarrow -k^2+4 = (k-2)(6-3)$

$\Longrightarrow -k^2+4 = 3k-6 \Longrightarrow k^2+3k-10=0 \Rightarrow (k+5)(k-2)=0$

$\Longrightarrow k \in \{2,-5\}$ ———— (☆☆) 10

We need conditions (2) and (3) to be satisfied in the same time $\Longrightarrow k \in \{1,2\} \cap \{2,-5\} = \{2\} \Longrightarrow \boxed{k=2}$ 10