Coincidences of Lipschitz Type Hybrid Maps and Invariant Approximation

A.R. Khan*
Department of Mathematical Sciences
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia

A.A. Domlo
Department of Mathematics
Taibah University, P.O. Box 30002
Madinah Munawarah, Saudi Arabia

N. Hussain
Department of Mathematics, Faculty of Science
King Abdul Aziz University, P.O. Box # 80203
Jeddah 21589, Saudi Arabia

Abstract

The aim of this paper is to obtain new coincidence and common fixed point theorems by using Lipschitz type conditions of hybrid maps (not necessarily continuous) on a metric space. As applications, we demonstrate the existence of common fixed points from the set of best approximations. Our work sets analogues, unifies and improves various known results existing in the literature.

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* Corresponding author.

E-mail: arahim@kfupm.edu.sa

1 Introduction

Common fixed point theorems for families of commuting contraction maps have been a popular area of research (see, e.g. Al-Thagafi [2], and Belluce and Kirk [4]). In 1982, Sessa [18] introduced the concept of weakly commuting maps to generalize commutativity. Jungck [10] generalized weak commutativity to the notion of compatible maps. In 1996, Jungck [11] further weakened compatibility to the concept of weak compatibility. Since then, many interesting fixed point theorems of compatible and weakly compatible maps under contractive conditions have been obtained by a number of authors. Singh and Mishra [23] considered the notion of (IT)-commuting hybrid maps to extend weak compatibility. Recently, Kamran [12] introduced the concept "f is T-weakly commuting" for hybrid maps f and T to generalize (IT)-commuting maps (see [12, Example 3.8]). Kamran [12] and Singh and Hashim [22] generalized the results in [1] for a hybrid pair of maps under strict contractive conditions.

The existence of fixed points from the set of best approximations has been studied by various authors; see Al-Thagafi [2], Hussain and Khan [7-8], Kamran [12], O'Regan and Shahzad [14], Sahab et al. [16], Shahzad [19-21], Singh [24] and Subrahamanyam [25].

In this paper, we establish new coincidence and common fixed point results for hybrid maps satisfying Lipschitz type conditions on a metric space. As applications, we obtain: the existence of common fixed points of the maps from the set of best approximations and solution of an eigenvalue problem for a multivalued map on a normed space.

2 Preliminaries

Throughout the paper, X denotes a metric space with metric d.

Suppose that $x \in X$ and $A \subseteq X$. Define $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $C(X)$, the class of all nonempty closed subsets of X. Let H be the Hausdorff metric with respect to d; that is,

$$H(A,B)=\max\left\{\sup_{x\in A}d(x,B),\sup_{y\in B}d(y,A)\right\},\text{ for every }A,B\in C(X).$$

Let $f:X\to X$ and $S:X\to C(X)$. A point $u\in X$ is a coincidence (common fixed) point of f and S if $fu\in Su(u=fu\in Su)$. Denote by $F(f)$, the set of fixed points of f. The maps f and S are: (1) weakly commuting if $fSx\in C(X)$ for all $x\in X$, and $H(Sfx,fSx)\leq d(fx,Sx)$; (2) compatible if $\lim_{n\to\infty}H(fSx_n,Sfx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Sx_n=A\in C(X)$ and $\lim_{n\to\infty}fx_n=t\in A$; (3) weakly compatible if they commute at their coincidence points; i.e., if $fu\in Su$ for some u in X, then $fSu=Sfu$; (4) (IT)-commuting at $x\in X$ if $fSx\subset Sfx$ (cf. [23]); (5) f is S-weakly commuting at $x\in X$ if $ffx\in Sfx$ (see [12]); (6) satisfying the property (E.A) (called tangential maps by Sastry and Murthy [17]) if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty}Sx_n=A\in C(X)$ and $\lim_{n\to\infty}fx_n=t\in A$.

Note that (i) weakly commuting maps are compatible, (ii) compatible maps are weakly compatible; (iii) weakly compatible maps are (IT)-commuting at their coincidence points; (iv) f and S are (IT)-commuting at the coincidence points implies that f is S-weakly commuting, but the converse in each case does not hold (for examples and counter examples, see [10-12], and [22-23]). We remark that commutativity, compatibility, and weak compatibility of f and S are equivalent at their coincidence points (cf. [22]). It is easy to see that two maps which are not compatible satisfy the property (E.A) (cf. [1]).

Let M be a subset of X and $u\in X$. We denote by $P_M(u)$, the set of best approximations to u from M; that is,

$$P_M(u)=\{y\in M:d(y,u)=d(u,M)\};$$

where $d(u,M)=\inf\{d(u,m):m\in M\}$. Let $\Im_0$ be the class of all closed convex subsets of X containing 0. For $M\in\Im_0$, we define $M_u=\{x\in M:\|x\|\leq 2\|u\|\}$. Clearly, $P_M(u)\subset M_u\in$

$\Im_0$ (see [2]). For $f : M \to X$, we follow Al-Thagafi [2] to define: $C_M^f(u) = \{x \in M : fx \in P_M(u)\}$.

Let E be a normed space. A real number λ is said to be an eigenvalue of a map $S : E \to C(E)$ if there exists a point $x \neq 0$ in E such that $\lambda x \in Sx$. Solutions of nonlinear eigenvalue problems for single-valued maps on a Banach space have been obtained by many authors (see, e.g., Kim [13]).

3 Coincidence and Common Fixed Points

We obtain some coincidence and common fixed point theorems for a hybrid pair of maps (not necessarily continuous) satisfying the property (E.A) and Lipschitz type conditions on a metric space X. We begin with a generalization of Theorems 3.4 and 3.10 of Kamran [12] and Theorem 3.1 due to Singh and Hashim [22]; the Lipschitz type condition we use, on the one hand, is simpler than their contractive conditions and on the other hand, contains as a special case the condition due to Pant [15].

Theorem 3.1 *Let $Y \subseteq X$, $S : Y \to C(X)$ and $f : Y \to X$ be such that:*

(i) f and S satisfy the property (E.A); i.e., there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} Sx_n = A \in C(X)$ and $\lim_{n\to\infty} fx_n = t \in A$;

(ii) fY is a complete subspace or SY is a complete subspace with $SY \subseteq fY$;

(iii) for all $x_n, n = 1, 2, 3, \ldots$ and $a \in Y$ with $fa = t$, the following Lipschitz type condition holds:

$$H(Sx_n, Sa) \leq (1 + u_n) \max\{r_n d(fx_n, fa), r_n d(Sx_n, fx_n) + \alpha_n d(Sa, fa), r_n d(Sx_n, fa) + \alpha_n d(Sa, fx_n)\} \quad (3.1)$$

where $\{u_n\}$, $\{r_n\}$ and $\{\alpha_n\}$ are sequences in $[0, +\infty)$ with $\lim_{n\to\infty} u_n = 0$, $\lim_{n\to\infty} r_n = r$

and $\lim_{n\to\infty}\alpha_n = \alpha$, *for some* $r \in [0,+\infty)$ *and* $\alpha \in [0,1)$. *Then* a *is a coincidence point of* S *and* f. *Moreover, if* $fa \in Y, f$ *is* S*-weakly commuting at* a *and* $ffa = fa$, *then* S *and* f *have a common fixed point.*

Proof. If fY is complete, then $\lim\limits_{n\to\infty} fx_n = t = fa$ for some $a \in Y$. We show that $fa \in Sa$ Suppose not; taking the limit as $n \to \infty$ in (3.1), we get

$$\begin{aligned} H(A, Sa) &\leq \max\{rd(fa, fa), r\,d(A, fa) + \alpha d(Sa, fa), r\,d(A, fa) + \alpha d(Sa, fa)\} \\ &= \alpha\, d(Sa, fa). \end{aligned}$$

Since $fa = t \in A$, it follows from the definition of the Hausdorff metric H that $d(fa, Sa) \leq H(A, Sa) \leq \alpha d(Sa, fa)$. Since $0 \leq \alpha < 1$, we get a contradiction. Thus $fa \in Sa$.

Now assume that $fa \in Y, f$ is S- commuting at a and $ffa = fa$. Thus $fa = ffa \in Sfa$ and so fa is a common fixed point of S and f. Similarly the case SY is complete and $SY \subseteq fY$ can be verified. ∎

The following example shows that our theorem extends substantially Theorems 3.4 and 3.10 of Kamran [12] and Theorem 3.1 of Singh and Hashim [22].

Example 3.2 Let X be the space of usual reals. Define $fx = x^2$ and

$$Sx = \begin{cases} \left[0, x^3\right] & \text{if} \quad x \geq 0 \\ \left[x^3, 0\right] & \text{if} \quad x < 0 \end{cases}$$

Note that the contractive condition of Theorem 3.1 in [22] is not satisfied; in particular, the contractive condition of Theorems 3.4 and 3.10 in [12] does not hold (take $x = 2$ and $y = 0$). Hence those theorems are not applicable here. Now, f and S satisfy the property (E.A) for the sequences $\left\{\frac{1}{n}\right\}$ and $\left\{1 - \frac{1}{n}\right\}$; in case of $\left\{\frac{1}{n}\right\}$; $t = 0$, $a = 0$ and (3.1) is satisfied because $H\left(S\left(\frac{1}{n}\right), S(0)\right) = \frac{1}{n^3} \leq \frac{1}{n^2} = d\left(f\left(\frac{1}{n}\right), f(0)\right)$. Same concerns the case of the sequence $\left\{1 - \frac{1}{n}\right\}$. All the conditions of Theorem 3.1 are satisfied and S and f have common fixed points 0 and 1.

Corollary 3.3 *Let $Y \subseteq X$ and $f, g : Y \to X$ be such that:*

(i) f and g satisfy the property (E.A); i.e., there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$;

(ii) fY is a complete subspace or gY is a complete subspace with $gY \subseteq fY$;

(iii) for all $x_n, n = 1, 2, 3, \ldots$ and $a \in Y$ with $fa = t$, the following condition holds:

$$d(gx_n, ga) \leq (1 + u_n) \max\{r_n d(fx_n, fa), r_n d(gx_n, fx_n) + \alpha_n d(ga, fa),$$
$$r_n d(gx_n, fa) + \alpha_n d(ga, fx_n)\}$$

where $\{u_n\}, \{r_n\}$ and $\{\alpha_n\}$ are as in the statement of Theorem 3.1. Then a is a coincidence point of f and g. Further, if $fa \in Y$, f and g are weakly compatible and $ffa = fa$, then f and g have a common fixed point.

Proof. By Theorem 3.1, a is a coincidence point of f and g. Since f and g are weakly compatible, it follows that $ffa = fga = gfa = gga$. Thus ga is a common fixed point of f and g. ■

Corollary 3.4 *Let $Y \subseteq X$ and $f, g : Y \to X$. Suppose that the conditions (i) and (ii) in Corollary 3.3 are satisfied and for all $x \neq y$ in Y, the following contractive condition holds:*

$$d(gx, gy) < \max\{d(fx, fy), r\, d(gx, fx) + \alpha\, d(gy, fy), \frac{1}{2}[d(gx, fy) + d(gy, fx)]\} \qquad (3.2)$$

where $r \in [0, +\infty)$ and $\alpha \in [0, 1)$. Then there exists a point $a \in Y$ such that a is a coincidence point of f and g. If $fa \in Y$, and f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. By Corollary 3.3, there exists $a \in Y$ such that $fa = ga$. Suppose that $fa \in Y$, and f and g are weakly compatible. Then $ffa = fga = gfa = gga$. We show that $gga = ga$. If not, then by (3.2), we get

$$
\begin{aligned}
d(ga, gga) &< \max\{d(fa, fga), rd(ga, fa) + \alpha d(gga, fga), \\
&\quad \frac{1}{2}[d(ga, fga) + d(gga, fa)]\} \\
&= d(ga, gga)
\end{aligned}
$$

a contradiction. Thus $gga = ga$ and so ga is a common fixed point of f and g. Now assume that $u \neq v$ are two common fixed points of f and g. By (3.2), we obtain

$$
\begin{aligned}
d(u, v) = d(gu, gv) &< \max\{d(fu, fv), rd(gu, fu) + \alpha d(gv, fv), \\
&\quad \frac{1}{2}[d(gu, fv) + d(gv, fu)]\} \\
&= d(u, v)
\end{aligned}
$$

which is a contradiction. Thus $u = v$. ■

Remark 3.5 If $r = \alpha = \frac{1}{2}$ in (3.2), then we obtain Corollary 3.6 of Singh and Hashim [22] which itself is an extension of Theorem 1 of Aamri and El Moutawakil [1].

The following result extends Theorem 4 of Sastry and Murthy [17] which is itself a generalization of the theorem of Pant [15].

Theorem 3.6 *Let $Y \subseteq X$ and $f, g : Y \to X$ be such that:*

(i) f and g satisfy the property (E.A);

(ii) fY is complete or gY is complete with $gY \subseteq fY$;

(iii) g is f-continuous; i.e., if $fx_n \to fx$, then $gx_n \to gx$ whenever $\{x_n\}$ is a sequence in Y and $x \in Y$.

Then f and g have a coincidence point. Further, if a is a coincidence point of f and g such that $fa \in Y$, f and g are weakly compatible and $fa \in Y$ such that $d(fa, ffa) \neq$

$\max\{d(fa, gfa), d(ffa, gfa)\}$ *whenever the right hand side is nonzero, then* f *and* g *have a common fixed point.*

Proof. By (i), there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. If fY is complete, then $\lim_{n\to\infty} fx_n = fa$ for some $a \in Y$. By (iii), $\lim_{n\to\infty} gx_n = ga$. Thus $fa = ga$. Weak compatibility of f and g implies that $fga = gfa$ and so $ffa = fga = gfa = gga$. Suppose $ffa \neq fa$, then

$$d(fa, ffa) \neq \max\{d(fa, gfa), d(gfa, ffa)\} = d(fa, ffa)$$

a contradiction. Thus $fa = ffa = gfa$; i.e., fa is a common fixed point of f and g. ■

The following theorem concerning four maps improves upon Theorem 3.2 [22] (compare the result with [1, Theorem 2]).

Theorem 3.7 *Let* $Y \subseteq X$, $S, T : Y \to C(X)$ *and* $f, g : Y \to X$ *be such that:*

(i) there exists a sequence $\{x_n\}$ *in* Y *such that* $\lim_{n\to\infty} Tx_n = A \in C(X)$ *and* $\lim_{n\to\infty} gx_n = t \in A$;

(ii) fY *or* TY *is a complete subspace,* gY *or* SY *is a complete subspace,* $SY \subseteq gY$ *and* $TY \subseteq fY$;

(iii) for any sequence $\{y_n\}$ *in* Y *with* $\lim_{n\to\infty} gy_n = t$ *and each* $x \in Y$ *with* $y_n \neq x$, *the following condition holds:*

$$H(Sx, Ty_n) \leq (1 + u_n) \max\{r_n d(fx, gy_n), \alpha_n[d(gy_n, Ty_n) + d(fx, Sx)], \alpha_n[d(fx, Ty_n) + d(gy_n, Sx)]\} \tag{3.3}$$

where $\{u_n\}, \{r_n\}$ *and* $\{\alpha_n\}$ *are sequences in* $[0, +\infty)$ *with* $\lim_{n\to\infty} u_n = 0$, $\lim_{n\to\infty} r_n = r$, $\lim_{n\to\infty} \alpha_n = \alpha$, *for some* $r \in [0, +\infty)$ *and* $\alpha \in [0, 1)$.

Then:

(a) f and S have a coincidence point and g and T have a coincidence point;

(b) if a is a coincidence point of f and S with $fa \in Y$, f is S-weakly commuting at a and $ffa = fa$, then f and S have a common fixed point;

(c) if b is a coincidence point of g and T with $gb \in Y$, g is T-weakly commuting at b and $ggb = gb$, then g and T have a common fixed point;

(d) S, T, f and g have a common fixed point provided that (b) and (c) hold.

Proof. (a) By (i) and $TY \subseteq fY$, there exists a sequence $\{y_n\}$ in Y such that $fy_n \in Tx_n$, for each n, and $\lim\limits_{n\to\infty} fy_n = t \in A = \lim\limits_{n\to\infty} Tx_n$. We show that $\lim\limits_{n\to\infty} Sy_n = A$. If not,then there exists a subsequence $\{Sy_k\}$ of $\{Sy_n\}$, a positive integer n and a real number $\epsilon > 0$ such that for $k \geq n$, we have $H(Sy_k, A) \geq \epsilon$. From (iii), we get

$$H(Sy_k, Tx_k) \leq (1+u_k)\max\{r_k d(fy_k, gx_k), \alpha_k\left[d(gx_k, Tx_k) + d(fy_k, Sy_k)\right],$$

$$\alpha_k\left[d(fy_k, Tx_k) + d(gx_k, Sy_k)\right]\}$$

$$\leq (1+u_k)\max\{r_k d(fy_k, gx_k), \alpha_k[d(gx_k, Tx_k) + d(fy_k, A) + H(A, Sy_k)],$$

$$\alpha_k[d(fy_k, Tx_k) + d(gx_k, A) + H(A, Sy_k)]\}.$$

Taking the limit as $k \to \infty$, we obtain $\lim_{k\to\infty} H(Sy_k, A) \leq \alpha \lim_{k\to\infty} H(A, Sy_k)$. Since $0 \leq \alpha < 1$, we get a contradiction. Thus $\lim\limits_{n\to\infty} Sy_n = A$. Consequently, f and S satisfy the property (E.A) for the sequence $\{y_n\}$. If fY or TY is complete, then there exists a point $a \in Y$ such that $\lim\limits_{n\to\infty} fy_n = t = fa$. We show that $fa \in Sa$. If not, then

$$H(Sa, Tx_n) \leq (1+u_n)\max\{r_n d(fa, gx_n), \alpha_n[d(gx_n, Tx_n) + d(fa, Sa)],$$

$$\alpha_n[d(fa, Tx_n) + d(gx_n, Sa)]\}.$$

Taking the limit as $n \to \infty$, we have $H(Sa, A) \leq \alpha\, d(fa, Sa)$. Thus, $d(Sa, fa) \leq H(Sa, A) \leq \alpha\, d(fa, Sa)$ a contradiction by virtue of $fa = t \in A$. Thus $fa \in Sa$. Since $SY \subseteq gY$, therefore there exists a sequence $\{z_n\}$ in Y such that $gz_n \in Sy_n$, for each n, and $\lim_{n\to\infty} gz_n = t \in A = \lim_{n\to\infty} Sy_n$. As above, we can show that $\lim_{n\to\infty} Tz_n = A$. If gY or SY is complete, then there exists a point $b \in Y$ such that $\lim_{n\to\infty} gz_n = t = gb$. Take the sequence $b_n = b$, for all n, so, $\lim_{n\to\infty} gb_n = t$. Suppose $gb \notin Tb$. Using (iii) and taking the limit as $n \to \infty$, we obtain $H(Sa, Tb) \leq \alpha\, d(gb, Tb)$. Hence $d(gb, Tb) = d(fa, Tb) \leq H(Sa, Tb) \leq \alpha\, d(gb, Tb)$; a contradiction. Thus $gb \in Tb$.

(b) Now, if $fa \in Y$, f is S-weakly commuting at a and $ffa = fa$, then $fa = ffa \in Sfa$ and so fa is a common fixed point of f and S.

(c) Similar to case (b).

(d) Immediate, in view of $fa = gb = t$.

■

We close this section with an application of Theorem 3.1 to solve an eigenvalue problem:

Theorem 3.8 *Let E be a normed space, $Y \subseteq E$ and $S : Y \to C(E)$ satisfy:*

(i) there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} Sx_n = A \in C(E)$ and $\lim_{n\to\infty} \lambda x_n = t \in A$ where λ is a real number;

(ii) λY is a complete subspace or SY is a complete subspace with $SY \subseteq \lambda Y$;

(iii) for all $x_n, n = 1, 2, 3, \ldots$ and $a \in Y$ where $a = t/\lambda$, the following condition holds:

$$H(Sx_n, Sa) \leq (1 + u_n) \max\{r_n||x_n - a||, r_n\, d(Sx_n, \lambda x_n) + \alpha_n\, d(Sa, \lambda a),$$

$$r_n\, d(Sx_n, \lambda a) + \alpha_n\, d(Sa, \lambda x_n)\}$$

where $\{u_n\}, \{r_n\}$ and $\{\alpha_n\}$ are as in Theorem 3.1. Then S has an eigenvalue.

Proof. Let $f : Y \to E$ be defined by $fx = \lambda x$. Then, by Theorem 3.1, $fa \in Sa$; i.e., $\lambda a \in Sa$. Thus λ is an eigenvalue of S and a is the corresponding eigenvector. ■

4 Approximation Results

In this section, we obtain common fixed points of the maps, considered in Section 3, from the set of best approximations. The following theorem extends Theorem 3.14 of Kamran [12].

Theorem 4.1 *Let $M \subset X$, $u \in X$ and $D = P_M(u)$ be nonempty. Suppose that $f : X \to X$ and $S : X \to C(X)$ satisfy:*

(i) there exists a sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} Sx_n = A \in C(D)$ and $\lim_{n\to\infty} fx_n = t \in A$;

(ii) fD is complete or SD is complete with $SD \subseteq fD$;

(iii) for all $x_n, n = 1, 2, 3, \ldots$ and $a \in D$ with $fa = t$, (3.1) holds.

If D is S-invariant and $fD = D$, then a is a coincidence point of f and S. Further, if f is S-weakly commuting at a and $ffa = fa$, then f and S have a common fixed point in $P_M(u)$.

Proof. Since $SD \subseteq D$, it follows that S maps D into $C(D)$. The result follows from Theorem 3.1. ■

The existence of common fixed points from the set of best approximations for four maps is established in the next result which can be easily verified on the basis of Theorem 3.7. It is remarked that study of best approximations in the context of four maps is a new one in the literature.

Theorem 4.2 *Let $M \subset X$, $u \in X$ and $D = P_M(u)$ be nonempty and complete. Assume that $f, g : X \to X$ and $S, T : X \to C(X)$ satisfy:*

(i) there exists a sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} Tx_n = A \in C(D)$ and $\lim_{n\to\infty} gx_n = t \in A$;

(ii) for any sequence $\{y_n\}$ in D with $\lim_{n\to\infty} gy_n = t$ and each $x \in D$, (3.3) holds.

(iii) D is S and T-invariant, $fD = D$ and $gD = D$.

Then:

(a) f and S have a coincidence point $a \in D$, and g and T have a coincidence point $b \in D$;

(b) if f is S-weakly commuting at a and $ffa = fa$, then $D \cap F(f) \cap F(S) \neq \phi$;

(c) if g is T-weakly commuting at b and $ggb = gb$, then $D \cap F(g) \cap F(T) \neq \phi$;

(d) S, T, f and g have a common fixed point from D provided that (b) and (c) hold.

Let D be a nonempty subset of a normed space E. The set D is called q-starshaped if there exists $q \in D$ such that the segment $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x, is contained in D for all $x \in D$. Suppose that D is q-starshaped, f and g are selfmaps of D with $q \in F(f)$, and $C_q(f, g) = \cup\{C(f, g_k) : 0 \leq k \leq 1\}$ where $C(f, g)$ denotes the set of coincidence points of f and g, and $g_k x = kgx + (1-k)q$. The maps f and g are called: (i) R-subweakly commuting if $\|gfx - fgx\| \leq Rd(fx, [q, gx])$ for all $x \in D$ and some $R > 0$; (ii) R-subcommuting if $\|gfx - fgx\| \leq \frac{R}{k}\|((1-k)q + kgx) - fx\|$ for all $x \in D$, $\ k \in (0, 1]$ and some $R > 0$. If $R = 1$, we get the concept of 1-subcommuting map [8]; (iii) C_q-commuting if $fgx = gfx$, for all $x \in C_q(f, g)$. Clearly, R-subweakly commuting and R-subcommuting selfmaps are C_q-commuting, and C_q-commuting selfmaps are weakly compatible, but the converse in each case does not hold (see [3] and [9]).

Recently, Hussain and Khan [8] obtained in Theorem 3.1, a generalization of Theorem 3 by Sahab et al. [16] for 1-subcommutative single-valued selfmaps of a Hausdorff locally convex space; an improvement of this result is given below for hybrid maps in the setup of a metric space.

Theorem 4.3 *Let $M \subset X$ and $D = P_M(u)$ be nonempty where u is a common fixed point of the maps $f, g : X \to X$. Suppose that:*

(i) f and g satisfy the property (E.A) on D;

(ii) fD or gD is complete;

(iii) $fD = D$ and $g(\partial M) \subseteq M$ (here ∂M denotes the boundary of M);

(iv) the pair $\{f, g\}$ satisfies for all $x \in D \cup \{u\}$,

$$d(gx, gy) < \begin{cases} d(fx, fu) & \text{if } y = u, \\ max\{d(fx, fy), rd(fx, gx) + \alpha d(fy, gy), & \\ \frac{1}{2}[d(fx, gy) + d(fy, gx)]\} & \text{if } y \in D, \end{cases}$$

Then f and g have a coincidence point in D. Further, if f and g are weakly compatible, then f and g have a unique common fixed point in D.

Proof. Let $y \in D$. Then $fy \in D$. By the definition of $P_M(u)$, $y \in \partial M$ and so $gy \in M$. By (iv), we have

$$d(gy, u) = d(gy, gu) \le d(fy, fu) = d(fy, u).$$

Now, $gy \in M$ and $fy \in D$ imply that $gy \in D$; consequently, f and g are selfmaps of D. The result follows from Corollary 3.4. ■

As an application of Corollary 3.4, we obtain a generalization of Theorems 4.1-4.2, about C_q-commuting maps, in [3] which themselves are extensions of Theorems 1.1, 4.1 and 4.2 in [2], Corollary 2.16 in [6], Theorems 2.3-2.4 in [19], Theorem 2.1 in [20] and Theorem 2.9 in [21]. Indeed, our class of maps is more general but needs to satisfy the property $(E.A)$, as indicated by the following:

Example 4.4 Let $X = R$ with usual norm and $M = [1, \infty)$. Let $f(x) = 2x-1$ and $g(x) = x^2$, for all $x \in M$. Let $q = 1$. Then M is q-starshaped with $fq = q$ and $C_q(f, g) = [1, \infty)$. Note that f and g are weakly compatible maps and satisfy the property $(E.A)$ but f and g are not C_q-commuting (and hence not R-subweakly commuting).

Theorem 4.5 *Let f and g be selfmaps of a normed space E with $u \in F(f) \cap F(g)$ and $M \in \Im_0$ such that $g(M_u\ ) \subset\ f(M) = M$. Suppose that $\|fx - u\| = \|x - u\|$ for all $x \in M$, $\|gx - u\| \leq \|fx - u\|$ for all $x \in M_u$, and one of the following two conditions is satisfied:*

(a) $clf(M_u)$ is compact (cl denotes the closure),

(b) $clg(M_u)$ is compact.

Then:

(i) $P_M(u)$ is nonempty, closed and convex,

(ii) $g(P_M(u)) \subset f(P_M(u)) = P_M(u)$,

(iii) $P_M(u) \cap F(f) \cap F(g) \neq \phi$ provided f and g are weakly compatible, satisfy the property (E.A) on $P_M(u)$, and (3.2) holds for all $x \neq y$ in $P_M(u)$.

Proof. (i) We follow the arguments used in [6] and [14]. We may assume that $u \notin M$. If $x \in M \setminus M_u$, then $\|x\| > 2\|u\|$. Note that

$$\|x - u\| \geq \|x\| - \|u\| > \|u\| \geq d(u, M_u).$$

Thus, $d(u, M_u) = d(u, M) \leq \|u\|$. Also $\|z - u\| = d(u, clf(M_u))$ for some $z \in clf(M_u)$. This implies that $d(u, M_u) \leq d(u, clf(M_u)) \leq d(u, f(M_u)) \leq \|fx - u\| \leq \|x - u\|$, for all $x \in M_u$. Hence $\|z - u\| = d(u, M)$ and so $P_M(u)$ is nonempty. Moreover it is closed and convex. The same conclusion holds whenever $clg(M_u)$ is compact where we replace f by g and utilize inequalities $\|gx - u\| \leq \|fx - u\|$ and $\|fx - u\| = \|x - u\|$ to obtain that $P_M(u)$ is nonempty.

(ii) Let $z \in P_M(u)$. Then $\|fz - u\| = \|fz - fu\| \leq \|z - u\| = d(u, M)$. This implies that $fz \in P_M(u)$ and so $f(P_M(u)) \subset P_M(u)$. For the converse assume that $y \in P_M(u)$, then $y \in M = f(M)$. Thus there is some $x \in M$ such that $y = fx$. Now $\|x - u\| = \|fx - u\| = \|y - u\| = d(u, M)$. This implies that $x \in P_M(u)$ and so $f(P_M(u)) = P_M(u)$.

Let $y \in g(P_M(u))$. Since $g(M_u) \subset f(M)$ and $P_M(u) \subset M_u$, there exist $z \in P_M(u)$ and $x_0 \in M$ such that $y = gz = fx_0$. Further, we have

$$\|fx_0 - u\| = \|gz - gu\| \leq \|fz - fu\| = \|fz - u\| \leq \|z - u\| = d(u, M).$$

Thus, $x_0 \in C_M^f(u) = P_M(u)$ and so (ii) holds. In both the cases (a) and (b), $f(P_M(u))$ is complete. Hence (iii) follows from Corollary 3.4. ■

The following result extends and improves [2, Theorem 4.2], [5, Theorem 8], [14, Corollary 2.10], [19, Theorems 2.3-2.4], [20, Theorem 2.1] and [21, Theorem 2.9].

Theorem 4.6 *Let f and g be selfmaps of a normed space E with $u \in F(f) \cap F(g)$ and $M \in \Im_0$ such that $g(M_u) \subset f(M) \subset M$. Suppose that $\|fx - u\| \leq \|x - u\|$ for all $x \in M_u$, $\|gx - u\| \leq \|fx - u\|$ for all $x \in M_u$, and one of the following conditions is satisfied:*

(a) $clf(M_u)$ is compact,

(b) $clg(M_u)$ is compact.

Then

(i) $P_M(u)$ is nonempty, closed and convex,

(ii) $g(P_M(u)) \subset f(P_M(u)) \subset P_M(u)$, provided that $\|fx - u\| = \|x - u\|$ for all $x \in C_M^f(u)$, and

(iii) $P_M(u) \cap F(f) \cap F(g) \neq \phi$ provided that $\|fx - u\| = \|x - u\|$ for all $x \in C_M^f(u)$, $g(P_M(u))$ is closed, f and g are weakly compatible, satisfy the property (E.A) on $P_M(u)$, and (3.2) holds for all $x \neq y$ in $P_M(u)$.

Proof. (i) and (ii) follow as in Theorem 4.5.

(iii)(a) By (ii), the compactness of $clf(M_u)$ implies that $g(P_M(u))$ is complete. The conclusion now follows from Corollary 3.4 applied to $P_M(u)$.

(iii)(b) Obviously, $g(P_M(u))$ is complete. Corollary 3.4 now guarantees that $P_M(u) \cap F(g) \cap F(f) \neq \emptyset$. ■

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