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A note on a symmetry analysis and exact solutions of a nonlinear fin equation

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Abstract

A similarity analysis of a nonlinear fin equation has been carried out by M. Pakdemirli and A.Z. Sahin [Similarity analysis of a nonlinear fin equation, Appl. Math. Lett. (2005) (in press)]. Here, we consider a further group theoretic analysis that leads to an alternative set of exact solutions or reduced equations with an emphasis on travelling wave solutions, steady state type solutions and solutions not appearing elsewhere.

Keywords: Nonlinear fin equation; Exact solutions; Similarity analysis

1. Introduction

A fin is an extended surface device attached to the surface of a structure protruding into the adjacent fluid, where its purpose is to increase the heat transfer between the solid surface and the fluid [1]. This device is used to enhance the transfer of heat from a surface to its surrounding medium. Because of the wide ranging applications, the analysis of fin heat transfer is of great significance. A large variety of contributions have been made in this direction. One of the early studies of temperature dependent straight fins with internal heat generation was done in [2]. Razelos and Imre [3] analyzed the conductive heat transfer of a convecting fin from the base to its tip. Jany and Bejan [4] have investigated the optimum shape for straight fins with temperature dependent conductivity. A critical review of different models of fins has been presented in [5].

The equations governing fins with temperature dependent conductivity are nonlinear diffusion type differential equations. Due to the mathematical complexity of these equations exact analytical solutions are not easily tractable. In most cases only numerical procedures have been presented. Aziz [6], Krane [7] have used perturbation techniques to find a numerical solution, while Muzzio [8] adopted the Galerkin method to obtain approximate solutions in the case of temperature dependent conductivity. Chiu and Chen [9] uses the Adomian decomposition method for solving the convective longitudinal fin equation with variable conductivity (see [10]).

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Finding analytic exact solutions to non-linear heat conduction (diffusion) equations arising from variable thermal conductivity is a challenging task. Recently Pakdemirli and Sahin [11] have used symmetry methods in an attempt to obtain some similarity solutions of a non-linear fin equation arising from temperature dependent thermal conductivity and a variable heat transfer coefficient. They present some interesting solutions which are valid in some special cases of conductivity and heat transfer coefficients. The symmetry method provides a powerful tool for the generation of the transformations that can be used to transform the given non-linear partial differential equation to a simpler equation while preserving the invariance of the original equation. Consequently, it enjoys widespread application and has attracted the attention of many researchers. An account of this method can be found in [12] and [13]. In this note we study the fin equation with temperature dependent thermal conductivity and present some new exact solutions and provide a direction for providing more solutions using symmetry techniques.

The reader can then utilize the exact solution by generating values of the constants that arise from a given set of boundary or initial values. The resultant graphical forms of the solution may be quite involved but obtainable through standard packages.

The non-dimensional non-linear fin equation in the case of temperature dependent thermal conductivity is [6]

$$\frac{\partial}{\partial x}(k(u)u_x) - N^2 f(x)u = u_t, \tag{1.1}$$

where u is the dimensionless temperature, k the thermal conductivity, f the heat transfer coefficient and N the fin parameter. The generator of point symmetry of (1.1) is of the form

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{1.2}$$

and its second prolongation on Eq. (1.1) modulo the solutions of (1.1) leads to $\xi = \xi(x,t)$ and $\tau = \tau(t)$ and a system of linear partial differential equations (pdes) in ξ, τ and ϕ, namely,

$$\begin{aligned}
&\frac{k_u}{k}\phi = 2\xi - \tau_t,\\
&2\frac{\partial}{\partial u}(k\phi_x) = k\xi_{xx} - \xi_t,\\
&\phi k_{uu} + k\phi_{uu} + k_u\phi_u - \phi\frac{k_u^2}{k} = 0,\\
&\frac{k_u}{k}\phi N^2 fu - \phi N^2 f + k\phi_{xx} + \phi_u N^2 fu - 2\xi_x N^2 fu - \xi N^2 f_x u - \phi_t = 0.
\end{aligned} \tag{1.3}$$

2. Results

We analyze and present exact solutions or reduced forms of (1.1) arising from symmetry—the reduced form is an ordinary differential equation (ODE).

2.1. Symmetry generators

In this section we summarize some of the generators of point symmetry that arise for specific cases of interest. In some cases, these produce a knowledge of new solutions.

2.1.1. $k = u^m$

In the first case, we consider a polynomial form of the thermal conductivity, namely, $k(u) = u^m$ for $m \neq 0$. From (1.3)(a), it is clear that $\phi = \frac{1}{m}(2\xi - \tau)u + \alpha(x,t)$ and (1.3)(c) implies $\alpha = 0$. Thus,

$$\phi = \frac{1}{m}(2\xi_x - \tau_t)u \tag{2.1}$$

and (1.3)(b), (1.3)(d) lead to, respectively,

$$\xi = Ax + B,$$
$$\tau = C + \frac{D}{mN^2}\mathrm{e}^{mN^2 t} - (Ax + B) f_x t, \tag{2.2}$$

where A, B, C and D are constants.

(a) The case $f = \mathcal{F}$, a constant, gives rise to a four-dimensional Lie algebra of point symmetries,

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} + \frac{2}{m}u\frac{\partial}{\partial u},$$
$$X_4 = \mathrm{e}^{mN^2 t}\frac{\partial}{\partial t} - mN^2\mathrm{e}^{mN^2 t}u\frac{\partial}{\partial u} \tag{2.3}$$

whose non-zero commutators are $[X_2, X_3] = X_1$ and $[X_1, X_4] = mN^2\mathcal{F}X_4$.

(b) With $f = 1/x^2$, (1.1) admits a two-dimensional algebra $X_1 = \frac{\partial}{\partial t}$ and $X_2 = t\frac{\partial}{\partial t} + \frac{1}{2}x\frac{\partial}{\partial x}$.

(c) If $f = \mathrm{e}^{nx}$, $n \neq 0$, the algebra of Lie point symmetries is generated by $X_1 = \frac{\partial}{\partial t}$ and $X_2 = t\frac{\partial}{\partial t} - (1/n)\frac{\partial}{\partial x} - (1/m)u\frac{\partial}{\partial u}$.

2.1.2. $k(u) = k$, a constant

(a) $f = x^n$ leads to the symmetry generators $X_1 = \frac{\partial}{\partial t}$, $X_2 = u\frac{\partial}{\partial u}$, $X_\alpha = \alpha(x,t)\frac{\partial}{\partial u}$, where α satisfies Eq. (1.1) for this choice of f.

(b) $f = \mathrm{e}^{nx}$ generates the algebra as in (a).

(c) $f = \frac{1}{x^2}$ admits the symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t},$$
$$X_3 = u\frac{\partial}{\partial u}, \quad X_4 = -4ktx\frac{\partial}{\partial x} - 4kt^2\frac{\partial}{\partial t} + (2kt + x^2)u\frac{\partial}{\partial u}, \tag{2.4}$$
$$X_\alpha = \alpha(x,t)\frac{\partial}{\partial u},$$

where α satisfies (1.1).

(d) $f = 1$ leads to the symmetry generators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = u\frac{\partial}{\partial u},$$
$$X_4 = -\frac{1}{2N^2}x\frac{\partial}{\partial x} - \frac{1}{N^2}t\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}, \quad X_5 = -2kt\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \tag{2.5}$$
$$X_6 = -4ktx\frac{\partial}{\partial x} - \frac{2k}{N^2}t(2N^2t - 1)\frac{\partial}{\partial t} + (4kN^2t^2 + x^2)u\frac{\partial}{\partial u}, \quad X_\alpha = \alpha(x,t)\frac{\partial}{\partial u},$$

where α satisfies (1.1).

2.2. Invariant solutions

Here, we consider some exact solutions which are symmetry invariant or invariant under a subalgebra of symmetries. In particular, we point out to the reader the cases that give rise to travelling wave solutions for the temperature function u which are of practical interest but seemed to be ignored in the literature regarding (1.1). Other reduced equations and/or solutions are also presented.

2.2.1. $f = \mathcal{F}$, a constant

(a) *Travelling wave solutions*

(i) For all $k(u)$, travelling wave solutions are obtained by the symmetry $c\partial_x + \partial_t$, where c is the wave speed. This generator yields the invariants $y = x - ct$ and $v = u$ with $v = v(y)$. That is, (1.1) becomes

$$k(v)v'' + k'v'^2 - N^2\mathcal{F}v + cv' = 0. \tag{2.6}$$

(ii) For $k = k(u^m)$, (2.6) becomes

$$v^m v'' + m v^{m-1} v'^2 - N^2 \mathcal{F} v + c v' = 0 \tag{2.7}$$

which for $m = 1$ $(k = u)$ is $v v'' + v'^2 - N^2 \mathcal{F} v + c v' = 0$.

(iii) For $k = 1$, the reduced form is

$$v'' - N^2 \mathcal{F} v + c v' = 0. \tag{2.8}$$

which has the solution

$$v(y) = \mathrm{e}^{\frac{(-c-\sqrt{c^2+4kN^2})y}{2k}} C_1 + \mathrm{e}^{\frac{(-c+\sqrt{c^2+4kN^2})y}{2k}} C_2. \tag{2.9}$$

(b) As regards $k = u^m$, from the list in (2.3), X_4 is also a symmetry which yields another exact solution from the invariants $y = x$ and $v = \mathrm{e}^{\mathcal{F} N^2 t} u$, namely,

$$v v'' + m v' = 0 \tag{2.10}$$

which is further reducible by X_1. In fact, the solution is obtainable from $Cmy = \int \frac{1}{\ln v} \mathrm{d}v$; C is a constant.

2.2.2. $f = f(x)$

(a) *Steady state solutions*

Since (1.1) admits ∂_t as a symmetry generator for all cases of $k(u)$, it generates the steady state solution

$$\frac{\mathrm{d}}{\mathrm{d}y}(k(v)v') - N^2 f(y) v = 0 \tag{2.11}$$

which is an ode (in $v = v(y)$, where $y = x$ and $v = u$). Its solution would usually be associated with conservation of energy. We consider some special cases for f and $k(u) = k$, a constant.

(i) $f(x) = x^2$ yields $k v'' - N^2 y^2 v = 0$ for (2.11) whose solution is

$$v(y) = \frac{\left(\frac{Ny^2}{\sqrt{k}}\right)^{1/4} \mathrm{Bessel}I\left[-\frac{1}{4}, \frac{Ny^2}{2\sqrt{k}}\right] C_2 \mathrm{Gamma}\left[\frac{3}{4}\right]}{\sqrt{2}} + \mathrm{e}^{-\frac{Ny^2}{2\sqrt{k}}} C_1 \mathrm{Hermite}H\left[-\frac{1}{2}, \frac{\sqrt{N}y}{k^{1/4}}\right]. \tag{2.12}$$

(ii) $f = \mathrm{e}^{nx}$ yields $k v'' - N^2 \mathrm{e}^{ny} v = 0$ whose solution is

$$v(y) = \mathrm{Bessel}I\left[0, \frac{2\sqrt{\mathrm{e}^{ny} N^2}}{\sqrt{k}n}\right] C_1 + 2\mathrm{Bessel}K\left[0, \frac{2\sqrt{\mathrm{e}^{ny} N^2}}{\sqrt{k}n}\right] C_2. \tag{2.13}$$

(iii) $f = 1/x^2$ yields $k v'' - N^2 (1/y^2) v = 0$ whose complex solution is

$$v(y) = y^{\frac{i\left(-i\sqrt{k}-\sqrt{-k-4N^2}\right)}{2\sqrt{k}}} C_1 + y^{\frac{i\left(-i\sqrt{k}+\sqrt{-k-4N^2}\right)}{2\sqrt{k}}} C_2. \tag{2.14}$$

2.2.3. *Some general cases*

(a) The reduced ode for the case 2.1.1(b) is, using X_2,

$$v^m(4yv'' + 2v') + 4m v^{m-1} y v'^2 - (1/y) N^2 v + y v' = 0,$$

where $y = x^2/t$ and $v = u$.

(b) The reduced ode for the case 2.1.1(c) is, using X_2,

$$n^2 v^m (v'' + (2/m) v' + (1/m^2) v) + m n^2 v^{m-1} (v' + (1/m))^2 - N^2 v - \mathrm{e}^{-y} v' = 0,$$

where $y = nx + \ln t$ and $v = \mathrm{e}^{-(n/m)x} u$.

(c) For $f = \mathcal{F}$, a constant, the symmetry ∂_x (translation in x), for all $k(u)$ yields the ode

$$u_t = -N^2 \mathcal{F} t \tag{2.15}$$

whose solution is $u = C\mathrm{e}^{-N^2 \mathcal{F} t}$, C a constant. Here, the temperature u tends to zero with time.

3. Conclusion

We have shown that a further probe into a symmetry analysis of the fin equation gives rise to some interesting solutions for various forms of the heat transfer coefficient and thermal conductivity. In particular, steady state and travelling wave solutions of the temperature have practical value—the latter may imply soliton type behaviour of the temperature. Where the reduced form of the ode is not ‘trivial’ one may resort to a numerical scheme for the reduced equation using given boundary and/or initial conditions.

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