# RESEARCH STATEMENT

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## 1. Overview

My mathematical research interests are generally based in one hand on the study of partial differential equation methods for the mathematical description of critical phenomena in Statistical Physics and Euclidean Field Theory, in the other hand on the study of analytical methods for problems in Riemannian geometry.

The leading theme of my current research in mathematical physics involves the study of direct methods for integrals and operators of the type that appear naturally in Equilibrium Statistical Mechanics and Euclidean Field Theory. The focus is on the phenomenon of phase transition and the techniques involved stems mostly from the theory of partial differential equations and analysis.

The use of the Witten Laplacians in my work in statistical physics has triggered my interest in the study of the topology and the geometry of Riemannian manifolds by means of differential operators that are naturally defined in terms of the differential structure of the manifold

This present document describes my short-term research goal, achievements and future plans.

## 2. Specific topics

**Mathematical Physics:** In my dissertation, I investigated direct methods based on the analysis of the Witten Laplacians for the decay of the correlation functions and the analyticity of the pressure for certain classical unbounded spin systems.

In the context of classical equilibrium Statistical Mechanics, one is interested in a natural mathematical description of an equilibrium state of a physical system which consists of a very large number of interacting components. Consider for example a piece of ferromagnetic metal (like iron, cobalt or nickel) in thermal equilibrium. The piece consists of a very large number of atoms which are located at the sites of a crystal lattice $\Lambda$. Each atom shows a magnetic moment which can be visualized as a vector in $\mathbb{R}^3$. This magnetic moment is called the *spin* of the atom and represents the position of the atom in the lattice. The set $S$ of all possible orientations of the spins, is called the *state space* of the system. Each element $i$ of $\Lambda$ is called a (lattice) *site.* A particular state of the total system will be described by an element $x = (x_i)_{i\in\Lambda}$ of the product space $\Omega = S^{\Lambda}$. This set $\Omega$ is called the configuration space.

The physical system considered above is characterized by a sharp contrast: the microscopic structure is enormously complex, and any measurement of microscopic quantities is subject to Statistical fluctuations. The macroscopic behavior, however can be described by means of a few parameters such as magnetization, temperature...and macroscopic measurement leads to apparently deterministic results. This

contrast between the microscopic and the macroscopic level is the starting point of Classical Statistical Mechanics as developed by Maxwell, Boltzman and Gibbs. Their basic idea may be summarized as follows: The microscopic complexity may be overcome by a statistical approach and the macroscopic determinism then may be regarded as a consequence of a suitable law of large numbers. According to this philosophy, it is not adequate to describe the state of the system by a particular element $x$ of the configuration space $\Omega$. The system's state should rather be described by a family of $S-$valued random variables or (if we pass to the joint distribution of these random variables), by a probability measure $\mu$ on $\Omega$ consistent with the available partial knowledge of the system. In particular, $\mu$ should take account of the a priory assumption that the system is in thermal equilibrium.
Which kind of probability measure on $\Omega$ is suitable to describe a physical system in equilibrium? The term equilibrium clearly refers to the notion of forces and energies that act on the system. Thus one needs to define a Hamiltonian $\Phi$ which assigns to each configuration $x$ a potential energy $\Phi(x)$. In the physical system above, the essential contribution to the potential energy comes from the interaction of the microscopic components of the system and a possible external force. As soon as a Hamiltonian $\Phi$ have been specified, the answer to the question is generally believed to be the probability measure

$$d\mu(x) = Z^{-1}e^{-\beta\Phi(x)}d\lambda(x)$$

Here $d\lambda$ refers to a suitable a priory measure (for example the counting measure if $\Omega$ is finite), $\beta$ is a positive number which is proportional to the inverse of the absolute temperature and $Z > 0$ is a normalization constant. The above measure $\mu$ is called *the Boltzmann-Gibbs distribution*.
As we have mentioned above the number of atoms in a ferromagnet is extremely large. Consequently, the set $\Lambda$ in our mathematical model should be very large. According to a standard rule of a mathematical thinking, the intrinsic properties of large objects can be made manifest by performing suitable limiting procedures. It is therefore a common practice in Statistical Physics to pass to the infinite volume limit $|\Lambda| \to \infty$. (This limit is also referred to as the thermodynamic limit). The Boltzmann-Gibbs distribution does not admit a direct extension to infinite systems. However, when dealing with infinite systems, we can still look at finite subsystems provided the rest is held fixed. Indeed, starting with a potential $\phi$ we can define for each finite subsystem $\Lambda$ a Hamiltonian $\Phi_\Lambda^\phi$ which includes the interaction of $\Lambda$ with its fixed environment.
In the above, we argued that the physical systems like ferromagnets in equilibrium are reasonably modelled by Gibbs measures. We then should expect the Gibbs measure to exhibit a certain kind of behavior which reflects the physical phenomenon of phase transition. In order to find out what should happen, we consider the spontaneous magnetization of a ferromagnet at low temperature. First we place the ferromagnet in an external magnetic field (which is oriented along one of the axes of the ferromagnetic crystal). Turning the field off and waiting until equilibrium, we find that the ferromagnet exhibits a macroscopic magnetic moment in the same direction as the stimulating external field. A second experiment with an external field in the opposite direction produces an equilibrium state with the opposite magnetization as before. The ferromagnet thus admits two distinct equilibrium states.We thus expect that the physical phenomenon of phase transition should be reflected by the non-uniqueness of Gibbs measures. In 1968 Roland Dobroshin

who is considered as one of the founders of modern rigorous Statistical Mechanics proposed a uniqueness condition which would imply the absence of phase transitions. The condition roughly stated that the total interaction of a given spin with all other spins should be very small. This has triggered some interest in the study of the exponential decay of the two-point correlation function. The study of the exponential decay of the correlation also gained much interest when Fröhlich and Spencer discovered in 1981 that the non-uniqueness of equilibrium state is not the only critical phenomenon of physical interest, but a different sort of transition is characterized by a change from an exponential decay of the correlation to a power law decay.

The methods for investigating the dynamical behavior of certain classical unbounded spin systems took an interesting direction when powerful and sophisticated PDE techniques were introduced in the mathematical technology, The methods are generally based on the analysis of suitable differential operators

$$\mathbf{W}_{\Phi}^{(0)} = \left( -\mathbf{\Delta} + \frac{|\mathbf{\nabla}\Phi|^2}{4} - \frac{\mathbf{\Delta}\Phi}{2} \right)$$

and

$$\mathbf{W}_{\Phi}^{(1)} = -\mathbf{\Delta} + \frac{|\mathbf{\nabla}\Phi|^2}{4} - \frac{\mathbf{\Delta}\Phi}{2} + \mathbf{Hess}\Phi$$

which are in some sense, deformations of the standard Laplace Beltrami operator. These operators commonly called Witten Laplacians were first introduced by Edward Witten [14] in 1982 in the context of Morse theory for the study of some topological invariants of compact Riemannian manifolds. In 1994, Bernard Helffer and Jöhannes Sjostrand [5] introduced two elliptic differential operators

$$A_{\Phi}^{(0)} := -\mathbf{\Delta} + \mathbf{\nabla}\Phi \cdot \mathbf{\nabla}$$

and

$$A_{\Phi}^{(1)} := -\mathbf{\Delta} + \mathbf{\nabla}\Phi \cdot \mathbf{\nabla} + \mathbf{Hess}\Phi$$

These later operators serve to get direct methods for the study of integrals and operators in high dimensions for problems of the type that appear in Statistical Mechanics and Euclidean field theory. In 1996, Jöhannes Sjostrand [10] observed that these so called Helffer-Sjostrand operators are in fact equivalent to Witten's Laplacians.

Numerous techniques have been developed in the study of Laplace integrals associated to the equilibrium Gibbs state for certain unbounded spins systems. One of the most striking result is an exact formula for the covariance of two functions in terms of the Witten Laplacian on one forms leading to sophisticated methods for estimating the correlation functions. This formula is in some sense a stronger and more flexible version of the Brascamp-Lieb inequality [1].The formula may be written as follow:

$$\mathbf{cov}(f, g) = \int \left( A_{\Phi}^{(1)^{-1}} \mathbf{\nabla} f \cdot \mathbf{\nabla} g \right) e^{-\Phi(x)} dx. \tag{2.1}$$

Let us briefly mention that new methods that are purely based on spectral analysis have been recently developed by Helffer-Bodineau [2], Sjostrand-Bach-Jecko[20]. In these papers, the authors studied a certain class of unbounded spin models by means of the spectrum of the Witten Laplacian. In [21], the asymptotics of the two point correlation function to leading order in $\beta^{-1}$ was obtained under under weaker assumptions on the Hamiltonian. In 2003 V.Bach and J.S. Moller [21] proposed a refined version of the results in [20] by introducing a new twisted Witten Laplacian to relax the convexity assumption.

The results obtained in my dissertation involves exponentially weighted estimates leading to the exponential decay of the correlation functions. I managed to remove certain artificial assumptions made on the Hamiltonian in [5] where similar results were obtained in the one dimensional case and provided a generalization in the higher dimensional case under weaker assumptions on the potential. Let us also point out that the proof given in [5] which follows from the exponential decay of the limit mean value relies on the one dimensionality assumption which is not the case in our proof. I also obtained a formula suitable for a direct proof of the analyticity of the Pressure for certain classes of unbounded spin systems. The motivation for the study of the differentiability or even the analyticity of the pressure with respect to some distinguished thermodynamic parameters such as temperature, chemical potential or external field comes from the fact that the analytic behavior of the pressure is the classical thermodynamic indicator for the absence or existence of phase transition. The relevancy of my result in this direction can be seen by its potential contribution towards the solvability of the dipole gas problem in Coulomb systems. The dipole gas and other gases of particles interacting through Coulomb forces are very important systems in Statistical Mechanics. In particular, for dipole gases, the lack of screening is well known [35], and the analyticity of the pressure in the high temperature and low activity region has been proved in an indirect way, by means of renormalization group methods (see [36] and [37]). A direct proof of the analyticity of the pressure based on estimating the coefficients of the Mayer (Taylor) series is still an open problem. The close relationship between this model and the Coulomb gas in the Kortelitz-Thouless phase ($\beta > 8\pi$), go along with the non-existence of any proof for the analyticity of the pressure in the Coulomb gas. Indirect arguments are attempted in [38],[39] and [40].
Motivated by the desire to bring some light to this question, we obtained the formula

$$\frac{d^n}{dt^n} P_\Lambda(t) = \frac{(n-1)! < A_g^{n-1} g >_{t,\Lambda}}{|\Lambda|},$$

for the $nth$ derivative of the pressure when regarded as function of some thermodynamic parameter $t$. Here,

$$P_\Lambda(t) = \frac{1}{|\Lambda|} \log \left[ \int dx e^{-\Phi^t(x)} \right]$$

where

$$\Phi^t(x) = \Phi_\Lambda(x) - t g(x)$$

$\Phi_\Lambda$, $g$ are suitable $C^\infty-$functions and

$$A_g f := \left(A^1_{\Phi^t}\right)^{-1} \nabla f \cdot \nabla g$$

We believe that after a suitable regularization of the Coulomb potential at short distances, one can fit the problem into the framework of the models discussed in my dissertation and hopefully get an estimate of the coefficients of the Mayer series through this new formula for the derivatives of the finite volume pressure above. This is one of our short term goal; however, the problem I am currently working on is to get a sharp estimate of $< A_g^{n-1} g >_{t,\Lambda}$ for the analyticity of the pressure in the thermodynamic limit.

## References


[1] Brascamp, H.J and Lieb, E.H: *On Extensions of the Brunn-Minkowski and Prekopa-Leindler theorems including inequalities for log concave functions, and with application to the diffusion equation, J. Funct. Analysis, 22 (1976), 366-389.*

[2] Bodineau, T, Helffer, B: *Correlations, Spectral gap and LogSobolev inequalities for unbounded spins sytems, Proc. UAB Conf. March 16-20 1999, AMS/IP stud. adv. math 16 (2000), 51-66.*

[3] [He3] B. Helffer: *Semiclassical Analysis, Witten Laplacians and Statistical Mechanics Series on Partial Differential Equations and Applications-Vol.1 - World Scientific (2002).*

[4] Helffer, B:*Remarks on decay of correlations and Witten Laplacians. II, Analysis of the dependence on the interaction. Rev. Math. Phys, 11 (1999), no. 3, 321–336*

[5] Helffer, B and Sjöstrand. J: *On the Correlation for Kac-like Models in the Convex Case. J. of Stat. phys, 74 Nos.1/2, 1994.*

[6] Helffer, B and Sjöstrand, J: *Semiclassical expansions of the thermodynamic limit for a Schrödinger equation. I. The one well case. Méthodes semi-classiques, Vol. 2 (Nantes, 1991). Astérisque No. 210 (1992), 7–8, 135–181.*

[7] Johnsen, Jon: *On the spectral properties of Witten-Laplacians, their range projections and Brascamp-Lieb's inequality. Integral Equations Operator Theory 36 (2000), no. 3, 288–324.*

[8] Naddaf, A and Spencer, T: *On Homogenization and Scaling limit of Gradient Perturbations of a Massless Free Field, Comm. Math. Physics 183 (1997), 55-84.*

[9] Sjöstrand, J: *Correlation asymptotics and Witten Laplacians, Algebra and Analysis 8 (1996), No 1, 160-191*

[10] Sjöstrand, J: *Exponential convergence of the first eigenvalue divided by the dimension, for certain sequences of Schrödinger operators. Méthodes semi-classiques, Vol. 2 (Nantes, 1991). Astérisque No. 210 (1992), 10, 303–326..*

[11] Sjöstrand, J: *Potential wells in high dimensions. II. More about the one well case. Ann. Inst. H. Poincaré Phys. Théor. 58 (1993), no. 1, 43–53.*

[12] Sjöstrand, J: *Potential wells in high dimensions. I. Ann. Inst. H. Poincaré Phys. Théor. 58 (1993), no. 1, 1–41.*

[13] Witten, E: *Supersymmetry and Morse theory, J. of Diff. Geom. 17 (1982) 661-692.*

[14] Bach. V, Jecko. T and Sjostrand. J: *Correlation Asymptotics of Classical Lattice Spin Systems with nonconvex Hamilton Function at Low Temperature. Ann. Henri Poincare (2000), 59-100*

[15] Bach. V and Møller. J. S: *Correlation at Low Temperature, Exponential Decay. Jour. funct. anal 203 (2003), 93-148., 313-321.*

[16] Park, Y.M: *Lack of Screening in the Continuous Dipole Systems, Comm.Math.Phys. 70, 161-167, (1979)*

[17] Gawedzki, K and Kupiainen: *Block Spin Renormalization Group for Dipole Gas and* $(\nabla\phi)^4$, *Ann.Phys, 147, 198 (1983)*

[18] Brydges, D and Yau, H.T: *Grad* $\phi$ *Perturbations of Massless Gaussian Fields, Comm.Math.Phys. 129, 351, (1990).*

[19] Fröhlich, J and Spencer, T: *On the statistical Mechanics of Classical Coulomb and Dipole Gases, J.Stat.Phys. 24, 617 (1981)*

[20] Fröhlich, J and Park, Y.M: *Correlation Inequalities in the Thermodynamic Limit for Classical and Quantum Systems. Comm.Math.Phys, 59, 235-266, (1991).*

[21] Marchetti, D.H and Klein, A: *Power Law Fall-off in the two dimensional Coulomb Gases at inverse temperature* $\beta > 8\pi$, *J.Stat.Phys. 64, 135 (1991)*


**Geometric Analysis:** In differential geometry, the only natural differential operator on a Riemannian manifold $\mathbf{M}$ is the exterior derivative, $\mathbf{d}$, taking $k-$forms to $(k+1)-$forms. This operator is defined purely in terms of the smooth structure of the manifold. Using $\mathbf{d}$, and a smooth function on $\mathbf{M}$, we can define a series of differential operators

$$\mathbf{W}_{\Phi}^{(k)} = \mathbf{d}_{\Phi}\mathbf{d}_{\Phi}^{*} + \mathbf{d}_{\Phi}^{*}\mathbf{d}_{\Phi}$$

where

$$\mathbf{d}_{\Phi} = \mathbf{e}^{-\frac{\Phi}{2}}\mathbf{d}\mathbf{e}^{\frac{\Phi}{2}} \quad \text{and} \quad \mathbf{d}_{\Phi}^{*} = \mathbf{e}^{\frac{\Phi}{2}}\mathbf{d}^{*}\mathbf{e}^{-\frac{\Phi}{2}}.$$

Here, $\mathbf{d}^{*}$ is the exterior coderivative. These operators acting on $k-$forms on $\mathbf{M}$ are called Witten Laplacians. They were first introduced by E.Witten [1] in 1982. When $\Phi \equiv 0$, $\mathbf{W}_{0}^{(k)} = \mathbf{\Delta}^{(k)}$, the standard Laplace Beltrami operator. It is well known that the sprectrum $\{\lambda^{(k)}\}$ of $\mathbf{\Delta}^{(k)}$ contains both topological and geometric information. In particular, by the Hodge theorem the dimension of the kernel of $\mathbf{\Delta}^{(k)}$ equals the $k^{th}$ betti number, and so the Laplacians $\mathbf{\Delta}^{(k)}$ determine the Euler characteristic which is a topological invariant of the manifold. On the other hand, if we consider the heat equation

$$\left(\partial_t + \mathbf{\Delta}^{(k)}\right) u = 0$$

on $k-$forms with solution given by the heat semigroup $e^{-t\mathbf{\Delta}^{(k)}}u_o$, $u_o$ being the initial $k-$form, the behavior of the trace of the heat semigroup

$$Tr(e^{-t\mathbf{\Delta}^{(k)}}) = \sum_i e^{-t\lambda_i^{(k)}}$$

as $t \to 0$ is controlled by an infinite sequence of geometric data, involving the volume of the manifold and the integral of the scalar curvature. Now in the case where the operators are given by the $\mathbf{W}_{\Phi}^{(k)}$'$s$ for a suitable nonzero smooth function $\Phi$, one can hope to get more general and flexible versions of the results already obtained in the case of the Laplace Beltrami operator where $\Phi \equiv 0$. This is already seen in analysis with the Helffer-Sjostrand formula for the covariance which provides a more general and flexible version of the Brascamp-Lieb inequality.
Now we propose to formulate the problem which we hope will contribute to the unification of Analysis and Geometry.
We start with the construction of $\Phi$ in terms of the differential structure of the manifold.
Let $\mathbf{M}$ be a compact Riemannian manifold of dimension $m$ with differential structure $(U_\alpha, \varphi_\alpha)_{\alpha\in\mathcal{A}}$, $\mathcal{A}$. being an index set. A function $u$ defined in $\mathbf{M}$ is said to be in $C^p(\mathbf{M})$ if for every $\alpha$, the composite function $\left(\varphi_\alpha^{-1}\right)^* u$ defined by

$$\left(\varphi_\alpha^{-1}\right)^* u(x) = u\left(\varphi_\alpha^{-1}(x)\right) \qquad x \in \tilde{U}_\alpha = \varphi_\alpha\left(U_\alpha\right) \subset \mathbb{R}^m$$

is in $C^p(\tilde{U}_\alpha)$. Let $(u_\alpha)_{\alpha\in\mathcal{A}}$ be a family of smooth functions on $\tilde{U}_\alpha$ satisfying

$$u_\beta = \left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)^* u_\alpha \qquad \text{in } \varphi_\beta\left(U_\alpha \cap U_\beta\right).$$

It is kown that there exists a unique smooth function $\Phi$ on $\mathbf{M}$ such that

$$\Phi \circ \varphi_\alpha^{-1} = u_\alpha \qquad \text{for every } \alpha \in \mathcal{A} \qquad \text{(see [2])}$$

We now ask the following questions:
1. Given a suitable family $(u_\alpha)_{\alpha\in\mathcal{A}}$ of smooth functions on $\tilde{U}_\alpha$ as above, In what sense can one generalize the results already obtained for the standard Laplace Beltrami operator if we replaced it by $\mathbf{W}_{\Phi}^{(k)}$?
2. For which differential structure(s) the generalization is optimal?

## References


[1] Witten, E: *Supersymmetry and Morse theory, J. of Diff. Geom. 17 (1982) 661-692..*

[2] Lars, Hormander: *The Analysis of Linear Partial Differential operators, Springer-Verlac 1983*
.

[3] Helffer, B: *Semiclassical Analysis for The Schrodinger Operator and Applications, Lecture Notes in Mathematics, Springer-Verlac 1988..*

[4] Helffer, B and Sjostrand J: *Puits Multiples en Mecanique Semi-Classique, Etude du Complexe de Witten. Comm.Par.diff.eq, 10 (3), 245-340 (1985).*

[5] Bueler, E and Prokhorenkov, I: *Hodge Theory and Cohomology with compact support, Soochow Journal of mathematics, vol 28, No 1, 33-55 (2002)*

[6] Bueler, E: *The Heat Kernel Weighted Hodge Laplacian on Noncompact Manifolds. Tran.Amer.math.soc, vol 351 No 2, 683-713 (1999)*

[7] Bismut, J.M: *The Witten Complex and the Degenerate Morse inequalities. J.Diff.Geom. 23, 207-240 (1986)*

[8] Wai, Hon-Kit: *Witten-Helffer-Sjostrand Theory for* $S^1-$*Equivariant Cohomology Tran.Amer.math.soc, vol 351 No 6, 2141-2182*



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