On Some Characteristics of the Bivariate T-Distribution

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Abstract The bivariate *t*-distribution is a natural generalization of the bivariate normal distribution as a derived sampling distribution. For broad spectrum of researchers, the paper emphasizes the bivariate *t*-distribution as a mixture of bivariate normal distribution and an inverted gamma distribution. Moments and related characteristics of the distribution are presented from this perspective.

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1. Introduction

It is well known that the bivariate *t*-distribution arises as a derived sampling distribution from the bivariate normal distribution and the chi-square distribution (Anderson, 2003, 289). In this paper we emphasize a scale mixture representation to calculate some product moments and standardized moments for the bivariate *t*-distribution. This compounds a nonnegative continuous distribution with the bivariate normal distribution. Though we have considered an 'inverted chi-square' distribution as the scaling or the compounding distribution, the generalization of the model to any other continuous distribution is evident. This section of the paper is a decent introduction of relevant materials for Section 2 to motivate those who are experts in statistical analysis by normal distributions but wish to check the robustness of their theories for a broader context of *t*-distributions. Wherever possible, matrix algebra has been avoided for broad spectrum of readers.

Since the use of the multivariate *t*-distribution is on the increase in business especially in stock returns, the paper will enlighten as well as stimulate research in business, econometrics and statistics. Interested readers may go through Joarder (1992), Kotz and Nadarajah (2004), Nadarajah and Kotz (2005) and Nadarajah and Kotz (2005a) and the references therein.

(i) The Univariate T-Distribution

A random variable Z is said to have a univariate *t*-distribution with ν degrees of freedom if its probability density function (pdf) is given by

$$f_1(z) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\,\Gamma(\nu/2)}\left(1+\frac{z^2}{\nu}\right)^{-(\nu+1)/2}, \quad -\infty < z < \infty .$$

It is denoted by t_ν. Also let T (read as tau) be a random variable taking value τ have an inverted chi-square distribution given in the following theorem..

Theorem 1.1 Let $Z \sim N(0,\tau^2)$ and $\nu \mathrm{T}^{-2} \underline{\underline{d}}\ W \sim \chi^2_\nu$ where the symbol $\underline{\underline{d}}$ means that both sides of it have the same distribution.

(a) $(\nu/W)^{1/2} Z \sim t_\nu$,

$$(b)\ h_{\mathrm{T}}(\tau) = \frac{2(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \tau^{-(\nu+1)} e^{-\nu/(2\tau^2)},\ \ 0 < \tau, \tag{1.1}$$

(c) the pdf (probability density function) of the univariate *t*-dsitribution with ν degrees of freedom has the following representation:

$$f_1(z) = \int_0^\infty \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{z^2}{2\tau^2}\right) h_{\mathrm{T}}(\tau) d\tau. \tag{1.2}$$

Proof. The proof of the first two parts are well known.

By plugging (1.1) in (1.2) we have

$$f_1(z) = \int_0^\infty \frac{(\nu/2)^{\nu/2}\sqrt{2/\pi}}{\tau\Gamma(\nu/2)} \exp\left[-\left(\frac{z^2}{2\tau^2} + \frac{\nu}{2\tau^2}\right)\right] \tau^{-(\nu+1)} d\tau.$$

With the transformation $\nu/\tau^2 = w$, we have

$$f_1(z) = \int_0^\infty \frac{(\nu/2)^{\nu/2}\sqrt{2/\pi}}{\Gamma(\nu/2)} \sqrt{w} \exp\left[-\left(\frac{z^2}{2\nu} + \frac{1}{2}\right) w\right] w^{(\nu+1)/2} \left(\frac{1}{2} w^{-3/2} \sqrt{\nu}\right) dw$$

$$= \int_0^\infty \frac{1}{2^{\nu/2}\Gamma(\nu/2)\sqrt{2\pi\nu}} \exp\left[-\left(\frac{z^2}{2\nu} + \frac{1}{2}\right) w\right] w^{(\nu-1)/2} dw$$

which simplifies to the pdf of t_ν proving part (c).

Corollary 1.1 Let T have the pdf given by (1.1). Then the a-th moment of T is given by

$$\gamma_a = E\left(\mathrm{T}^a\right) = \frac{(\nu/2)^{a/2}\Gamma(\nu/2 - a/2)}{\Gamma(\nu/2)},\ \ \nu > a.$$

(ii) The Standard Bivariate T-Distribution

A distribution is said to have the standard bivariate *t*-distribution if its pdf is given by

$$f_2(z_1, z_2) = \frac{1}{2\pi} \left[1 + \frac{1}{\nu}\left(z_1^2 + z_2^2\right)\right]^{-(\nu/2+1)}, \nu > 0. \tag{1.3}$$

It can be proved that the components Z_1 and Z_2 in (1.3) are uncorrelated but they are not independent unless $\nu \to \infty$. Product moments of the above distribution are given by Corollary 3.3. Following the univariate *t*-distribution, the quantity ν may also be called the degrees of freedom though it is just a shape parameter here. Since the pdf in (1.3) is constant on the circle $z_1^2 + z_2^2 = r^2$, for any fixed r, the distribution is also called the circular *t*-distribution. The pdf in (1.3) was used by Landenna and Marasini (1981) for testing equality with flexible type I error control.

(iii) Correlated Bivariate T-Distribution

The following is the pdf of a correlated bivariate t-distribution

$$f_3(y_1,y_2)=\frac{1}{2\pi\sqrt{1-\rho^2}}\left(1+\frac{y_1^2+y_2^2-2\rho y_1 y_2}{\nu(1-\rho^2)}\right)^{-(\nu/2+1)} \tag{1.4}$$

which is a special case of the well known bivariate t distribution (Anderson, 2003, 289). The above distribution is permutation-symmetric as the components have common mean 0, common variance γ_2 and correlation coefficient ρ where $\gamma_2(\nu-2)=\nu,\ \nu>2$ (See Tong, 1990, 202-203). Product moments of the above distribution are given by Corollary 3.2.
The following theorem is similar to a theorem in El-Bassiouni, Sultan and Moshref (2006).

Theorem 1.2 Let $Z=(Z_1,Z_2)'\sim N_2(0,\Sigma),\Sigma=(\sigma_{ik}),\ \sigma_{11}=\sigma_{22}=1,\ \sigma_{12}=\rho=\sigma_{21},\ (-1<\rho<1)$ independent of W where $\nu\mathrm{T}^{-2}\underline{\underline{d}}\ W\sim\chi_\nu^2$. Then

$(a)\ Y=(\nu/W)^{1/2}Z\sim T_2(0,\Sigma;\nu),$

(b) The pdf of a bivariate t-distribution with pdf in (1.4) can be written as

$$f_4(y)=(2\pi)^{-1}\int_0^\infty|\tau^2\Sigma|^{-1/2}\exp\left[y'(\tau^2\Sigma)^{-1}y\right]h_{\mathrm{T}}(\tau)d\tau \tag{1.5}$$

which is the scale mixture of bivariate normal and the 'inverted' chi-square distribution of T with pdf in (1.1), i.e., symbolically $(X\mid\mathrm{T}=\tau)\sim N_2(\theta,\tau^2\Sigma)$.

Proof. (a) The pdf of Z_1,Z_2 and W is given by

$$f_5(z_1,z_2,w)=\frac{1}{2\pi}\exp\left(-\frac{z_1^2+z_2^2-2\rho z_1z_2}{2(1-\rho^2)}\right)\frac{w^{\nu/2-1}}{2^{\nu/2}\Gamma(\nu/2)}\exp(-w/2).$$

Letting $y_1=(\nu/w)^{1/2}z_1,\ y_2=(\nu/w)^{1/2}z_2,\ w=w$ with Jacobian w/ν, we have

$$f_6(y_1,y_2,w)=\frac{2^{-(\nu/2+1)}w^{\nu/2}}{\pi\nu\Gamma(\nu/2)\sqrt{1-\rho^2}}\exp\left[-\left(\frac{y_1^2+y_2^2-2\rho y_1y_2}{2\nu(1-\rho^2)}+\frac{1}{2}\right)w\right].$$

Integrating out w, we have the pdf of bivariate t distribution given by (1.4).

(b) The pdf in (1.5) can be written as

$$f_7(y_1,y_2)$$
$$=\int_0^\infty\frac{1}{2\pi\tau^2\sqrt{1-\rho^2}}\exp\left[-\frac{y_1^2+y_2^2-2\rho y_1y_2}{2\tau^2(1-\rho^2)}\right]\frac{2(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}\tau^{-(\nu+1)}e^{-\nu/(2\tau^2)}d\tau,$$

which can be simplified as

$$f_7(y_1,y_2)$$
$$=\frac{(\nu/2)^{\nu/2}}{\pi\Gamma(\nu/2)\sqrt{1-\rho^2}}\int_0^\infty\exp\left[-\left(\frac{y_1^2+y_2^2-2\rho y_1y_2}{2(1-\rho^2)\tau^2}+\frac{\nu}{2\tau^2}\right)\right]\tau^{-(\nu+3)}d\tau.$$

With the transformation $\nu/\tau^2=w$, we have

$$f_7(y_1,y_2)$$
$$=\frac{(\nu/2)^{\nu/2}}{\pi\Gamma(\nu/2)\sqrt{1-\rho^2}}\int_0^\infty \exp\left[-\left(\frac{y_1^2+y_2^2-2\rho y_1y_2}{2(1-\rho^2)}+\frac{\nu}{2}\right)w\right](\nu/w)^{-(\nu+3)/2}\left(\frac{1}{2}(\nu/w^3)^{1/2}\right)dw$$
$$=\frac{1}{2^{\nu/2+1}\pi\nu\Gamma(\nu/2)\sqrt{1-\rho^2}}\int_0^\infty \exp\left[-\left(\frac{y_1^2+y_2^2-2\rho y_1y_2}{2(1-\rho^2)\nu}+\frac{1}{2}\right)w\right]w^{\nu/2}dw$$
$$=\frac{\Gamma(\nu/2+1)}{2^{\nu/2+1}\pi\nu\Gamma(\nu/2)\sqrt{1-\rho^2}}\left[\left(\frac{y_1^2+y_2^2-2\rho y_1y_2}{2(1-\rho^2)\nu}+\frac{1}{2}\right)\right]^{-(\nu/2+1)},$$

which simplifies to the pdf of the bivariate *t*-distribution given by (1.4).

(iv) A General Bivariate T-Distribution

The pdf of the bivariate normal distribution is given by

$$f_8(x)=(2\pi)^{-1}|\Sigma|^{-1/2}\exp\left(-(x-\theta)'\Sigma^{-1}(x-\theta)/2\right) \quad (1.6)$$

where $X=(X_1,X_2)'$, $\theta=(\theta_1,\theta_2)'$ is unknown vector of location parameters and Σ is the 2×2 unknown positive definite matrix of scale parameters. It is known that $E(X)=\theta$ and $Cov(X)=\Sigma$. The pdf in (1.6) can be rewritten as

$$f_8(x_1,x_2)=\frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2}\exp\left(\frac{-q(x_1,x_2)}{2}\right) \quad (1.7)$$

where

$$(1-\rho^2)q(x_1,x_2)=\left(\frac{x_1-\theta_1}{\sigma_1}\right)^2+\left(\frac{x_2-\theta_2}{\sigma_2}\right)^2-\frac{2\rho(x_1-\theta_1)(x_2-\theta_2)}{\sigma_1\sigma_2}. \quad (1.8)$$

The pdf of the general (location-scale) bivariate t-distribution is given by

$$f_9(x)=(2\pi)^{-1}|\Sigma|^{-1/2}\left(1+(x-\theta)'(\nu\Sigma)^{-1}(x-\theta)\right)^{-\nu/2-1} \quad (1.9)$$

where the scalar ν is assumed to be a known positive constant (Anderson, 2003, 289). The probability density function will be denoted by $T_2(\theta,\Sigma;\nu)$ in contrast to the bivariate normal by $N_2(\theta,\Sigma)$. The pdf of the bivariate *t*-distribution in (1.9) can be written as a mixture of the bivariate normal distribution and the inverted chi-square distribution as follows:

$$f_9(x)=(2\pi)^{-1}\int_0^\infty|\tau^2\Sigma|^{-1/2}\left(1+(x-\theta)'(\nu\tau^2\Sigma)^{-1}(x-\theta)\right)^{-\nu/2-1}h_{\mathrm{T}}(\tau)d\tau \quad (1.10)$$

which can be written as

$$f_9(x_1,x_2)=\int_0^\infty\frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2\tau^2}\exp\left[\frac{-q(x_1,x_2)}{2\tau^2}\right]h_{\mathrm{T}}(\tau)d\tau. \quad (1.11)$$

where $q(x_1,x_2)$ is given by (1.8). The pdf in (1.11) can further be written as

$$f_9(x_1,x_2)=\frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2}\left(1+q(x_1,x_2)/\nu\right)^{-(\nu+2)/2}, \quad (1.12)$$

where $q(x_1,x_2)$ is defined by (1.8) and is well known to be the pdf of the general bivariate *t*-distribution (cf. Anderson, 2003, 289). Since the pdf in (1.12) is constant on the ellipse

$$\left(\frac{x_1-\theta_1}{\sigma_1}\right)^2+\left(\frac{x_2-\theta_2}{\sigma_2}\right)^2-\frac{2\rho(x_1-\theta_1)(x_2-\theta_2)}{\sigma_1\sigma_2}=c^2,$$

for any fixed c , the distribution is also called elliptical *t*-distribution. Note that even if $\rho=0$ in the pdf in (1.12), the components X_1 and X_2 do not become independent unless $\nu\to\infty$.

The pdf in (1.10) can be expressed by the scale mixture representation as

$$(X \mid \mathrm{T}=\tau)\sim N_2(\theta,\tau^2\Sigma). \tag{1.13}$$

It is interesting to note that the transformation

$$x_1=\theta_1+\sigma_1 y_1,\ x_2=\theta_2+\sigma_2 y_2 \tag{1.14}$$

in the general bivariate *t*-distribution with pdf in (1.12) yields the pdf of the correlated bivariate *t*-distribution as in (1.4). Also the following transformation

$$\begin{aligned} x_1&=\theta_1+\sigma_1\sqrt{(1+\rho)/2}\ z_1+\sigma_1\sqrt{(1-\rho)/2}\ z_2,\\ x_2&=\theta_2+\sigma_2\sqrt{(1+\rho)/2}\ z_1-\sigma_2\sqrt{(1-\rho)/2}\ z_2 \end{aligned} \tag{1.15}$$

in the pdf in (1.12) yields that of the standard *t*-distribution given by (1.3) which is obvious by virtue of

$$\begin{aligned} &(1-\rho^2)q(x_1,x_2)\\ &=\left(\sqrt{(1+\rho)/2}\ z_1+\sqrt{(1-\rho)/2}\ z_2\right)^2+\left(\sqrt{(1+\rho)/2}\ z_1-\sqrt{(1-\rho)/2}\ z_2\right)^2\\ &-2\rho\left(\sqrt{(1+\rho)/2}\ z_1+\sqrt{(1-\rho)/2}\ z_2\right)\left(\sqrt{(1+\rho)/2}\ z_1-\sqrt{(1-\rho)/2}\ z_2\right)\\ &=(1-\rho^2)(z_1^2+z_2^2), \end{aligned}$$

and $J((x_1,x_2)\to(z_1,z_2))=\mathrm{mod}\left(\frac{\partial x_1}{\partial z_1}\frac{\partial x_2}{\partial z_2}-\frac{\partial x_1}{\partial z_2}\frac{\partial x_2}{\partial z_1}\right)=\sigma_1\sigma_2\sqrt{1-\rho^2}$. Conversely, the transformation

$$z_1=\frac{1}{\sqrt{2(1+\rho)}}\left(\frac{x_1-\theta_1}{\sigma_1}+\frac{x_2-\theta_2}{\sigma_2}\right),\ z_2=\frac{1}{\sqrt{2(1-\rho)}}\left(\frac{x_1-\theta_1}{\sigma_1}-\frac{x_2-\theta_2}{\sigma_2}\right)$$

in (1.3) also yields the pdf in (1.12) since $z_z^2+z_2^2=q(x_1,x_2)$ and

$$J((z_1,z_2)\to(x_1,x_2))=\left(\sigma_1\sigma_2\sqrt{1-\rho^2}\right)^{-1}.$$

2. Moments of the Bivariate T-Distribution

In the rest of the paper we concentrate mostly on the general bivariate *t*-distribution discussed above. It follows from (1.13) that the expected value and the covariance matrix of the distribution are given by

$$\begin{aligned} E(X)&=E[E(X\mid\mathrm{T})]=E(\theta)=\theta \text{ and}\\ Cov(X)&=E[Cov(X\mid\mathrm{T})]+Cov[E(X\mid\mathrm{T})]\\ &=E\left(\mathrm{T}^2\Sigma\right)+Cov(\theta)\\ &=\gamma_2\Sigma \end{aligned} \tag{2.1}$$

where $\gamma_2=\nu/(\nu-2),\ \nu>2$.

The characteristic function of the general bivariate *t*-distribution is given by

$$\phi_{X(t)} = E\left(e^{it'X}\right) = e^{it'\theta}\psi_\nu\left(\sqrt{\nu}\,|\Sigma^{1/2}t\,|\right), \tag{2.2}$$

with

$$\psi_\nu\left(|t\,|\right) = \frac{|t\,|^{\nu/2}}{2^{\nu/2-1}\Gamma(\nu/2)} K_{\nu/2}(|t\,|), \tag{2.3}$$

where $K_{\nu/2}(|t\,|)$ is the Macdonald function with order $\nu/2$ and argument $|t\,|$. The Macdonald function admits numerous integral and series representations (See Spainer and Oldham, 1987, Chapter 51). The characteristic function can be derived (Joarder and Alam, 1995) by

$$\begin{aligned} E\left(e^{it'X}\right) &= E\left[E\left(e^{it'X}\ |\,\mathrm{T}\right)\right] \\ &= E\left(e^{it'\theta}e^{-t'\mathrm{T}^2\Sigma t/2}\right) \\ &= e^{it'\theta}E[\exp(-\alpha\mathrm{T}^2)], \end{aligned}$$

where $\alpha = t'\Sigma t/2$. Since

$$E\left(e^{-\alpha\mathrm{T}^2}\right) = \int_0^\infty e^{-\alpha\tau^2}h_{\mathrm{T}}(\tau)d\tau = \frac{(\alpha\nu)^{\nu/4}}{2^{\nu/4-1}\Gamma(\nu/2)}K_{\nu/2}\left(\sqrt{2\nu\alpha}\right),$$

where $K_\alpha(w) = K_{-\alpha}(w) = \dfrac{w^{-\alpha}}{2^{1-\alpha}}\displaystyle\int_0^\infty u^{\alpha-1}\exp\left(-u - \frac{w^2}{4u}\right)du$ (Lebedev, 1965), the characteristic function is given by (2.2).

Setting $t_2 = 0$ in the characteristic function (2.2), we immediately have $E\left(e^{it_2X_2}\right) = e^{it_2\theta_2}\psi_\nu\left(\sigma_1\sqrt{\nu}\,|t_1|\right)$. Hence the marginal probability density function of X_1 is given by

$$f_{10}(x_1) = \frac{\Gamma((\nu+1)/2)}{\sigma\sqrt{\nu\pi}\ \Gamma(\nu/2)}\left(1 + \frac{1}{\nu\sigma_1^2}(x_1-\theta_1)^2\right)^{-\nu/2-1}, \nu > 0,$$

which is denoted by $X_1 \sim t\left(\theta_1, \sigma_1^2; \nu\right)$.

The (a,b)-th raw product moment of any two variables X_1 and X_2 is defined by $E(X_1^aX_2^b)$ while the (a,b)-th centered product moment between X_1 and X_2 is defined by

$$E[(X_1-\theta_1)^a(X_2-\theta_2)^b], \tag{2.4}$$

where $\theta_1 = E(X_1), E(X_2) = \theta_2$.

3. Centered Product Moments of Bivariate T-Distribution

The scale mixture representation of the bivariate *t*-distribution with pdf in (1.13) can be represented by

$$(X\ |\,\mathrm{T}=\tau) \sim N_2(\theta, \tau^2\Sigma) \text{ where } \Sigma = \begin{pmatrix} \mu(2,0) & \mu(1,1) \\ \mu(1,1) & \mu(0,2) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

The pdf's between (1.9) to (1.12) are equivalent. The following theorem is due to Joarder (2006).

Theorem 3.1 Let X_1 and X_2 have the bivariate *t*-distribution with pdf in (1.12). Then the product moments between X_1 and X_2 are given by

$$\mu(a,b;\nu) = (a+b-1)\rho\sigma_1\sigma_2\mu(a-1,b-1)\gamma_2 + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,b-2)\gamma_4,$$

$$\mu(2a,2b;\nu) = \sigma_1^{2a}\sigma_2^{2b}\frac{(2a)!(2b)!}{2^{a+b}}\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}\gamma_{2a+2b},$$

$$\mu(2a+1,2b+1;\nu) = \sigma_1^{2a+1}\sigma_2^{2b+1}\frac{(2a+1)!(2b+1)!}{2^{a+b}}\rho\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}\gamma_{2a+2b+2},$$

$$\mu(2a,2b+1;\nu) = \mu(2a+1,2b;\nu) = 0$$

where γ_a is the a-th moment of T.

Proof. By applying the scale mixture representation in (1.5) to Kendal and Stuart (1969, 91) we have

$$\begin{aligned}\mu(a,b;m) &= E[E[(X_1-\theta_1)^a(X_2-\theta_2)^b \mid \mathrm{T}]]\\ &= E[(a+b-1)\rho(\sigma_1\mathrm{T})(\sigma_2\mathrm{T})\mu(a-1,b-1)\\ &\quad + (a-1)(b-1)(1-\rho^2)\upsilon^2(\sigma_1^2\mathrm{T}^2)(\sigma_2^2\mathrm{T}^2)\mu(a-2,b-2)]\\ &= (a+b-1)\rho\sigma_1\sigma_2\mu(a-1,b-1)E(\mathrm{T}^2)\\ &\quad + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,b-2)E(\mathrm{T}^4).\end{aligned}$$

Similarly we have

$$\mu(2a,2b;\nu) = \sigma_1^{2a}\sigma_2^{2b}\frac{(2a)!(2b)!}{2^{a+b}}\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}E(\mathrm{T}^{2a+2b})$$

$$\mu(2a+1,2b+1;\nu) = \sigma_1^{2a+1}\sigma_2^{2b+1}\frac{(2a+1)!(2b+1)!}{2^{a+b}}\rho\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}E(\mathrm{T}^{2a+2b+2})$$

$$\mu(2a,2b+1;\nu) = \mu(2a,2b+1)E(\mathrm{T}^{2a+2b+1}) = 0,$$

$$\mu(2a+1,2b;\nu) = \mu(2a+1,2b)E(\mathrm{T}^{2a+2b+1}) = 0.$$

Corollary 3.1 Let X_1 and X_2 have the bivariate *t*-distribution with pdf in (1.12). Then the product moment correlation between X_1 and X_2 is given by $\rho_{X_1X_2} = \rho$.

Proof. By virtue of

$E(X_1-\theta_1)^2 = \mu(2,0;\nu) = \gamma_2\sigma_1^2,\ E(X_2-\theta_2)^2 = \mu(0,2;\nu) = \gamma_2\sigma_2^2,$

$E(X_1-\theta_1)(X_2-\theta_2) = \mu(1,1;\nu) = \gamma_2\rho\sigma_1\sigma_2,$

the corollary follows from

$$\left[\left(E(X_1-\theta_1)^2\right)\left(E(X_2-\theta_2)^2\right)\right]^{1/2}\rho_{X_1,X_2} = E(X_1-\theta_1)(X_2-\theta_2).$$

We conclude this section by providing centered product moments of the bivariate correlated *t*-distribution and the standard *t*-distribution. Corollary 3.2 follows from Theorem 3.1 for $\sigma_1 = \sigma_2 = 1$.

Corollary 3.2 The product moments of correlated bivariate *t*-distribution with pdf in (1.4) are given by

$\mu(a,b;\nu) = (a+b-1)\rho\mu(a-1,b-1)\gamma_2 + (a-1)(b-1)(1-\rho^2)\mu(a-2,b-2)\gamma_4,$

$\mu(2a,2b;\nu) = \dfrac{(2a)!(2b)!}{2^{a+b}} \displaystyle\sum_{j=0}^{\min(a,b)} \dfrac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}\gamma_{2a+2b},$

$\mu(2a+1,2b+1;\nu) = \dfrac{(2a+1)!(2b+1)!}{2^{a+b}}\rho \displaystyle\sum_{j=0}^{\min(a,b)} \dfrac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}\gamma_{2a+2b+2},$

$\mu(2a,2b+1;\nu) = \mu(2a+1,2b;\nu) = 0,$

where γ_a is the a-th moment of T.

The following corollary follows from Theorem 3.1 for $\sigma_1 = \sigma_2 = 1$ and $\rho = 0$.

Corollary 3.3 The product moments of standard bivariate *t*-distribution with pdf in (1.3) are given by

$\mu(a,b;\nu) = (a-1)(b-1)\mu(a-2,b-2)\gamma_4,$

$\mu(2a,2b;\nu) = \dfrac{(2a)!(2b)!}{2^{a+b}a!b!}\gamma_{2a+2b},$

$\mu(2a+1,2b+1;\nu) = 0,\ \mu(2a,2b+1;\nu) = \mu(2a+1,2b;\nu) = 0,$

where γ_a is the a-th moment of T.

4. The Standardized Moments of the Bivariate T-Distribution

Let $\mu(a,b) = E[(X_1-\theta_1)^a(X_2-\theta_2)^b]$ and $\mu(a,b;\nu) = E[(X_1-\theta_1)^a(X_2-\theta_2)^b]$ be the (a,b)-th centered product moments of the bivarite normal distribution with pdf in (1.6) and the bivariate *t*-distribution with pdf in (1.9) respectively. Since the covariance matrix for the bivariate *t*-distribution is given by

$\Lambda = \begin{pmatrix} \mu(2,0;\nu) & \mu(1,1;\nu) \\ \mu(1,1;\nu) & \mu(0,2;\nu) \end{pmatrix} = \begin{pmatrix} \mu(2,0) & \mu(1,1) \\ \mu(1,1) & \mu(0,2) \end{pmatrix}\gamma_2 = \gamma_2\Sigma,$

the quantity

$(X-\theta)'\Sigma^{-1}(X-\theta) = \|\Sigma^{-1/2}(X-\theta)\|^2 = Z'Z = R^2$ is not the standardized distance. The standardized distance for the bivariate *t*-distribution is defined by

$Q = (X-\theta)'(\gamma_2\Sigma)^{-1}(X-\theta).$ (4.1)

Let $\beta_a = E(Q^a),\ (a = 1,2,\cdots)$ be the a-th standardized moment of Q (Joarder, 2006). Some properties of standardized moments are discussed below.

Theorem 4.1 Let $X = (X_1, X_2)'$ have the general bivariate *t*-distribution, and $Q = (X-\theta)'(\gamma_2\Sigma)^{-1}(X-\theta)$ be the standardized distance of the distribution. Then

$\beta_a = (\nu-2)^a\ \dfrac{\Gamma(a+1)\Gamma(\nu/2-a)}{\Gamma(\nu/2)},\ \nu > 2a.$

Proof. The quantity Q can be written as $Q = (X-\theta)'(\gamma_2\Sigma)^{-1}(X-\theta) = \gamma_2^{-1}Z'Z = \gamma_2^{-1}R^2$ where $\Sigma^{-1/2}(X-\theta) = Z$. Then from (1.3), the pdf of Z is

$f_2(z) = \dfrac{1}{2\pi}\left(1 + \dfrac{z'z}{\nu}\right)^{-\nu/2-1}.$ (4.2)

By making the polar transformation
$z_1 = r\cos\theta,\ z_2 = r\sin\theta,\ \ r \in (0,\infty),\ \ \theta \in (0,2\pi)$ in (4.2) with Jacobian r, it follows that the probability density functions of R and Θ are given by
$g_1(r) = r(1+r^2/\nu)^{-(\nu/2+1)},\ r \in (0,\infty)$ and
$g_2(\theta) = (2\pi)^{-1}\ ,\theta \in (0,2\pi)$
respectively. Then it can be checked easily that $R^2/2 \sim F(2,\nu),$ and consequently $W = \gamma_2 Q/2 \sim F(2,\nu),\ \nu > 2$ (cf. Muirhead, 1982, 37). Then the a-th standardized moment is given by

$$\begin{aligned}\beta_a &= E(Q^a)\\ &= 2^a\gamma_2^{-a}E(W^a)\\ &= 2^a\gamma_2^{-a}(\nu/2)^a\ \frac{\Gamma(a+1)\Gamma(\nu/2-a)}{\Gamma(\nu/2)}\\ &= (\nu-2)^a\ \frac{\Gamma(a+1)\Gamma(\nu/2-a)}{\Gamma(\nu/2)},\ \ \nu > 2a.\end{aligned}$$

Notice that as $\nu \to \infty$, the standardized moments of the bivariate *t*-distribution, as expected, coincide with that of the bivariate normal distribution. In case a is a nonnegative integer

$$\beta_a = 2^a a!\frac{((\nu/2)-1)^a}{((\nu/2)-1)^{\{a\}}},\quad \nu > 2(a-1)$$

where $c^{\{a\}} = c(c-1)\cdots(c-a+1),\ (c \neq 0).$

Corollary 4.1 The first three standardized moments of the general bivariate *t*-distribution are given by

$$\beta_1 = 2,$$
$$\beta_2 = \frac{8(\nu-2)}{\nu-4},\ \ \nu > 4,$$
$$\beta_3 = \frac{48(\nu-2)^2}{(\nu-4)(\nu-6)},\ \ \nu > 6.$$

5. Shanon Entropy

The Shanon Entropy for any bivariate density function $f(x_1,x_2)$ is defined by $H(f) = -E[\ln f(X_1,X_2)]$. Let us calculate the Shanon entropy for the bivariate normal distribution. It follows from (1.7) and (1.8) that

$$-\ln f_6(x_1,x_2) = \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) + \frac{q(x_1,x_2)}{2}\ \text{ and}$$

$$E[q(X_1,X_2)] = \frac{1}{1-\rho^2}(1+1-2\rho^2) = 2$$

respectively so that the Shanon entropy for the bivariate normal distribution with pdf in (1.7) is given by

$$\begin{aligned}-E(\ln f_8(X_1,X_2)) &= \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)+\frac{E[q(X_1,X_2)]}{2}\\ &= \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)+1.\end{aligned} \tag{5.1}$$

Theorem 5.1 Let the bivariate *t*-distribution have the pdf in (1.11). Then the Shanon entropy for the bivariate *t*-distribution has the following equivalent representations:

$$(i)\, H(f_9)=\ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)+E(\ln \mathrm{T}^2)+1$$

$$(ii)\, H(f_9)=\ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)+\frac{\nu+2}{2}\,E\left[\ln\left(1+q(X_1,X_2)/\nu\right)\right],$$

$$(iii)\, H(f_9)=\ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)+\ln\left[\frac{\sqrt{\pi}\,\Gamma(\nu/2+1)}{\Gamma((\nu+1)/2)}\right]+\frac{\nu+2}{2}\left[\Psi(\nu/2+1)-\Psi(\nu/2)\right],$$

where $\nu\mathrm{T}^{-2}\sim\chi^2_\nu$ and $\Psi(t)=d\,\ln\Gamma(t)/dt$ denotes the digamma function (Nadarajah and Kotz, 2005).

Proof. By virtue of (1.10), (1.11) or (1.13), it follows from (5.1) that

$$\begin{aligned}H(f_9) &= E\left[\left(\ln\left(2\pi(\sigma_1\mathrm{T})(\sigma_2\mathrm{T})\sqrt{1-\rho^2}\right)+1\right)\mid \mathrm{T}\right]\\ &= \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)+E(\ln\mathrm{T}^2)+1,\end{aligned}$$

which is part (i) of the theorem. Alternatively, it follows from (1.1) that

$$E[\ln f_9(X_1,X_2)]=-\ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)-\frac{\nu+2}{2}\,E\left[\ln\left(1+q(X_1,X_2)/\nu\right)\right],$$

which yields (ii) in the theorem. Note that $E(\ln\mathrm{T}^2)=\ln\nu-E(\ln W)$, $W\sim\chi^2_\nu$. For part (iii), see Nadarajah and Kotz (2005).

6. Distribution of the Sample Variances and Correlation Coefficient

For any bivariate random vector, the mean vector is $\bar{X}'=(\bar{X}_1,\bar{X}_2)$ and the sums of squares and cross product matrix is given by $\sum_{j=1}^{N}(X_j-\bar{X})(X_j-\bar{X})'=A$. The symmetric bivariate matrix A can be written as $A=(a_{ik}), i=1,2; k=1,2$ where $a_{ii}=mS_i^2=\sum_{j=1}^{N}(X_{ij}-\bar{X}_i)^2$, $m=N-1, (i=1,2)$ and $a_{12}=\sum_{j=1}^{N}(X_{1j}-\bar{X}_1)(X_{2j}-\bar{X}_2)=mRS_1S_2$. Fisher (1915) derived the distribution of A for $p=2$ in order to study the distribution of the correlation coefficient from a normal sample.

Let $X' = (X_1, X_2)$ be bivariate t-random vector with probability density function in (1.9). Now consider a sample $X_1, X_2, \cdots X_N$ $(N > 2)$ having the joint probability density function

$$f_{11}(x_1, x_2, \cdots, x_N) = \frac{\Gamma(\nu/2 + N)}{(\nu\pi)^N \, \Gamma(\nu/2} \; |\Sigma|^{-N/2} \left(1 + \sum_{j=1}^{N} (x_j - \theta)'(\nu\Sigma)^{-1}(x_j - \theta)\right)^{-\nu/2-1}, \qquad (6.1)$$

which is the bivariate *t*-model for the sample. Note that the observations in the sample are uncorrelated and not independent unless $\nu \to \infty$.

The pdf of sample sum of squares and sum of products based on bivariate *t*-model (6.1) can be written as

$$f_{12}(a_{11}, a_{22}, a_{12}) = C_\nu(m,2) \; \left(1-\rho^2\right)^{-m/2} (\sigma_1\sigma_2)^{-m} \left(a_{11}a_{22} - a_{12}^2\right)^{(m-3)/2}$$
$$\times \left(1 + \frac{1}{\nu(1-\rho^2)} \left(\frac{a_{11}}{\sigma_1^2} + \frac{a_{22}}{\sigma_2^2} - \frac{2\rho a_{12}}{\sigma_1\sigma_2}\right)\right)^{-\nu/2-m}$$

where $C_\nu(m,2) = \dfrac{2^{m-2} \;\; \nu^{-m} \;\; \Gamma(m+\nu/2)}{\pi \; \Gamma(m-1)\Gamma(\nu/2)}$

and $a_{11} > 0, a_{22} > 0, -\infty < a_{12} < \infty, -1 < \rho < 1,\; m > 2, \sigma_1 > 0, \sigma_2 > 0$ (Sutradhar and Ali, 1989).

Under the transformation $a_{11} = ms_1^2, a_{22} = ms_2^2, a_{12} = mrs_1s_2$ with Jacobian $J(a_{11}, a_{22}, a_{12} \to r, s_1^2, s_2^2) = m^3 s_1 s_2$, the pdf of S_1^2, S_2^2 and R is given by

$$f_{13}\left(s_1^2, s_1^2, r\right) = m^m \;\; C_\nu(m,2)(\sigma_1\sigma_2)^{-m} \;\; \left(1-\rho^2\right)^{-m/2} \left(1-r^2\right)^{(m-3)/2} (s_1 s_2)^{m-2}$$
$$\times \left(1 + \frac{1}{\nu(1-\rho^2)} \left(\frac{ms_1^2}{\sigma_1^2} + \frac{ms_1^2}{\sigma_2^2} - 2\rho r \, \frac{ms_1s_2}{\sigma_1\sigma_2}\right)\right)^{-\nu/2-m} . \qquad (6.2)$$

By integrating out sample variances, the distribution of the correlation coefficient comes out to be what was obtained by Fisher (1915). An elegant proof is due to Joarder (2007). The null and non-null distribution robustness was proved, among others, by Ali and Joarder (1991).

Theorem 6.1 The probability density function of the correlation coefficient R based on the joint pdf in (6.1) is given by

$$h(r) = \frac{2^{m-2}\left(1-\rho^2\right)^{m/2}}{\pi \; \Gamma(m-1)} \;\; \left(1-r^2\right)^{(m-3)/2} \sum_{k=0}^{\infty} \frac{(2\rho r)^k}{k\,!} \; \Gamma^2\left(\frac{m+k}{2}\right), \; -1 < r < 1$$

where $m > 2, -1 < \rho < 1$.

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