Research Statement

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My primary research interest is in computability and complexity theory. More specifically, I have an active research program in the area of effective dimension theory and randomness. Computability theory was first developed in the 1930s and '40s by Alan Turing who has come to be seen, along with Gödel and Tarski, as one one of the founders of modern mathematical logic. The subject developed over the 20th century and has branched into many different sub disciplines including the study of resource-bounded complexity (in which the famous P≠NP problem is formulated), randomness, and descriptive complexity. It is in these last two fields that I am currently concentrating my research energies.

Descriptive complexity begins with the concept of Kolmogorov complexity. This defines the complexity of a finite mathematical object to be the length of the shortest computer program that will output the object. It is meant to capture the intuitive notion that a simple finite object is more succinctly described than a complex one. This idea is quickly extended to infinite objects (characterised as infinite binary sequences) by *effective packing dimension*, which is defined to be

$$\dim_p X = \limsup_n \frac{C(X \restriction n)}{n},$$

where X is an infinite binary sequence and $C(X \restriction n)$ is the Kolmogorov complexity of the initial segment of X of length n.

My research, which has been developed with the aid of a significant research grant from KFUPM, has used these ideas in a variety of ways. At the moment I am interested in relativising this definition of effective packing dimension to

$$\limsup_n \frac{C(X \restriction n \mid Y \restriction n)}{n},$$

where $C(X \restriction n \mid Y \restriction n)$ is the length of the shortest program that outputs $X \restriction n$ given $Y \restriction n$ as input. This definition induces a novel metric on the space of all binary sequences and has suggestive geometric and algebraic properties.

The study of this metric and its geometrical properties forms the core of a current ongoing research program. I have published three papers in this area - one in collaboration with a student (see [2], [3], [13]) - and am currently working on a fourth. In these papers I demonstrated that the defined metric produces a complete, path-connected, non-compact topological space, within which we can define concepts such as angle, projection and scalar multiplication. We also can define a notion of data compression in which every infinite binary sequence with sufficient regularity is able to be maximally compressed.

I intend in future work to proceed in two distinct directions. The first is to see if these ideas can be fruitfully applied to other areas of mathematics. Presently I am investigating with a colleague from the University of Connecticut whether we can find applications in the field of group presentations. We are looking at a variant of the so-called "temperature model of random groups". In this model a quotient of the free group in n generators is produced by a set of relators. This set is given a measure between 0 and 1 by giving a length-dependent weight to each possible relator. This measure can then induce a measure on the set of group presentations.

Our approach is to look at the effective packing dimension of the characteristic function of the set of relators. A high dimension will suggest a "typical" group while a low dimension an atypical one. We can then address questions such as "do most group presentations produce hyperbolic groups?" or "what is the probability that a randomly chosen group presentation yields an Abelian group?". For an overview on the background of these ideas see [10]. Such research has the possibility of contributing to both computability theory and group theory as well as suggesting future research in model theory and algebra.

A second focus of my research is whether geometric or topological methods can be applied to contribute to the field of classical computability theory. In this context I study the nature of continuous paths in the metric space, whether geodesics always exist, and whether the space can be extended to a Banach space. In the three papers mentioned above I have shown that there exist certain natural algebraic operations in the space that cohere well with the geometric structure. Future work will investigate the topological dimension of the space and relationships between points in the space and Martin-Löf randomness.

Also in this direction, I am continuing an investigation into the nature of the relationship of packing dimension to Π^0_1 classes. A Π^0_1 class is the set of infinite paths through some computable binary tree, and the study of these classes is a well established research direction in which much has been published in the last decade (see [5] [12] [4] for overviews). My research continues to uncover the nature of the connection between the size of an Π^0_1 class and the complexity-theoretic nature of its paths. Work by Kucera [8] showed that a Π^0_1 class has positive measure if and only if it contains a Martin-Löf random element. I developed these ideas in [1] and showed that the existence of a path with a certain amount of complexity is equivalent to a Π^0_1 class's being large in another well-defined sense. This I also showed was related to the class's having a universal set of paths (in the sense that any real is wtt-reducible to an element of the set). This generalised and extended the well-known Kučera-Gács Theorem and has had applications in computable model theory and in reverse mathematics. There are initial suggections that this has connections to the previously defined metric.

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