

On a conjecture of Dobrinen and Simpson concerning almost everywhere domination

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September 20, 2005

1 Introduction

Dobrinen and Simpson [4] introduced the notions of almost everywhere domination and uniform almost everywhere domination to study recursion theoretic analogues of results in set theory concerning domination in generic extensions of transitive models of ZFC and to study regularity properties of the Lebesgue measure on 2^ω in reverse mathematics. In this article, we examine one of their conjectures concerning these notions.

Throughout this article, \leq_T denotes Turing reducibility and μ denotes the Lebesgue (or “fair coin”) probability measure on 2^ω given by $\mu(\{X \in 2^\omega \mid X(n) = i\}) = 1/2$. A property holds **almost everywhere** or **for almost all** $X \in 2^\omega$ if it holds on a set of measure 1. For $f, g \in \omega^\omega$, f **dominates** g if $\exists m \forall n > m (f(n) > g(n))$.

Definition 1.1 (Dobrinen, Simpson). A set $A \in 2^\omega$ is **almost everywhere (a.e.) dominating** if for almost all $X \in 2^\omega$ and all functions $g \leq_T X$, there is a function $f \leq_T A$ such that f dominates g . A is **uniformly almost everywhere (u.a.e.) dominating** if there is a function $f \leq_T A$ such that for almost all $X \in 2^\omega$ and all functions $g \leq_T X$, f dominates g .

There are several trivial but useful observations to make about these definitions. First, although these properties are stated for sets, they are also properties of Turing degrees. That is, a set is (u.)a.e. dominating if and only if every other set of the same degree is (u.)a.e. dominating. Second, both properties are closed upwards in the Turing degrees. Third, u.a.e. domination implies a.e. domination. Finally, if A is u.a.e. dominating, then there is a function $f \leq_T A$ which dominates every computable function.

Dobrinen and Simpson [4] introduced these notions to study the following two regularity properties of μ in reverse mathematics: for each G_δ set $Q \subseteq 2^\omega$ and each $\epsilon > 0$, there is a

Solomon’s research was partially supported by an NSF Grant DMS-0400754. The authors thank Joe Miller for helpful comments on a draft of this article.

closed set $F \subseteq Q$ such that $\mu(F) \geq \mu(Q) - \epsilon$, and for each G_δ set $Q \subseteq 2^\omega$, there is an F_σ set $S \subseteq Q$ such that $\mu(Q) = \mu(S)$. ACA_0 is strong enough to prove these statements, so as the first step toward establishing reversals, they proved the following two theorems. (Reverse mathematics plays only a motivational role here, but the reader who is not familiar with this subject is referred to Simpson [18].)

Theorem 1.2 (Dobrinen, Simpson). *For $A \in 2^\omega$, the following are equivalent.*

1. A is a.e. dominating.
2. For every Π_2^0 set $Q \subseteq 2^\omega$ and $\epsilon > 0$, there is a $\Pi_1^{0,A}$ set $F \subseteq Q$ such that $\mu(F) \geq \mu(Q) - \epsilon$.

Theorem 1.3 (Dobrinen, Simpson). *For $A \in 2^\omega$, the following are equivalent.*

1. A is u.a.e. dominating.
2. For every Π_2^0 set $Q \subseteq 2^\omega$, there is a $\Sigma_2^{0,A}$ set $S \subseteq Q$ such that $\mu(Q) = \mu(S)$.

Given these connections, it is reasonable to think that results in computability theory concerning a.e. domination and u.a.e. domination will have implications for the reverse mathematics content of the regularity properties stated above. At the time of Dobrinen and Simpson [4], several facts about u.a.e. domination were already known:

$$A \geq_T 0' \Rightarrow A \text{ is u.a.e. dominating} \Rightarrow A' \geq_T 0''.$$

The first implication follows from a result of Kurtz [10] that $0'$ is u.a.e. dominating and the second implication follows from a result of Martin [13] that A computes a function which dominates every computable function if and only if $A' \geq_T 0''$. (A set A for which $A' \geq_T 0''$ is called **high**.) Furthermore, Dobrinen and Simpson [4] presented an unpublished proof by Martin that no computable set is a.e. dominating.

Several questions arise naturally from these implications. Does every u.a.e. dominating set compute $0'$? Is every high degree u.a.e. dominating or at least a.e. dominating? Is every a.e. dominating degree high? Does a.e. domination imply u.a.e. domination?

Cholak, Greenberg and Miller [2] recently answered the first question in the negative by showing that there is a c.e. set $A <_T 0'$ which is u.a.e. dominating. They also used their methods to show a number of results in reverse mathematics concerning the regularity property that for every G_δ set Q there is an F_σ set $S \subseteq Q$ such that $\mu(Q) = \mu(S)$. In particular, this property does not imply ACA_0 even over WKL_0 . The fourth question remains open. Concerning the second and third questions, Dobrinen and Simpson made the following conjecture.

Conjecture 1.4 (Dobrinen, Simpson). $A' \geq_T 0'' \Leftrightarrow A$ is a.e. dominating.

This conjecture is our main focus. The strongest results appear in Section 5 where we show that if $A \leq_T 0'$ is a.e. dominating, then A is high (giving a partial answer to the \Leftarrow direction of Conjecture 1.4) and that for any a.e. dominating set Z , every set which is 1-random relative to Z is actually 2-random. As a corollary (applying work of Nies [15]), we obtain the stronger

property that every a.e. dominating set $A \leq_T 0'$ satisfies $0'' \leq_{tt} A'$, where \leq_{tt} denotes truth table reducibility. (Such sets are called **super high**.) Because there are Δ_2^0 (even Σ_1^0) sets which are high but not super high, this result refutes the \Rightarrow direction of Conjecture 1.4.

Before arriving at Section 5, we follow a meandering path to explore the connections between a.e. domination and notions such randomness and genericity. Because relatively little is known about a.e. domination, we approach this property from different angles and occasionally offer more than one proof of our results. Hopefully, others will see additional connections and push this work towards a more complete understanding of this property.

In Section 2, we give a direct construction of a high computably enumerable (c.e.) set H which is not a.e. dominating. The construction combines Martin's technique for showing the computable sets are not a.e. dominating with a standard technique for constructing high c.e. degrees. Noam Greenberg and Joe Miller independently obtained a similar (although not c.e.) result using a different method. In Section 3, we show that if A is 2-random, then A is not a.e. dominating and hence the measure of all a.e. dominating sets is 0. Furthermore, we show that almost every degree is bounded by a high degree which is not a.e. dominating. It follows that there are 2^{\aleph_0} many counterexamples to the \Rightarrow direction of Conjecture 1.4. In Section 4, we prove that if A is 2-generic (with respect to Cohen forcing), then A is not a.e. dominating. Furthermore, we show that for any a.e. dominating A , there is a 2-random R whose degree is c.e. in A .

In Section 6, we approach Conjecture 1.4 from the viewpoint of Turing ideals. Suppose that an ideal \mathcal{I} satisfies $\forall X \in \mathcal{I} \exists Y \in \mathcal{I} (X <_T Y \wedge X'' \leq_T Y')$. Must \mathcal{I} be a Scott set (that is, contain a path through each infinite subtree of $2^{<\omega}$ contained in \mathcal{I})? In other words, must such an ideal be the second order part of an ω -model of WKL_0 ? We show that for any computable tree $T \subseteq 2^{<\omega}$ without a computable infinite path, there is an ideal closed under the highness property above that does not contain a path through T . Hence, this ideal is not a Scott set and does not give an ω -model of WKL_0 . Along the same lines, we show that there is an ideal closed under this highness property that does not contain an a.e. dominating set.

Our notation is standard and mostly follows Soare [19]. Φ_e denotes the list of partial computable functionals and we fix a computable bijection $\langle x, y \rangle$ between ω^2 and ω . For $A \subseteq \omega$, let $A^{[e]} = \{x \mid \langle e, x \rangle \in A\}$ and for $m \in \omega$, let $[m, \infty) = \{n \mid n \geq m\}$. We sometimes equate sets with reals from the interval $[0, 1]$ by viewing a set $B \subseteq \omega$ as the real $\sum_{n \in B} 2^{-n}$. For strings σ and τ , we write $\sigma \sqsubseteq \tau$ to indicate that σ is an initial substring of τ . Similarly, if F_1 is a finite set and F_2 is a set, then we write $F_1 \sqsubseteq F_2$ if F_1 is an initial segment of F_2 .

2 Computably enumerable example

In this section, we give a direct construction of a high c.e. set which is not a.e. dominating. In Section 5, we give an alternate proof of Theorem 2.1 using index sets.

Theorem 2.1. *There is a high c.e. set which is not a.e. dominating.*

To prove this theorem, we build a c.e. set H such that $\emptyset'' \leq_T H'$ and H is not a.e. dominating. By negating the property of a.e. domination, it suffices to prove there is a set $T \subseteq 2^\omega$ of positive measure such that $\forall X \in T \exists g \leq_T X \forall f \leq_T H (g \text{ is not dominated by } f)$.

Fix any rational number δ such that $0 < \delta < 1$. We build a partial computable functional Φ such that $\mu(\{X \mid \Phi(X) \text{ is not total}\}) \leq \delta$ and we let $T = \{X \mid \Phi(X) \text{ is total}\}$. Therefore, $\mu(T) \geq 1 - \delta$ as required. Furthermore, we ensure that for every $e \in \omega$ and for every $X \in T$, if $\Phi_e(H)$ is total, then $\Phi(X)$ is not dominated by $\Phi_e(H)$. These properties suffice to prove the theorem.

There are two types of requirements: R_e requirements which guarantee that H is high and M_e requirements which define the functional Φ and the set T . Our construction takes place on a tree of strategies which is described below. If α is an R_e or M_e strategy, then we let $A^{[\alpha]}$ and $H^{[\alpha]}$ denote $A^{[e]}$ and $H^{[e]}$, $\langle \alpha, x \rangle$ denote $\langle e, x \rangle$ and Φ_α denote Φ_e .

To make H high, we use the following standard trick. Let Tot denote the index set of total computable functions and let A be a c.e. set such that for all e , if $e \in \text{Tot}$, then $A^{[e]} = \omega$ and if $e \notin \text{Tot}$, then $A^{[e]}$ is a finite initial segment of ω . To make H high, it suffices to make $H \subseteq A$ such that if $A^{[e]}$ is infinite then $A^{[e]} - H^{[e]}$ is finite. It follows that if $e \in \text{Tot}$, then $H^{[e]}$ is cofinite (and hence $\lim_x H(\langle e, x \rangle) = 1$) and if $e \notin \text{Tot}$, then $H^{[e]}$ is finite (and hence $\lim_x H(\langle e, x \rangle) = 0$). Because $\text{Tot} = \lim_x H(\langle e, x \rangle)$, the Limit Lemma gives $\emptyset'' \equiv_T \text{Tot} \leq_T H'$.

Let R_e denote the requirement that $H^{[e]} \subseteq A^{[e]}$ and $A^{[e]}$ infinite implies $A^{[e]} - H^{[e]}$ is finite. An R_e strategy α operates under a finite restraint and maintains a parameter o_α which is larger than this restraint and which is only changed when some higher priority strategy raises its restraint and initializes α . α acts as follows.

1. When α first acts or has been initialized, define o_α to be large and set $n = o_\alpha$.
2. Wait for n to enter $A^{[\alpha]}$.
3. When n enters $A^{[\alpha]}$, put n into $H^{[\alpha]}$, increase the value of n by 1 and return to Step 2.

Clearly this strategy makes $H^{[\alpha]} \subseteq A^{[\alpha]}$. If α is on the true path, then the higher priority strategies initialize α only finitely often. Therefore, the parameter o_α reaches a limit and every number in $A^{[\alpha]}$ larger than the final value of o_α enters $H^{[\alpha]}$.

An R_e strategy α has infinitely many possible outcomes: the numbers in ω (which denote the current value of o_α) and Fin . These outcomes are ordered by $n <_L \text{Fin}$ for every $n \in \omega$ and $n <_L m$ if $n > m$. (That is, Fin is the rightmost outcome and the numerical outcomes increase in value as they move to the left.) The strategy takes outcome o_α each time it acts in Step 3 and it takes outcome Fin otherwise. If $A^{[\alpha]}$ is finite, then the strategy is eventually stuck in Step 2 forever and cofinitely often takes the Fin outcome. On the other hand, as long as α is initialized only finitely often, if $A^{[\alpha]}$ is infinite, then there is a final value of o_α for which α takes outcome o_α infinitely often.

The second type of requirement concerns building the partial computable functional Φ and the set T . Globally, we need to make sure that $\mu(T) \geq 1 - \delta$, and locally we let M_e denote the requirement that if $\Phi_e(H)$ is total, then $\Phi(X)$ is not dominated by $\Phi_e(H)$ for any $X \in T$. To avoid domination, it is enough to make sure that for each e and each $X \in T$, there is at least one value x such that $\Phi(X; x) > \Phi_e(H; x)$. (See Lemma 2.7 for a proof that this condition is sufficient.) The action for a single M_e strategy α proceeds as follows.

1. Pick a small value $\epsilon_\alpha = 1/2^{p_\alpha}$ for some large p_α . (We discuss below how to choose this number, but in particular $\epsilon_\alpha < \delta$.)

2. Divide 2^ω into 2^{p_α} many disjoint clopen sets $U_1^\alpha, \dots, U_{2^{p_\alpha}}^\alpha$ each of size ϵ_α .
3. Cycle through the U_i^α sets beginning with $i = 1$.
 - (a) Pick a large value x_i^α and define $\Phi(X; x_i^\alpha) = 0$ for all $X \notin U_i^\alpha$.
 - (b) Wait for $\Phi_\alpha(H; x_i^\alpha)$ to converge.
 - (c) If $\Phi_\alpha(H; x_i^\alpha)$ converges, then define $\Phi(X; x_i^\alpha) > \Phi_\alpha(H; x_i^\alpha)$ for all $X \in U_i^\alpha$, increase i by 1 and return to Step 3(a). To preserve the computation $\Phi_\alpha(H; x_i^\alpha)$, restrain H from changing below the use of this computation.
4. If i eventually runs through Step 3 for all the numbers $1, \dots, 2^{p_\alpha}$ then stop the action for α and declare it satisfied.

Consider what such a strategy does in isolation. If $\Phi_\alpha(H)$ is total, then it runs through the cycle in Step 3 for each i between 1 and 2^p and defines Φ such that $\forall X \exists x (\Phi(X; x) > \Phi_\alpha(H; x))$. This action wins M_e . If $\Phi_\alpha(H)$ is not total, then there may be an i between 1 and 2^{p_α} for which $\Phi(H; x_i^\alpha)$ does not halt. In this case, α gets stuck in Step 3(b) during the i^{th} cycle. Consequently, $\Phi(X; x_i^\alpha)$ does not converge for any $X \in U_i^\alpha$ and so $\Phi(X)$ is not total for any $X \in U_i^\alpha$. However, because α does not progress past Step 3(b) of the i^{th} cycle, the sets $X \in U_i^\alpha$ are the only sets on which α causes Φ to be partial. Therefore, M_e is won trivially (since $\Phi_\alpha(H)$ is not total) and α contributes a set of measure ϵ_α on which Φ is not total. In each of the two cases described here, α imposes only finitely much restraint since it has only finitely many cycles to run through.

The action of α when $\Phi_\alpha(H)$ is not total tells us how to pick the values of ϵ during the construction. Each strategy which acts for an M_e requirement potentially contributes a set of measure ϵ on which Φ is not total. Therefore, we have to choose the values of ϵ as we go through the construction so that the sum of these values (over all e) is less than δ .

The outcomes for an M_e strategy α are the numbers in ω (indicating the current restraint imposed by α). These outcomes are ordered by $n <_L m$ if $n > m$. (That is, the numerical values of the outcomes increase as they move left.)

We define the tree of strategies by induction. The empty string λ is assigned to R_0 . If α is assigned to R_e , then $\alpha * \text{Fin}$ and $\alpha * n$ (for $n \in \omega$) are assigned to M_e . If α is assigned to M_e , then $\alpha * n$ (for $n \in \omega$) is assigned to R_{e+1} . In both cases, the outcomes are ordered as described above.

How do the strategies interact? Suppose α is an M strategy and β is an R strategy. If α is to the left of β , then β is initialized whenever α acts. In particular, when β next acts, it picks a new value for o_β which is larger than the restraint (if any) imposed by α . If α is an M strategy such that $\alpha * m \sqsubseteq \beta$, then β is not eligible to act until α has imposed restraint up to m . Therefore, o_β is chosen $> m$ and β respects α 's restraint. On the other hand, if $\beta * \text{Fin} \sqsubseteq \alpha$, then each time β puts a number into $H^{[\beta]}$, α is initialized. If $\beta * \text{Fin}$ is on the true path, then eventually β always takes outcome $\beta * \text{Fin}$. Therefore, β causes only finitely much injury to α .

The one nontrivial interaction is when $\beta * m \sqsubseteq \alpha$. In this case, α is only eligible to act if β sets o_β to be m , and α guesses that β will eventually put every number greater than

$o_\beta = m$ into $H^{[\beta]}$. Therefore, when α sees a convergent computation $\Phi_\alpha(H; x_i^\alpha)$ with use u , it only believes the computation if every number $\langle \beta, x \rangle \leq u$ with $o_\beta = m \leq x$ is in H . Because α believes that β will place every such x into $H^{[\beta]}$, α believes that any computation missing such a number will eventually be destroyed by the enumeration of x into $H^{[\beta]}$. Therefore, the general construction contains this minor modification for an M strategy β .

We now present the formal construction. At stage 0, let $H_0 = \emptyset$. At stage $s > 0$, we let strategies act beginning with the R_0 strategy λ until we reach a strategy of length s . Once we reach a strategy of length s , end the stage and initialize all strategies of lower priority than the last strategy eligible to act. Initializing an R strategy α means canceling o_α and n_α . Initializing an M strategy α means canceling r_α (the current restraint imposed by α), ϵ_α and p_α , canceling the partition U_i^α and canceling all witnesses x_i^α . Any parameters not canceled by initialization retain their values at the next stage. Once the initialization is done, we define $\Phi(X; y) = 0$ for all sets X and all numbers $y \leq s$ which are not currently witnesses of the form x_i^α for some M strategy α . (Formally, we choose a large value k and let $\Phi(\sigma, y) = 0$ for all strings σ of length k .)

When an R strategy α is eligible to act, it acts as follows. If s is the first stage at which α is eligible to act or if α has been initialized since it was last eligible to act, define o_α to be large and set $n_\alpha = o_\alpha$. Check if $n_\alpha \in A_s^{[\alpha]}$. (We begin at this step if n_α is already defined.) If not, then let $\alpha * \text{Fin}$ be the next strategy eligible to act. If so, then enumerate n_α into $H^{[\alpha]}$, increase n_α by 1 and let $\alpha * o_\alpha$ be the next strategy eligible to act.

When an M strategy α is eligible to act, it acts as follows. If s is the first stage at which α is eligible to act or if α has been initialized since it was last eligible to act then we need to define r_α , ϵ_α and p_α . Set $r_\alpha = 0$. (The parameter r_α denotes α 's current level of restraint.) Let q be the sum of all ϵ_γ parameters defined by all M strategies γ that have been eligible to act at any time during the construction so far. In the verification below, we prove that $q < \delta$. Define p_α to be a large number so that $\epsilon_\alpha = 1/2^{p_\alpha}$ satisfies $q + \epsilon_\alpha < \delta$. Partition 2^ω into 2^{p_α} many disjoint clopen sets $U_1^\alpha, \dots, U_{2^{p_\alpha}}^\alpha$ each of size ϵ_α .

Begin the cycles for α with this choice of p_α . (If p_α was already defined, we start the action of α wherever it left off in this cycle procedure.) Run cycles beginning with $i = 1$ and proceeding through $i = 2^{p_\alpha}$. The i^{th} cycle acts as follows. Pick a large value for the witness x_i^α when the cycle begins and define $\Phi(X; x_i^\alpha) = 0$ for all $X \notin U_i^\alpha$. Let $\beta_0, \dots, \beta_{k-1}$ denote the R strategies such that $\beta_j * m_j \sqsubseteq \alpha$ for some $m_j \in \omega$. Check if $\Phi_{\alpha, s}(H_s; x_i^\alpha)$ converges. If not, let $\alpha * r_\alpha$ be the next strategy eligible to act. If it does converge, then let u be the use of the computation. For each $0 \leq j \leq k-1$, check if for every number $\langle \beta_j, y \rangle \leq u$ with $m_j \leq y$, we have $y \in H_s^{[\beta_j]}$. If not, then let $\alpha * r_\alpha$ be eligible to act (and α remains in the i^{th} cycle when it is next eligible to act). If so, then define $\Phi(X; x_i^\alpha) > \Phi_\alpha(H_s; x_i^\alpha)$ for all $X \in U_i^\alpha$. Redefine r_α to be the maximum of its old value and u , increase i by 1 (so that α will begin the $(i+1)^{\text{st}}$ cycle when it is next eligible to act) and let $\alpha * r_\alpha$ be eligible to act. If i_α reaches the value $2^{p_\alpha} + 1$, then α performs no further actions (unless it is initialized) and takes outcome $\alpha * r_\alpha$ at all future stages.

This completes the description of the formal construction. A strategy α is on the true path if α is the leftmost strategy of length $|\alpha|$ which is eligible to act infinitely often. A stage at which α is eligible to act is called an α stage.

Lemma 2.2. $H \subseteq A$.

Proof. Numbers are enumerated into H only after they have entered A . \square

Lemma 2.3. *If α is an R strategy on the true path, then o_α reaches a limit \hat{o}_α . Furthermore, if $\alpha * \hat{o}_\alpha$ is on the true path, then $A^{[\alpha]} = \omega$ and $A^{[\alpha]} - H^{[\alpha]}$ is finite, while if $\alpha * \text{Fin}$ is on the true path, then $A^{[\alpha]}$ is finite.*

Proof. Let s be the first α stage such that no strategy to the left of α is eligible to act after stage s and hence α is never initialized after stage s . Because values of o_α are canceled only by initialization, α defines the final value for o_α at stage s .

Once o_α has reached its limit \hat{o}_α , there are only two possible outcomes for α to take at any future α stage: \hat{o}_α and Fin . Because \hat{o}_α is greater than the restraint imposed on α by any M strategy of higher priority, α is free to place any number bigger than \hat{o}_α which enters $A^{[\alpha]}$ into $H^{[\alpha]}$. Recall that $A^{[\alpha]}$ is either ω or a finite initial segment of ω , that α places n_α into $H^{[\alpha]}$ and increments n_α (beginning at \hat{o}_α) whenever it sees n_α enter $A^{[\alpha]}$, and that α only takes outcome $\alpha * \hat{o}_\alpha$ when it puts the current value of n_α into $H^{[\alpha]}$.

Assume that $\alpha * \hat{o}_\alpha$ is eligible to act infinitely often. This situation implies the interval $I = [\hat{o}_\alpha, \infty)$ is contained in $A^{[\alpha]}$ and hence $A^{[\alpha]} = \omega$. Furthermore, each element of I is placed into $H^{[\alpha]}$ and hence $A^{[\alpha]} - H^{[\alpha]}$ is finite. On the other hand, assume that $\alpha * \text{Fin}$ is on the true path. In this case, there must be a value of n_α for which α never sees n_α enter $A^{[\alpha]}$ and hence $A^{[\alpha]}$ is a finite initial segment of ω . \square

It follows from the previous two lemmas that each requirement R_e is met by the construction, so H has high degree.

Lemma 2.4. *Let α be an M strategy that is eligible to act at stage s . Let q_s^α be the sum of all ϵ_γ parameters for all M strategies γ that have been eligible to act at any point in the construction before α is eligible to act at stage s . Then $q_s^\alpha < \delta$.*

Proof. This lemma follows by induction on s , and for each s by a subinduction on the strategies which are eligible to act at stage s . If α defines ϵ_α at stage s , then by induction $q_s^\alpha < \delta$, so α can define ϵ_α such that that $q_s^\alpha + \epsilon_\alpha < \delta$. \square

Let $T = \{X \mid \Phi(X) \text{ is total}\}$ and let $q_s = q_s^\alpha$ where α is the last M strategy eligible to act at stage s . By Lemma 2.4, $q_s < \delta$ and hence $\lim_s q_s \leq \delta$. In other words, the sum of all parameters ϵ_α chosen by M strategies during the construction is $\leq \delta$.

Lemma 2.5. $\mu(T) \geq 1 - \delta$.

Proof. At the end of each stage s , if $x \leq s$ and x is not equal to the current value of an x_i^α parameter for some M strategy α , then we define $\Phi(X; x) = 0$ for all X . Fix a number y and calculate $\mu(\{X \mid \Phi(X; y) \uparrow\})$. If $\Phi(X; y) \uparrow$, then for each stage $s \geq y$, y must be equal to the current value of some x_i^α parameter. Because new values for these parameters are always chosen large (and are never reused), there must be a fixed M strategy α and a fixed cycle number i such that $y = x_i^\alpha$ at all stages $s \geq y$. (In particular, α is never initialized after stage y .) When α chose $x_i^\alpha = y$, it defined $\Phi(Y; y) = 0$ for all $Y \notin U_i^\alpha$. Therefore, if

$\Phi(X; y) \uparrow$, then $X \in U_i^\alpha$. Because $\mu(U_i^\alpha) = \epsilon_\alpha$, we have that either $\Phi(X; y)$ converges for all X or $\mu(\{X \mid \Phi(X; y) \uparrow\}) = \epsilon_\alpha$. Summing over all y , we see that $\mu(\{X \mid \Phi(X) \text{ not total}\})$ is bounded by the sum of all values of ϵ_β chosen over the course of the construction by all M strategies β . As noted above, this sum is $\leq \delta$. \square

Lemma 2.6. *If α is an M strategy on the true path such that $\Phi_\alpha(H)$ is total, then*

$$\forall X \exists x (\Phi(X; x) > \Phi_\alpha(H; x)).$$

Proof. Assume α is on the true path and $\Phi_\alpha(H)$ is total. Let $\beta_0, \dots, \beta_{k-1}$ be the R strategies such that $\beta_j * m_j \sqsubseteq \alpha$ for some $m_j \in \omega$. Because each $\beta_j * m_j$ is on the true path, if \hat{o}_{β_j} denotes the final value of the parameter o_{β_j} , then $\hat{o}_{\beta_j} = m_j$. By the proof of Lemma 2.3, the interval $[m_j, \infty)$ is contained in $H^{[\beta_j]}$ for each $0 \leq j \leq k-1$.

Let s be the first α stage after which α is never initialized. At stage s , α defines the parameters p_α and ϵ_α , sets $r_\alpha = 0$ and defines the partition U_i^α . After these definitions, α begins its first cycle for defining Φ . It chooses x_1^α and defines $\Phi(X, x_1^\alpha) = 0$ for all $X \notin U_1^\alpha$. At each α stage $t \geq s$, α checks whether $\Phi_{\alpha,t}(H_t; x_1^\alpha)$ converges and if so whether each number $\langle \beta_j, y \rangle$ below the use with $m_j \leq y$ is in H_t . Because $\Phi_\alpha(H)$ is total and each interval $[m_j, \infty) \subseteq H^{[\beta_j]}$, α must eventually see a convergent computation which meets this criterion. When α sees an appropriate computation at stage $t \geq s$, it defines $\Phi(X; x_1^\alpha) > \Phi_{\alpha,t}(H_t; x_1^\alpha)$ and redefines its restraint r_α to be greater than the use of $\Phi_{\alpha,t}(H_t; x_1^\alpha)$.

Since $t \geq s$, no strategy to the left of α is ever eligible to act after t , so none of these strategies can place a number into H which will destroy the $\Phi_{\alpha,t}(H_t; x_1^\alpha)$ computation. The β_j strategies have already placed all the numbers below the use into $H^{[\beta_j]}$ so they will not destroy this computation. Any R strategy β with $\beta * \text{Fin} \sqsubseteq \alpha$ never places any more elements into H since if it did, the path would move to the left of α contradicting the fact that $t \geq s$. Finally, all strategies of lower priority than α respect α 's new restraint. Therefore, $\Phi_\alpha(H; x_1^\alpha) = \Phi_{\alpha,t}(H_t; x_1^\alpha)$ and we have met the condition of this lemma for all $X \in U_1^\alpha$.

We repeat the same argument for α 's remaining cycles to see that for each U_i^α , there is a witness x_i^α such that $\Phi(X; x_i^\alpha) > \Phi_\alpha(H; x_i^\alpha)$ for all $X \in U_i^\alpha$. Since the U_i^α partition 2^ω , we have established the lemma. \square

Lemma 2.7. *Each requirement M_e is met.*

Proof. Assume that some requirement M_e is not met. M_e is not met means that $\Phi_e(H)$ is total and that for some set $X \in T$, $\Phi(X)$ is dominated by $\Phi_e(H)$. Fix n such that for all $x > n$, $\Phi(X; x) < \Phi_e(H; x)$. Let e' be an index for a partial computable functional such that for all sets Z and all numbers x , if $x < n$, then $\Phi_{e'}(Z; x) = \Phi(X; x) + 1$ and if $n \leq x$, then $\Phi_{e'}(Z; x) = \Phi_e(Z; x)$. (Since $X \in T$, the computations $\Phi(X; x)$ for $x < n$ are defined, so we are just fixing the same finite initial segment of $\Phi_{e'}(Z)$ for every Z .) Let α' be the $M_{e'}$ strategy on the true path. We have that $\Phi_{\alpha'}(H)$ is total and for all x , $\Phi(X; x) < \Phi_{\alpha'}(H; x)$. These facts directly contradict Lemma 2.6. \square

3 Random examples

In this section, we show that almost every degree is not a.e. dominating and that almost every degree is bounded by a high degree that is not a.e. dominating. In contrast with the c.e. set H of the last section, all of the examples here satisfy some degree of randomness and hence have DNR (diagonally nonrecursive) degree.

Definition 3.1. A **Martin-Löf test** relative to a set A is a sequence $\langle U_n : n \in \omega \rangle$ of $\Sigma_1^{0,A}$ classes which is uniform in A such that $\mu(U_n) \leq 2^{-n}$ for each n . R is **n - A -random** (**n -random relative to A**) if for each Martin-Löf test relative to $A^{(n-1)}$ (the $n - 1^{\text{st}}$ jump of A), we have $R \notin \bigcap_n U_n$. If A is computable we say that R is **n-random**. R is **weakly n - A -random** if for each $\Sigma_n^{0,A}$ class C of measure 1, we have $R \in C$.

Notice two consequences of these definitions: if R is n - A -random then R is weakly n - A -random and if R is n - A -random for some A then R is n -random. For more information about randomness (including various equivalent definitions), see Kautz [8], Kurtz [10] or the online manuscript of Downey and Hirschfeldt [5]. One of the fundamental results about randomness that we will use repeatedly is the following.

Theorem 3.2 (Martin-Löf [14]). *For any A and $n \geq 1$, the measure of the class of all n - A -random sets is 1.*

We now state the main result of this section which we will prove at the end of this section.

Theorem 3.3. *Each 4-random degree is bounded by a high 2-random degree that is not almost everywhere dominating.*

Corollary 3.4. *Almost every degree is bounded by a high degree which is not a.e. dominating.*

Towards this theorem, as we would like to find degrees that are not almost everywhere dominating, we need examples of functions that are hard to dominate but are nevertheless computable by a sufficiently (to be specified later) random oracle. That is, suppose we fix functions $f_R \leq_T R$ for each sufficiently random set R . By Theorem 3.2, the measure of such R is 1. Let A be any a.e. dominating set and let S be a class of sets of measure 1 such that every function computable from an element of S is dominated by some function computable from A . Because S has measure 1 and the collection of sufficiently random R has measure 1, some such R must be in S . Therefore, the a.e. dominating set A must compute a function g which dominates some f_R function. If we can make the f_R functions hard to dominate, we can use them to construct examples of sets which are not a.e. dominating. We begin with the following theorem. (Kurtz [10] proved that the class of sets R such that there is a set $B <_T R$ for which R is c.e. in B has measure 1 and Kautz [8] later strengthen this result to Theorem 3.5.)

Theorem 3.5 (Kautz [8]). *If R is 2-random, then there is a set B such that $B <_T R$ and R is c.e. in B .*

We will combine this theorem with the following simple observation. Let R be any set and suppose that R is c.e. in B . For any fixed index e such that $R = W_e^B$, we can define the computation function c for R relative to this index e by

$$c(x) = \mu s(W_{e,s}^B \upharpoonright x = W_e^B \upharpoonright x = R \upharpoonright x).$$

Typically, we will abuse notation by suppressing the index e and referring to c as “the” computation function for R as a c.e. set in B .

Lemma 3.6. *If R is c.e. in B and f dominates the computation function for R as a c.e. set in B , then $f \oplus B \geq_T R$.*

Proof. Assume that e is the index relative to which the computation function is defined. Because f dominates the computation function, for sufficiently large x we have $R \upharpoonright x = W_e^B \upharpoonright x = W_{e,f(x)}^B \upharpoonright x$. \square

For any 2-random set R , fix B_R and f_R such that $B_R <_T R$, R is c.e. in B_R and f_R is the computation function for R as a set c.e. in B_R . Since $f_R \leq_T B_R \oplus R$ and $B_R <_T R$, we have $f_R \leq_T R$. Therefore, any a.e. dominating set A must be able to compute a function g which dominates some f_R . Hence, for some 2-random R , we must have $A \oplus B_R \geq_T R$. In other words, any a.e. dominating set must join some predecessor of some 2-random R above R . Stillwell [20] showed that sufficiently random sets do not have this property.

Lemma 3.7 (Stillwell [20]). *For any X, Y, G , if $X \not\leq_T Y$ and G is weakly 2- $X \oplus Y$ -random, we have $X \not\leq_T Y \oplus G$.*

From Lemma 3.7 and the comments above, it follows that if G is 2-random relative to every 2-random set R , then G is not a.e. dominating. Unfortunately, there is no such set G . (Suppose there is such a G . Let $R = G$ and notice that R is 2 random, but G cannot be 2-random relative to $R = G$.) However, van Lambalgen [11] showed that a set X can be random relative to every set that is random relative to X , and this turns out to be enough to prove Theorem 3.9.

Theorem 3.8 (van Lambalgen [11]). *Let $n \geq 1$. If A is n -random relative to B , and B is n -random, then B is n -random relative to A and $A \oplus B$ is n -random.*

Theorem 3.9. *If G is 2-random, then G is not a.e. dominating.*

Proof. Suppose G is a.e. dominating. There is a set S of measure 1 such that for all partial computable functionals Φ and all $X \in S$, if $\Phi(X)$ is total then $\Phi(X)$ is dominated by a function recursive in G . Because the collection of all sets which are 2-random relative to G has measure 1 (by Theorem 3.2) and because S has measure 1, S must contain some X that is 2-random relative to G . In particular, X is 2-random, so by Theorem 3.5, there is a set B such that $B <_T X$ and X is c.e. in B . Let Φ be such that $\Phi(X)$ is the computation function for X as a set c.e. in B . As X is in S , G computes a function dominating $\Phi(X)$. By Lemma 3.6, $G \oplus B \geq_T X$. By Lemma 3.7, G is not weakly 2- $X \oplus B$ -random. However, $X \oplus B \equiv_T X$ since $B <_T X$, so G is not weakly 2- X -random and hence G is not 2- X -random.

X is 2-random relative to G and that G is not 2-random relative to X . Suppose for a contradiction that G is 2-random. By Theorem 3.8, G is 2-random and X is 2-random relative to G implies that G is 2-random relative to X , giving the desired contradiction. \square

Corollary 3.10. *Almost every set is not a.e. dominating.*

Proof. This corollary follows from Theorems 3.2 and 3.9. □

Given Theorem 3.9, we can ask whether a 2-random set can be high. Kautz [8] showed that 3-random sets cannot be high, in fact $R^{(n)} \equiv_T R \oplus 0^{(n)}$ holds for each $n + 1$ -random set R , $n \geq 1$. Also, the argument of Theorem 3.9 does not generalize to all 1-random degrees, as $0'$ is a 1-random degree which is a.e. dominating. Nevertheless we get a positive answer.

Definition 3.11. A Turing machine U is called **prefix-free** if for all finite strings σ , $U(\sigma) \downarrow$ implies that $U(\tau) \uparrow$ for all proper extensions τ of σ . For any universal prefix-free Turing machine U , the **halting probability** of U is

$$\Omega_U = \sum_{U(\sigma) \downarrow} 2^{-|\sigma|}.$$

This notion relativizes to any oracle X and the following lemma lists three properties of Ω_U^X which will be useful for us later. The first two properties are due to Chaitin [1] and the third is due to Kurtz [10]. (For more information about Ω numbers see [1] and [6].) A real R is called c.e. in A if the set of rational numbers $q < R$ is c.e. in A .

Lemma 3.12. *The following properties hold for any universal prefix-free Turing machine U and any set X .*

1. Ω_U^X is a c.e. in X real.
2. Ω_U^X is 1-random relative to X .
3. $\Omega_U^X \oplus X \equiv_T X'$.

Theorem 3.13. *There is a high 2-random set below $0''$.*

Proof. Fix a universal prefix-free Turing machine U . For any set A , let $R = \Omega_U^A$. By Lemma 3.12, R is 1-random relative to A and satisfies $A' \equiv_T R \oplus A$. Now let $A = 0'$. Then R is 2-random and $0'' \equiv_T R \oplus 0' \leq_T R'$. □

Proof of Theorem 3.3. Let R_1 be 4-random. Let R_0 be a 2-random set with $0'' \leq_T R_0'$ (which exists by Theorem 3.13) and let $R = R_0 \oplus R_1$. We claim that R is the set we are looking for. Clearly $R \geq_T R_1$. As $R \geq_T R_0$, R is high. Since $R_0 \leq_T 0''$, R_1 is 2-random relative to R_0 , so by Theorem 3.8, R is 2-random and hence by Theorem 3.9, R is not a.e. dominating. □

4 Generic examples

In this section, we show that every 2-generic degree is not a.e. dominating.

Definition 4.1. Let V be any universal Turing machine, and let g be a computable function. **The time-bounded Kolmogorov complexity with time bound g** is the function C^g given by

$$C^g(x) = \min\{|p| : V(p) = x \text{ in } g(|x|) \text{ steps}\}.$$

(If there is no such p , then $C^g(x) = \infty$.) Z is **Kolmogorov random with time bound g** if there is a constant b such that

$$(\exists^\infty n)[C^g(Z \upharpoonright n) \geq n - b].$$

(We count $\infty > n$ for all $n \in \omega$, so the relation $C^g(Z \upharpoonright n) \geq n - b$ is computable in Z .)

Theorem 4.2 (Nies, Stephan and Terwijn [16]). *For each computable function g with $g(n) \geq n^2 + O(1)$ and each set Z , the following are equivalent.*

1. Z is 2-random.
2. Z is Kolmogorov random with time bound g .

Definition 4.3. Let A and B be sets. We say that A is **hyperimmune-free relative to B** , denoted by A is HIF(B), if for each function $f \leq_T A$ there is a function $g \leq_T B$ such that f is dominated by g .

The next proposition is a variation on Proposition 2.15 in [16].

Proposition 4.4. *Let A be a set. If there is a 2-random set Z such that Z is HIF(A), then there is a nonempty $\Pi_1^{0,A}$ class consisting entirely of 2-random sets.*

Proof. By Theorem 4.2, Z is Kolmogorov random with some time bound g and constant b . Let

$$f(m) = f_{g,b}^Z(m) = \mu n (\exists p_0, \dots, p_m \leq n) (\forall i \leq m) [C^g(Z \upharpoonright p_i) \geq p_i - b].$$

Note that $f \leq_T Z$ is a total function. Hence there exists $h \leq_T A$ such that h dominates f . In fact, we can assume that h majorizes f . (That is, $h(n) \geq f(n)$ for all n .) Consider the A -recursive tree

$$T = \{\sigma : (\forall m)[|\sigma| \geq h(m) \rightarrow (\exists p_0, \dots, p_m \leq |\sigma|) (\forall i \leq m) [C^g(\sigma \upharpoonright p_i) \geq p_i - b]]\}.$$

Since h majorizes f , Z is a path on T and so the set of paths of T is nonempty. Moreover, each path is time-bounded Kolmogorov random and hence 2-random by Theorem 4.2. Therefore, the set of paths through T is our desired $\Pi_1^{0,A}$ class. \square

Theorem 4.5 (Jockusch and Soare [7], relativized). *Let A be any set. Each nonempty $\Pi_1^{0,A}$ class P has a member R whose degree is c.e. in A .*

Proof. The $\Pi_1^{0,A}$ class P can be represented as the set of infinite paths through an A -computable tree $T_P \subseteq 2^{<\omega}$. Let R be the leftmost infinite path in T_P and we show that the degree of R is c.e. in A . Consider the set N of all finite binary strings which are either on R or to the left of R in $2^{<\omega}$. Because R is the leftmost path in T_P , N is c.e. in A , and clearly we have that $N \leq_T R$. To see that $R \leq_T N$, notice that $\sigma \in N$ is an initial segment of R if and only if it is the rightmost node of length $|\sigma|$ in N . \square

Alternately, we can view the elements of the $\Pi_1^{0,A}$ class P as reals, in which case the proof of Theorem 4.5 says that R is a c.e. in A real contained in P . (The set N represents the rational numbers $q < R$.) This perspective will be useful later when we want to view such a real as Ω_U^A for some universal prefix-free Turing machine U .

Proposition 4.6. *Let A be a set. If there is a 2-random set Z such that Z is HIF(A), then there is a 2-random R whose degree is c.e. in A .*

Proof. By Proposition 4.4, there is a nonempty $\Pi_1^{0,A}$ class P consisting of 2-random sets. By Theorem 4.5, there is a path R in P whose degree is c.e. in A . \square

As above, if we view the elements of P as reals, then Proposition 4.6 says that R is a c.e. in A real which is 2-random.

Theorem 4.7. *If A is a.e. dominating then there is a 2-random R whose degree is c.e. in A .*

Proof. Suppose A is a.e. dominating. Let C denote the class of all sets Z such that every $f \leq_T Z$ is dominated by some $g \leq_T A$ and let D denote the class of all 2-random sets. By definition, every $Z \in C$ is HIF(A) and because A is a.e. dominating, the measure of C is 1. Furthermore, since D has measure 1 (by Theorem 3.2), the intersection $C \cap D$ is nonempty. Therefore, there is a 2-random Z which is HIF(A) and we can apply Proposition 4.6. \square

In fact, Theorem 4.7 also follows from Theorem 5.5 below by considering Ω^A . Such a proof avoids the notion of time bounded Kolmogorov complexity. However, this approach does not give the stronger result of Proposition 4.6.

Definition 4.8. Let A and B be sets. We say that A is **diagonally nonrecursive in B** , denoted by $\text{DNR}(B)$, if there is a function $f \leq_T A$ such that for all e , $f(e) \neq \Phi_e(B; e)$.

Notice that no set A can compute a function which is $\text{DNR}(A)$ and that under this definition the $\text{DNR}(A)$ degrees are closed upwards trivially. (This definition is not the only way to relativize the property of diagonally nonrecursiveness.) The following lemma is a relativized version of the result of Kučera [9] that every 1-random R is $\text{DNR}(\emptyset)$.

Lemma 4.9. *For any A and any 1- A -random R , R is $\text{DNR}(A)$.*

Proof. Define a partition of ω by $I_0 = \{0\}$, $I_1 = \{1, 2\}$, $I_3 = \{3, 4, 5\}$, \dots , so that $|I_n| = n + 1$. Let $f \leq_T R$ be defined by $f(n) = R \upharpoonright I_n$. (That is, $f(n)$ is the canonical index for the finite set $R \upharpoonright I_n$.) Let $U_k = \{X \mid \exists s \exists e \geq k (X \upharpoonright I_e = \Phi_{e,s}(A; e))\}$. Because $|I_e| = e + 1$, the measure of all sets X for which $X \upharpoonright I_e = \Phi_e(A; e)$ is $2^{-(e+1)}$. Therefore, $\mu(U_k) \leq \sum_{e \geq k} 2^{-(e+1)} = 2^{-k}$. So the U_k classes form a Martin-Löf test relative to A . Since R is 1- A -random, there is a k such that $R \notin U_k$. Hence there are only finitely many e for which $f(e) = \Phi_e(A; e)$, and so R computes a $\text{DNR}(A)$ function. \square

Corollary 4.10. *Every 2-random R is $\text{DNR}(0')$.*

Proof. If R is 2-random, then R is 1- $0'$ -random. By Lemma 4.9, R is $\text{DNR}(0')$. \square

We will apply these results to Cohen generic sets. The forcing partial order for Cohen generics is $2^{<\omega}$ ordered by $\tau \leq \sigma$ (or τ is an extension of σ) if $\sigma \sqsubseteq \tau$.

Definition 4.11. A set G is called **n-A-generic** for forcing with a partial order P if for each $\Sigma_n^{0,A}$ set $D \subseteq P$, A either meets D or A meets the set of conditions in P having no extension in D . A subset $D \subset P$ is called **dense** if every $p \in P$ is extended by some $d \in D$. G is **weakly n-A-generic** if for each $\Sigma_n^{0,A}$ dense set $D \subseteq P$, G meets D .

Lemma 4.12. *If G is 2-generic, then G' is not $\text{DNR}(0')$.*

Proof. For a contradiction, assume that G' is $\text{DNR}(0')$. Each 2-generic is 1-generic and hence $G' \equiv_T G \oplus 0'$ (see for example [12]). So we can fix an index i such that $\Phi_i(G \oplus 0')$ is total and for all e , $\Phi_i(G \oplus 0'; e) \neq \Phi_e(0'; e)$. Consider the Σ_2^0 set $S = \{\sigma \in 2^{<\omega} \mid \exists e, \tau, s (\sigma \sqsubseteq \tau \wedge \Phi_{i,s}(\tau \oplus 0'; e) \downarrow = \Phi_{e,s}(0'; e) \downarrow)\}$. (Whenever we deal with computations such as $\Phi_{i,s}(\tau \oplus 0'; e) \downarrow$ in which the oracle has a finite component, we assume that the computation does not query any number in a finite component of the oracle which is larger than the length of that component.) Because G is 2-generic and by choice of i , there must be an initial segment ρ of G such that no extension of ρ is an element of S . However, since $\Phi_i(G \oplus 0')$ is total, we know that for every e , there is some $\sigma \sqsupseteq \rho$ such that $\Phi_i(\sigma \oplus 0'; e)$ converges. Notice that $0'$ can find such a σ by searching. Furthermore, since any such σ is not in S , we know that $\Phi_i(\sigma \oplus 0'; e)$ is not equal to $\Phi_e(0'; e)$. Therefore, $0'$ can compute a function that is $\text{DNR}(0')$ giving the desired contradiction. \square

Theorem 4.13. *If G is 2-generic then G is not a.e. dominating.*

Proof. Suppose that G is 2-generic and a.e. dominating. Because G is a.e. dominating, Theorem 4.7 implies that there is a 2-random R whose degree is c.e. in G . Therefore, $R \leq_T G'$. On the other hand, by Corollary 4.10, R is $\text{DNR}(0')$. Because the $\text{DNR}(0')$ degrees are closed upwards, G' is $\text{DNR}(0')$ which contradicts Lemma 4.12. \square

5 Degrees below $0'$

In this section, we give two proofs that that every a.e. dominating set below $0'$ is high. The first proof builds on Theorem 4.7 while the second proof uses the notion of being “low for random” to establish the stronger result that every a.e. dominating set is super-high. We begin with the following lemma which states that any real which satisfies the first two properties of Ω_U^X in Lemma 3.12 is actually an Ω number for some prefix-free universal machine relative to X .

Lemma 5.1 (Downey, Hirschfeldt, Miller and Nies [6]). *For any set A and real R , the following are equivalent:*

1. R is a c.e. real relative to A and 1-random relative to A ;
2. $R = \Omega_U^A$ for some universal prefix-free Turing machine U .

Lemma 5.2. *If A is a.e. dominating and $A \leq_T 0'$ then there exist universal prefix-free machines U, V with $\Omega_U^A = \Omega_V^{0'}$.*

Proof. By Theorem 4.7 there exists a real R that is 2-random (that is, 1-random in $0'$) and is c.e. in A . Since $A \leq_T 0'$, R is also 1-random in A and c.e. in $0'$. Hence by Lemma 5.1, there exist U, V such that $R = \Omega_U^A = \Omega_V^{0'}$. \square

Theorem 5.3. *If A is a.e. dominating and $A \leq 0'$, then $A' \equiv_T 0''$.*

Proof. By Lemma 5.2 and Property 3 of Lemma 3.12, $A' \geq_T \Omega_U^A \oplus 0' = \Omega_V^{0'} \oplus 0' \equiv_T 0''$. \square

This implication can be strengthened using the following theorem from Kautz [8]. (This theorem is a relativized form of a result first proved by Kučera [9].) For any string σ and set A , let $\sigma * A$ denote the set whose characteristic function is $\chi(n) = \sigma(n)$ for $n < |\sigma|$ and $\chi(n) = A(n - |\sigma|)$ if $n \geq |\sigma|$.

Theorem 5.4 (Kautz [8]). *Let Z be a set and let C be a $\Pi_1^{0,Z}$ class of positive measure. For every 1- Z -random R , there is a string σ and a set $A \in C$ such that $R = \sigma * A$.*

Theorem 5.5. *If Z is a.e. dominating then each 1- Z -random is 2-random.*

Proof. Let $P = \overline{U_1^{0'}} = 2^\omega - U_1^{0'}$ where $U_n^{0'}$, $n \in \omega$, is a universal Martin-Löf test relative to $0'$. Note that P is a Π_2^0 class of positive measure consisting entirely of 2-random reals. Suppose Z is a.e. dominating. By Theorem 1.2, P has a $\Pi_1^{0,Z}$ subclass C of positive measure.

Let R be 1- Z -random. By Theorem 5.4, there is a string σ and an $A \in C$ such that $R = \sigma * A$. A is 2-random because it is in C and we claim that R is 2-random. For a contradiction, suppose that R is not 2-random. Fix a $0'$ Martin-Löf test V_n , $n \in \omega$, such that $R \in \bigcap_n V_n$. Let $\hat{V}_n = \{\tau \mid \sigma * \tau \in V_{n+|\sigma|}\}$. Because $2^{-|\sigma|}\mu(\hat{V}_n) \leq \mu(V_{n+|\sigma|})$, we have $\mu(\hat{V}_n) \leq 2^{|\sigma|}2^{-(n+|\sigma|)}$ and hence $\mu(\hat{V}_n) \leq 2^{-n}$. Therefore, \hat{V}_n is also a $0'$ Martin-Löf test and $A \in \bigcap_n \hat{V}_n$ contradicting the fact that A is 2-random. \square

Because 2-random is the same as 1- $0'$ -random, we can restate Theorem 5.5 by saying that if Z is a.e. dominating, then every 1- Z -random set is 1- $0'$ -random. This characterization fits the following definition from Nies [15].

Definition 5.6. Low-for-random reducibility \leq_{LR} is defined by $A \leq_{LR} B$ iff every 1- B -random set is 1- A -random.

We can now restate Theorem 5.5 as $0' \leq_{LR} Z$ for every a.e. dominating Z . Notice that if $A \leq_T B$, then $A \leq_{LR} B$ because every Martin-Löf test relative to A is also a Martin-Löf test relative to B . Nies [15] also proved the following property of LR reducibility. (See Theorem 8.1 and the remarks before Proposition 8.3 in Nies [15].) In this statement, \leq_{tt} denotes truth table reducibility.

Theorem 5.7 (Nies [15]). $A \oplus B \leq_{LR} B \Rightarrow A' \leq_{tt} B'$.

Definition 5.8. If $A' \geq_{tt} 0''$ then we say that A is **super-high**.

Corollary 5.9. *If $Z \leq_T 0'$ is a.e. dominating then $Z' \equiv_{tt} 0''$ and hence Z is super-high.*

Proof. Let $Z \leq_T 0'$ be a.e. dominating. Since $Z \leq_T 0'$, we have $Z' \leq_{tt} 0''$ (in fact even $Z' \leq_1 0''$, see for example [19]). On the other hand, since Z is a.e. dominating, $0' \leq_{LR} Z$ by Theorem 5.5. Combining this reduction with $0' \oplus Z \leq_T 0'$ gives $0' \oplus Z \leq_{LR} Z$. By Theorem 5.7, $0'' \leq_{tt} Z'$, so we conclude $0'' \equiv_{tt} Z'$. \square

Using Corollary 5.9, we can give an alternate proof for Theorem 2.1 using index sets. Let $H_T = \{x \mid 0'' \leq_T W'_x\}$ be the index set for high c.e. sets and $H_{tt} = \{x \mid 0'' \leq_{tt} W'_x\}$ be the index set for super-high c.e. sets. A proof of the following theorem can be found in Soare [19].

Theorem 5.10 (Schwarz [17]). *H_T is Σ_3^0 complete.*

Lemma 5.11. *H_{tt} is a Σ_4^0 set.*

Proof. Let σ_n , $n \in \omega$ be a list of the well-formed formulas of sentential logic with sentential letters \mathbf{A}_n , $n \geq 1$. Let B be a set and let v be a truth assignment such that $v(\mathbf{A}_n) = T$ (true) iff $n \in B$. Let \bar{v} be the extension of v to all well-formed formulas. Write $B \models \sigma_n$ if $\bar{v}(\sigma_n) = T$. Then $A \leq_{tt} B$ iff there is a computable function f such that for all x , $x \in A$ iff $B \models \sigma_{f(x)}$. Hence $0'' \leq_{tt} W'_x \Leftrightarrow (\exists e)(\forall x)R(e, x)$ where

$$R(e, x) \Leftrightarrow \Phi_e(x) \downarrow \ \& \ [x \in 0'' \iff W'_x \models \sigma_{\Phi_e(x)}].$$

So H_{tt} is Σ_2^0 in $0''$, or in other words it is Σ_4^0 . \square

Corollary 5.12. *There exists a high, not super-high c.e. set.*

Proof. By Theorem 5.10, H_T is not a Σ_4^0 set, so $H_{tt} \neq H_T$. As clearly $H_{tt} \subseteq H_T$, we conclude that $H_{tt} \subsetneq H_T$. Let $x \in H_T - H_{tt}$; then W_x is high but not super-high. \square

Corollary 5.13. *There is a high c.e. set which is not a.e. dominating.*

Proof. Immediate from Corollaries 5.9 and 5.12. \square

6 High-above ideals

Definition 6.1. An **ideal** is a set $\mathcal{I} \subseteq 2^\omega$ such that if $X \in \mathcal{I}$ and $Y \leq_T X$, then $Y \in \mathcal{I}$ and if $X, Y \in \mathcal{I}$, then $X \oplus Y \in \mathcal{I}$. An ideal \mathcal{I} is called a **high-above ideal** if

$$(\forall A \in \mathcal{I})(\exists B \in \mathcal{I})(A <_T B \text{ and } A'' \leq_T B').$$

A trivial example of a high-above ideal is $\{A \mid (\exists n)(A \leq_T 0^{(n)})\}$. In this section, we construct two more examples of high-above ideals. In Proposition 6.8, we use Mathias forcing to show that for any infinite computable tree $T \subseteq 2^{<\omega}$ with no computable path, there is a high-above ideal which does not contain a path through T . Such an ideal is not the second order part of an ω -model of WKL_0 . (To see a very different application of Mathias forcing in recursion theory, the reader is referred to Cholak, Jockusch and Slaman [3].) In Proposition

6.9, we use the fact that 2-random sets are not a.e. dominating to construct a high-above ideal which does not contain an a.e. dominating set.

We begin with a relativized version of Martin's characterization of high degrees in terms of dominating functions. The proof in Chapter XI of Soare [19] relativizes to give the following theorem.

Theorem 6.2 (Martin [13], relativized). *For any sets A, B , we have $(A \oplus B)' \geq_T B''$ iff there is a single function computable in $A \oplus B$ which dominates all functions computable in B .*

Definition 6.3. Let H be any set. An **H-computable Mathias condition** is a pair $P = (F, C)$ where F is a finite subset of ω and C is an infinite H -computable set with $\max(F) < \min(C)$. We say that P_1 **extends** P_2 if $F_2 \sqsubseteq F_1 \subseteq F_2 \cup C_2$ and $C_1 \subseteq C_2$. We say that a set G **extends** a condition P if $F \sqsubseteq G$ and $G \subseteq F \cup C$.

We view H -computable Mathias conditions as pairs (e, i) where e is a canonical index for the finite set F and i is an index such that $C = W_i^H$. Using this notation, the set of H -computable Mathias conditions is Σ_3^H . Furthermore, if (F_1, C_1) and (F_2, C_2) are conditions, then the statement that (F_1, C_1) extends (F_2, C_2) is Π_2^H . Therefore, when discussing H -computable Mathias forcing, we will not discuss objects which are less than 3-generic, since merely describing the forcing conditions and their relationships requires statements which are Σ_3^H .

Lemma 6.4. *Let T be an infinite computable subtree of $2^{<\omega}$ and let A be a set such that A does not compute any path through T . If G is 3- A -generic for A -computable Mathias forcing, then $G \oplus A$ does not compute a path through T .*

Proof. We begin by defining what it means for a Mathias condition (F, C) to force various statements. We say $(F, C) \Vdash \Phi_e(G \oplus A; n) \downarrow$ if $\exists s(\Phi_{e,s}(F \oplus A; n) \downarrow)$ and

$$(F, C) \Vdash \Phi_e(G \oplus A; n) \uparrow \Leftrightarrow \forall \text{finite } \hat{F} \forall s (F \sqsubseteq \hat{F} \subseteq F \cup C \rightarrow \Phi_{e,s}(\hat{F} \oplus A; n) \uparrow).$$

The offset statement is equivalent to saying that no extension of (F, C) forces $\Phi_e(G \oplus A; n) \downarrow$. Given a condition (F, C) , this statement is Π_1^A and the statement that says (F, C) forces $\Phi_e(G \oplus A; n) \uparrow$ for some n is Σ_2^A . Therefore, the set S_e of all conditions (F, C) for which $\exists n[(F, C) \Vdash \Phi_e(G \oplus A; n) \uparrow]$ is Σ_3^A .

Assume that G is 3- A -generic for A -computable Mathias forcing and that $\Phi_e(G \oplus A)$ is total. Because the set S_e of conditions defined above is Σ_3^A , there must be a condition (F, C) such that G extends (F, C) and (F, C) has no extension in S_e . We say that such a condition forces $\Phi_e(G \oplus A)$ to be total.

There are two important features of conditions (F, C) which force $\Phi_e(G \oplus A)$ to be total. First, for every n and every (F', C') extending (F, C) , there is a condition (F'', C'') extending (F', C') which forces $\Phi_e(G \oplus A; n)$ to converge. Second, we can take the condition (F'', C'') to be a finite modification of (F', C') . That is, we can add a finite number of elements of C' to F' to get F'' and subtract a finite number of elements from C' to get C'' . In particular, if (F, C) forces $\Phi_e(G \oplus A)$ to be total, then there is a set $\hat{G} \leq_T A$ for which $\Phi_e(\hat{G} \oplus A)$ is total.

We construct \hat{G} by starting with $(F_0, C_0) = (F, C)$ and choosing conditions (F_n, C_n) such that (F_{n+1}, C_{n+1}) is a finite modification of (F_n, C_n) which extends (F_n, C_n) and which forces $\Phi_e(G \oplus A; n)$ to converge. These choices can be made using only the oracle A since A can compute C and A allows us to search for convergent computations of the form $\Phi_e(\hat{F} \oplus A; n)$ for finite extensions \hat{F} of F_n . The set $\hat{G} = \cup_n F_n$ clearly satisfies $\Phi_e(\hat{G} \oplus A)$ is total.

Next, we consider conditions which force $\Phi_e(G \oplus A)$ to not compute a path in T . Let $[T]$ denote the set of paths in T . We say $(F, C) \Vdash \Phi_e(G \oplus A) \notin [T] \Leftrightarrow$

$$\exists n ((F, C) \Vdash \Phi_e(G \oplus A; n) \uparrow) \vee \exists n (\Phi_e(F \oplus A) \upharpoonright n \downarrow \wedge \Phi_e(F \oplus A) \upharpoonright n \notin [T]).$$

That is, (F, C) forces $\Phi_e(G \oplus A) \notin [T]$ if (F, C) either forces that $\Phi_e(G \oplus A)$ is not total or it forces that some initial segment of $\Phi_e(G \oplus A)$ converges to a string not in T . As above, we want to say that $(F, C) \Vdash \Phi_e(G \oplus A) \in [T]$ if there is no extension of (F, C) which forces $\Phi_e(G \oplus A) \notin [T]$. If (F, C) already forces $\Phi_e(G \oplus A)$ to be total, then we can write this condition as $(F, C) \Vdash \Phi_e(G \oplus A) \in [T] \Leftrightarrow$

$$\forall \text{finite } \hat{F} \forall n (F \sqsubseteq \hat{F} \sqsubseteq F \cup C \rightarrow (\Phi_{e,s}(\hat{F} \oplus A) \upharpoonright n \downarrow \rightarrow \Phi_e(\hat{F} \oplus A) \upharpoonright n \in [T])).$$

Because the set of conditions (F, C) which force $\Phi_e(G \oplus A) \notin [T]$ is a Σ_3^A set, we know that for any 3- A -generic G , there is a condition (F, C) such that G extends (F, C) and either (F, C) forces $\Phi_e(G \oplus A) \notin [T]$ or (F, C) has no extension that forces $\Phi_e(G \oplus A) \notin [T]$. In other words, $\Phi_e(G \oplus A)$ is either forced into or out of $[T]$.

For a contradiction, suppose that $\Phi_e(G \oplus A)$ is a path in $[T]$. There is a condition (F, C) which is extended by G and which forces $\Phi_e(G \oplus A)$ to be total and $\Phi_e(G \oplus A) \in [T]$. Because (F, C) forces $\Phi_e(G \oplus A)$ is total, there is a set $\hat{G} \leq_T A$ such that \hat{G} extends (F, C) and $\Phi_e(\hat{G} \oplus A)$ is total. Furthermore, because (F, C) forces $\Phi_e(G \oplus A) \in [T]$, each initial segment $\Phi_e(\hat{G} \oplus A) \upharpoonright n$ must be an element of T . Therefore, $\Phi_e(\hat{G} \oplus A) \in [T]$. However, $\hat{G} \oplus A \leq_T A$, so we have a contradiction to the fact that A does not compute a path in T . \square

Definition 6.5. For any set X , the **principal function** p_X is defined by $p_X(n) =$ the $(n+1)^{\text{st}}$ element of X .

Lemma 6.6. *Let G be weakly 3- A -generic for forcing with A -computable Mathias conditions. The principal function p_G of G dominates all functions recursive in A .*

Proof. Let e be any index for which $\Phi_e(A)$ is total. For any condition (F, C) , we can A computably thin out C to $C' \subseteq C$ such that $p_{F \cup C'}$ dominates $\Phi_e(A)$. Furthermore, (F, C') will be an extension of (F, C) . Therefore, the set of conditions (\hat{F}, \hat{C}) for which $p_{\hat{F} \cup \hat{C}}$ dominates $\Phi_e(A)$ is dense and is also a Σ_3^A set. Therefore, G must meet each such set of conditions. \square

Corollary 6.7. *If G is weakly 3- A -generic for forcing with A -computable Mathias conditions, then $A'' \leq_T (G \oplus A)'$.*

Proof. This corollary follows immediately from Lemmas 6.2 and 6.6. \square

Proposition 6.8. *For any infinite computable tree $T \subseteq 2^{<\omega}$ with no computable paths, there is a high-above ideal \mathcal{I} such that no element of \mathcal{I} can compute a path through T .*

Proof. We define a sequence of sets $I_0 <_T I_1 <_T \dots$ such that I_n does not compute a path through T and $I''_n \leq_T I'_{n+1}$. $\mathcal{I} = \{X \mid \exists n (X \leq_T I_n)\}$ has the required properties.

Let $I_0 = \emptyset$ and notice that I_0 does not compute a path through T . Assume that I_n has been defined and does not compute a path through T . Let \hat{I}_n be a 3- I_n -generic with respect to computable I_n Mathias forcing and let $I_{n+1} = I_n \oplus \hat{I}_n$. By Lemma 6.4, I_{n+1} does not compute a path through T and by Corollary 6.7, $I''_n \leq_T I'_{n+1}$. \square

Proposition 6.9. *There is a high-above ideal that includes no a.e. dominating set.*

Proof. We define a chain $Q_0 \leq_T Q_1 \leq_T \dots$ and let $\mathcal{I} = \{X \mid (\exists n)(A \leq_T Q_n)\}$. To ensure $Q_n \leq_T Q_{n+1}$ we define first a sequence R_0, R_1, \dots and let $Q_0 = R_0$, $Q_{n+1} = R_{n+1} \oplus Q_n$.

Let $R_0 = \Omega^{0'}$, with respect to an arbitrary universal prefix-free machine. Let $R_{n+1} = \Omega^{Q'_n}$. Note that R_0 is 2-random and each R_{n+1} is 2-random relative to Q_n . Hence by van Lambalgen's Theorem, each R_n and Q_n is 2-random. Furthermore, Kautz [8] proved that 2-randoms are GL_1 , so we have that $Q'_n \equiv_T Q_n \oplus 0'$ for all n . Using this fact and Property 3 of Lemma 3.12, it follows that

$$Q'_{n+1} \equiv_T Q_{n+1} \oplus 0' = R_{n+1} \oplus Q_n \oplus 0' \equiv_T R_{n+1} \oplus Q'_n \equiv_T Q''_n.$$

Since each Q_n is 2-random (and hence not a.e. dominating by Theorem 3.9), the ideal generated by the Q_n 's contains no a.e. dominating set. \square

References

- [1] G.J. Chaitin, *Algorithmic information theory*, Cambridge University Press, Cambridge, 1987.
- [2] P. Cholak, N. Greenberg & J.S. Miller, Uniform almost everywhere domination, to appear.
- [3] P. Cholak, C.G. Jockusch & T.A. Slaman, On the strength of Ramsey's Theorem for pairs, *Journal of Symbolic Logic* **66** (2001), 1-55.
- [4] N.L. Dobrinen & S.G. Simpson, Almost everywhere domination, *Journal of Symbolic Logic*, **69** (2004), 914-922.
- [5] R. Downey and D.R. Hirschfeldt, *Algorithmic randomness and complexity*, online manuscript available at www.mcs.vuw.ac.nz/~downey
- [6] R. Downey, D.R. Hirschfeldt, J. Miller and A. Nies, Relativizing Chaitin's halting probability, submitted.
- [7] C.G. Jockusch Jr. and R.I. Soare, Degrees of members of Π_1^0 classes, *Pacific Journal of Mathematics*, **40** (1972), 605-616.
- [8] S. Kautz, *Degrees of Random Sets*, Ph. D. thesis, Cornell University, 1991.

- [9] A. Kučera, Measure, Π_0^1 -classes and complete extensions of PA, *Recursion theory week (Oberwolfach, 1984)*, Lecture Notes in Mathematics **1141**, 245–259, Springer, Berlin, 1985.
- [10] S. Kurtz, *Randomness and genericity in the degrees of unsolvability*, Ph. D. Thesis, University of Illinois at Urbana-Champaign, 1981.
- [11] M. van Lambalgen, *Random sequences*, Ph. D. Thesis, University of Amsterdam, 1987.
- [12] M. Lerman, *Degrees of unsolvability*, Springer-Verlag, Berlin, 1983.
- [13] D.A. Martin, Classes of recursively enumerable sets and degrees of unsolvability, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, **12** (1966), 295–310.
- [14] P. Martin-Löf, The definition of random sequences, *Information and Control*, **9** (1966), 602–619.
- [15] A. Nies, Lowness properties and randomness, to appear in *Advances in Mathematics*.
- [16] A. Nies, F. Stephan and S. Terwijn, Randomness, relativization and Turing degrees, to appear in *Journal of Symbolic Logic*.
- [17] S. Schwarz, *Index Sets of Recursively Enumerable Sets, Quotient Lattices, and Recursive Linear Orderings*, Ph.D. Dissertation, University of Chicago, 1982.
- [18] S.G. Simpson, *Subsystems of second order arithmetic*, Springer-Verlag, Berlin, 1999.
- [19] R.I. Soare, *Recursively enumerable sets and degrees*, Springer-Verlag, Berlin, 1987.
- [20] J. Stillwell, Decidability of the “almost all” theory of degrees, *Journal of Symbolic Logic*, **37** (1972), 501–506.

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