Shortest Path Algorithm

- What is the Shortest Path Problem?
- Is the shortest path problem well defined?
- The Dijkstra's Algorithm for Shortest Path Problem.
- Implementation Dijkstra's Algorithm

What is the shortest path problem?

- In an edge-weighted graph, the weight of an edge measures the cost of traveling that edge.

- For example, in a graph representing a network of airports, the weights could represent: distance, cost or time.

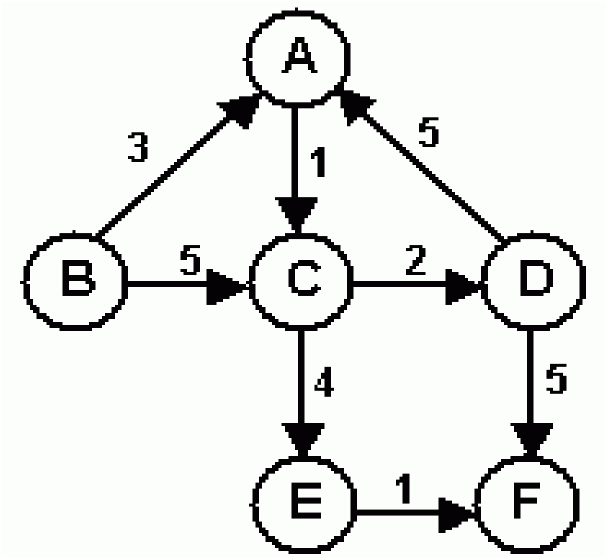
- Such a graph could be used to answer any of the following:
  - What is the fastest way to get from A to B?
  - Which route from A to B is the least expensive?
  - What is the shortest possible distance from A to B?

- Each of these questions is an instance of the same problem:

  The shortest path problem!

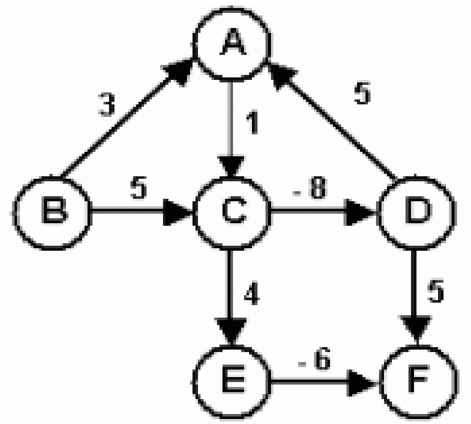
Is the shortest path problem well defined?

- If all the edges in a graph have non-negative weights, then it is possible to find the shortest path from any two vertices.
- For example, in the figure below, the shortest path from B to F is { B, A, C, E, F } with a total cost of nine.
- Thus, the problem is well defined for a graph that contains non-negative weights.



Is the shortest path problem well defined? - Cont'd

- Things get difficult for a graph with negative weights.
- For example, the path D, A, C, E, F costs 4 even though the edge (D, A) costs 5 -- the longer the less costly.
- The problem gets even worse if the graph has a negative cost cycle. e.g. {D, A, C, D}
- A solution can be found even for negative-weight graphs but not for graphs involving negative cost cycles.



{D, A, C, D, A, C, E, F} = 2

{D, A, C, D, A, C, D, A, C, E, F} = 0

The Dijkstra's Algorithm

- Dijkstra's algorithm solves the single-source shortest path problem for a non-negative weights graph.

- It finds the shortest path from an initial vertex, say s, to all the other vertices.

The Dijkstra's Algorithm Cont'd

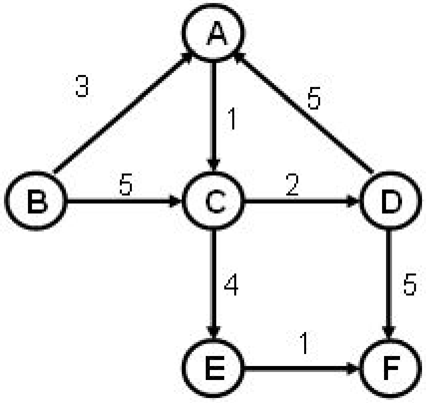
```
// Let V be the set of all vertices in G, and s the start vertex.
for(each vertex v){
   currentDistance(s-v) = ∞;
   predecessor(v) = undefined;
}
currentDistance(s-s) = 0;
T = V;
while(T ≠ ϕ){
   v = a vertex in T with minimal currentDistance from s;

      T= T – {v};
    for(each vertex u adjacent to v and in T){
      if(currentDistance(s-u) > currentDistance(s-v) + weight(edge(vu)){
         currentDistance(s-u) = currentDistance(s-v) + weight(edge(vu));
         predecessor(u) = v;
      }
    }
}
```

For each vertex, the algorithm keeps track of its current distance from the starting vertex and the predecessor on the current path

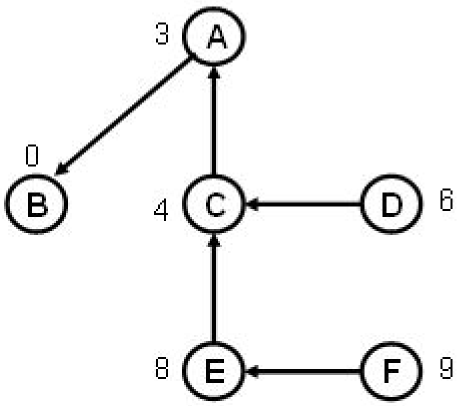
Example

Tracing Dijkstra's algorithm starting at vertex B:



Pass:	initially	1	2	3	4	5	6	Shortest distance	Predecessor
Active vertex:		B	A	C	D	E	F		
A	∞	3						3	B
B	0							0	-
C	∞	5	4					4	A
D	∞	∞	∞	6				6	C
E	∞	∞	∞	8	8			8	C
F	∞	∞	∞	∞	11	9		9	E

The resulting vertex-weighted graph is:



Data structures required

- The implementation of Dijkstra's algorithm uses the Entry structure, which contains the following three fields:
  - **know**: a boolean variable indicating whether the shortest path to v is known, initially false for all vertices.
  - **distance** : the shortest known distance from s to v, initially infinity for all vertices except that of s which is 0.
  - **predecessor** : the predecessor of v on the path from s to v, initially unknown for all vertices.

```
public class Algorithms{
    static final class Entry{
        boolean known;
        int distance;
        Vertex predecessor;

        Entry(){
            known = false;
            distance = Integer.MAX_VALUE;
            predecessor = null;
        }
    }
```

Implementation of Dijkstra's Algorithm

- The dijkstrasAlgorithm method shown below takes two arguments, a directed graph and the starting vertex.
- The method returns a vertex-weighted Digraph from which the shortest path from s to any vertex can be found.
- Since in each pass, the vertex with the smallest known distance is chosen, a minimum priority queue is used to store the vertices.

```
public static Graph dijkstrasAlgorithm(Graph g, Vertex start){
   int n = g.getNumberOfVertices();
   Entry table[] = new Entry[n];
   for(int v = 0; v < n; v++)
      table[v] = new Entry();

      table[g.getIndex(start)].distance = 0;
      PriorityQueue queue = new BinaryHeap(
                                          g.getNumberOfEdges());
      queue.enqueue(new Association(new Integer(0), start));
```

Implementation of Dijkstra's Algorithm - Cont'd

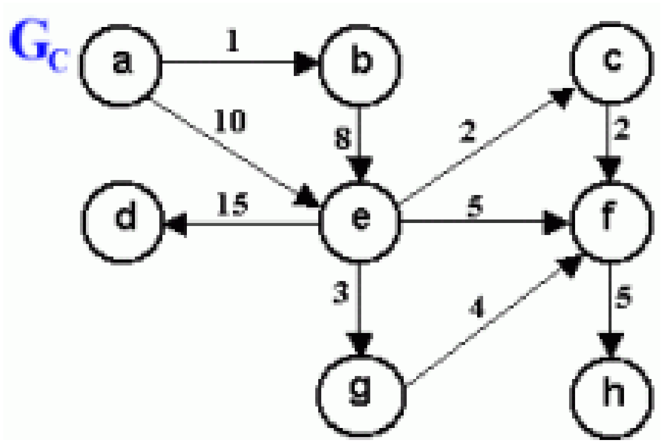
```
while(!queue.isEmpty()) {
   Association association = (Association)queue.dequeueMin();
   Vertex v1 = (Vertex) association.getValue();
   int n1 = g.getIndex(v1);
   if(!table[n1].known){
      table[n1].known = true;
      Iterator p = v1.getEmanatingEdges();
      while (p.hasNext()){
         Edge edge = (Edge) p.next();
         Vertex v2 = edge.getMate(v1);
         int n2 = g.getIndex(v2);
         Integer weight = (Integer) edge.getWeight();
         int d = table[n1].distance + weight.intValue();
         if(table[n2].distance > d){
            table[n2].distance = d;
            table[n2].predecessor = v1;
            queue.enqueue(new Association(d, v2));
         }
      }
   }
}
```

Implementation of Dijkstra's Algorithm Cont'd

```
    Graph result = new GraphAsLists(true);//Result is Digraph
    Iterator it = g.getVertices();
    while (it.hasNext()){
       Vertex v = (Vertex) it.next();
       result.addVertex(v.getLabel(),
                   new Integer(table[g.getIndex(v)].distance));
    }

    it = g.getVertices();
    while (it.hasNext()){
       Vertex v = (Vertex) it.next();
       if (v != start){
          String from = v.getLabel();
          String to = table[g.getIndex(v)].predecessor.getLabel();
          result.addEdge(from, to);
       }
    }
    return result;
}
```

Review Questions



- Use the graph Gc shown above to trace the execution of Dijkstra's algorithm as it solves the shortest path problem starting from vertex a.

- Dijkstra's algorithm works as long as there are no negative edge weights. Given a graph that contains negative edge weights, we might be tempted to eliminate the negative weights by adding a constant weight to all of the edges. Explain why this does not work.

- Dijkstra's algorithm can be modified to deal with negative edge weights (but not negative cost cycles) by eliminating the known flag and by inserting a vertex back into the queue every time its tentative distance decreases. Implement this modified algorithm.