MATH 301.02/04 - QUIZ I - SOLUTION

Q1. Find the directional derivative of the function *f(x, y) = arctan(y/x)* at (1, 3) in the direction of <1, –3>.

Solution. *If f(x, y) = arctan(y/x)= tan⁻¹(y/x) then*

$$f_x = \frac{1}{1+(y/x)^2}.(-y/x^2) = \frac{-y}{x^2+y^2},$$

and

$$f_y = \frac{1}{1+(y/x)^2}.(1/x) = \frac{1}{1+(y/x)^2}.(x/x^2) = \frac{x}{x^2+y^2}.$$

Thus

$$\nabla f\Big|_{(1,3)} = \left[\frac{-y}{x^2+y^2}\underline{i} + \frac{x}{x^2+y^2}\underline{j}\right]\Bigg|_{(1,3)} = \frac{1}{10}(-3\underline{i}+\underline{j}),$$

and since function u = (i –3j)/$\sqrt{10}$ *, it follows that*

$$\nabla f \bullet \underline{u}\Big|_{(1,3)} = \frac{1}{10}(-3\underline{i}+\underline{j}) \bullet \frac{1}{\sqrt{10}}(\underline{i}-3\underline{j}) = \frac{1}{10\sqrt{10}}(-3-3) = \frac{-6}{10\sqrt{10}} = \frac{-3\sqrt{10}}{50}$$

Q2. Find a vector that gives the direction in which

$$F(x, y, z) = x^2 + 4xz + 2yz^2$$

increases most rapidly at (2, –1, 2). Find also the maximum rate of increase.

Solution. *If* $F(x, y, z) = x^2 + 4xz + 2yz^2$ *then*

$$F_x = 2x + 4z,\ F_y = 2z^2 \text{ and } F_z = 4x + 4yz$$

and so

$$\nabla F\Big|_{(2,-1,2)} = \left[(2x+4z)\underline{i} + 2z^2\underline{j} + (4x+4yz)\right]\Big|_{(2,-1,2)} = 12\underline{i} + 8\underline{j} = <12,8,0>.$$

Hence the direction of maximum rate of increase is

$$\|\nabla F(2,-1,2)\| = \sqrt{12^2+8^2+0^2} = \sqrt{208} = 4\sqrt{13}.$$

Q3.If $f(x,y) = x^3 - 12x + y^2 - 14y$, find all points at which $\|\nabla f\| = 0$.

Solution. *If* $f(x,y) = x^3 - 12x + y^2 - 14y$ *then*

$$f_x = 3x^2 - 12 \text{ and } f_y = 2y - 14.$$

Therefore $\|\nabla f\| = 0$ *implies*

$$\sqrt{(3x^2-12)^2 + (2y-14)^2} = 0$$

$\Rightarrow \quad (3x^2-12)^2 + (2y-14)^2 = 0$

$\Rightarrow \quad (3x^2-12)^2 = 0 \text{ and } (2y-14)^2 = 0$

$\Rightarrow \quad x = 2, -2$ *and* $y = 7$.

So the points are: (2, 7) and (-2, 7).