Complexity Analysis (Part I)

- Motivations for Complexity Analysis.
- Example of Basic Operations
- Average, Best, and Worst Cases.
- Simple Complexity Analysis Examples.

Motivations for Complexity Analysis

- There are often many different *algorithms* which can be used to solve the same problem. Thus, it makes sense to develop techniques that allow us to:
  - compare different algorithms with respect to their “efficiency”
  - choose the most efficient algorithm for the problem

- The ***efficiency*** of any algorithmic solution to a problem is a measure of the:
  - **Time efficiency**: the time it takes to execute.
  - **Space efficiency**: the space (primary or secondary memory) it uses.

- We will focus on an algorithm’s efficiency with respect to time.

Machine independence

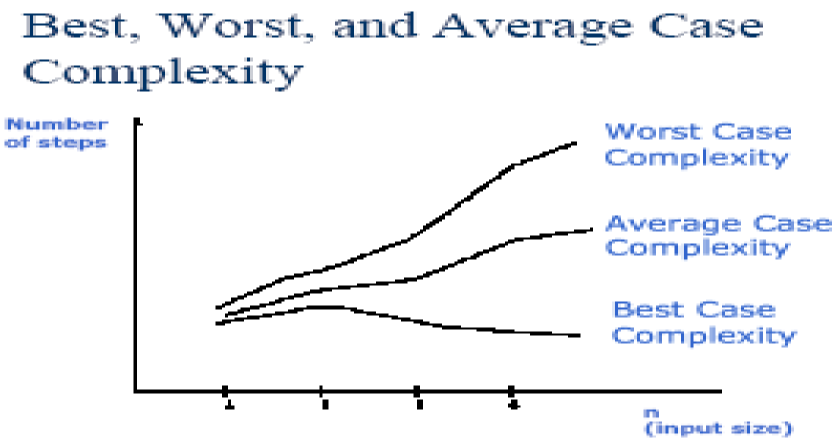
- The evaluation of efficiency should be as machine independent as possible.
- It is not useful to measure how fast the algorithm runs as this depends on which particular computer, OS, programming language, compiler, and kind of inputs are used in testing
- Instead,
  - we count the number of *basic operations* the algorithm performs.
  - we calculate how this number depends on the size of the input.
- A basic operation is an operation which takes a constant amount of time to execute.
- Hence, <u>the efficiency of an algorithm is the number of basic operations it performs</u>. This number is a function of the input size **n**.

Example of **Basic** Operations:

- Arithmetic operations: *, /, %, +, -
- Assignment statements of simple data types.
- Reading of primitive types
- writing of a primitive types
- Simple conditional tests: if (x < 12) ...
- method call (Note: the execution time of the method itself may depend on the value of parameter and it may not be constant)
- a method's return statement
- Memory Access
- We consider an operation such as ++ , += , and *= as consisting of two basic operations.
- **Note**: To simplify complexity analysis we will not consider memory access (fetch or store) operations.

Best, Average, and Worst case complexities

- We are usually interested in the **worst case** complexity: what are the most operations that might be performed for a given problem size. We will not discuss the other cases -- **best** and **average case**.



- Best case depends on the input
- Average case is difficult to compute
- So we usually focus on worst case analysis
  - Easier to compute
  - Usually close to the actual running time
  - Crucial to real-time systems (e.g. air-traffic control)

Best, Average, and Worst case complexities

- Example: Linear Search Complexity
- Best Case : Item found at the beginning: ***One comparison***
- Worst Case : Item found at the end: ***n comparisons***
- Average Case :Item may be found at index 0, or 1, or 2, . . . or n - 1
  - Average number of comparisons is: (1 + 2 + . . . + n) / n = **(*n*+1) / 2**

- Worst and Average complexities of common sorting algorithms

Method	Worst Case	Average Case
Selection sort	n^2	n^2
Inserstion sort	n^2	n^2
Merge sort	n log n	n log n
Quick sort	n^2	n log n

Simple Complexity Analysis: Loops

- We start by considering how to count operations in **for**-loops.
  - We use integer division throughout.

- First of all, we should know the number of iterations of the loop; say it is **x**.
  - Then the loop condition is executed **x + 1** times.

  - Each of the statements in the loop body is executed **x** times.

  - The loop-index update statement is executed **x** times.

Simple Complexity Analysis: Loops (with <)

- In the following for-loop:

```
for (int i = k; i < n; i = i + m){
       statement1;
       statement2;
 }
```

The number of iterations is: (**n – k) / m**

- The initialization statement, **i = k**, is executed **one** time.
- The condition, **i < n**, is executed (**n – k) / m + 1** times.
- The update statement, **i = i + m**, is executed (**n – k) / m** times.
- Each of **statement1** and **statement2** is executed (**n – k) / m** times.

Simple Complexity Analysis : Loops (with <=)

- In the following for-loop:

```
for (int i = k; i <= n; i = i + m){
        statement1;
        statement2;
 }
```

- The number of iterations is: (**n – k) / m + 1**
- The initialization statement, **i = k**, is executed **one** time.
- The condition, **i <= n**, is executed (**n – k) / m + 2** times.
- The update statement, **i = i + m**, is executed (**n – k) / m + 1** times.
- Each of **statement1** and **statement2** is executed (**n – k) / m + 1** times.

Simple Complexity Analysis: Loop Example

- Find the exact number of basic operations in the following program fragment:

```
double x, y;
x = 2.5 ; y = 3.0;
for(int i = 0; i < n; i++){
    a[i] = x * y;
    x = 2.5 * x;
    y = y + a[i];
}
```

- There are 2 assignments outside the loop => 2 operations.
- The **for** loop actually comprises
- an assignment (i = 0) => 1 operation
- a test (i < n) => n + 1 operations
- an increment (i++) => 2 n operations
- the loop body that has three ***assignments***, two ***multiplications***, and an ***addition*** => 6 n operations

Thus the total number of basic operations is 6 * n + 2 * n + (n + 1) + 3

= **9n + 4**

Simple Complexity Analysis: Examples

- Suppose **n** is a multiple of 2. Determine the number of basic operations performed by of the method myMethod():

```
static int myMethod(int n){
    int sum = 0;
    for(int i = 1; i < n; i = i * 2)
          sum = sum + i + helper(i);
     return sum;
 }
```

```
static int helper(int n){
     int sum = 0;
     for(int i = 1; i <= n; i++)
            sum = sum + i;
     return sum;
}
```

- Solution: The number of iterations of the loop:

  for(int i = 1; **i < n**; i = i * 2)
  sum = sum + i + *helper(i)*;

  is $\mathbf{log_2 n}$ (A Proof will be given later)

  Hence the number of basic operations is:

  1 + 1 + (1 + log2 n) + *log2 n[2 + 4 + 1 + 1 + (n + 1) + n[2 + 2] + 1]* + 1

  = 3 + log2 n + log2 n[10 + 5n] + 1

  = 5 n log2 n + 11 log2 n + 4

Simple Complexity Analysis: Loops With Logarithmic Iterations

- In the following for-loop: (with <)

```
for (int i = k; i < n; i = i * m){
        statement1;
        statement2;
 }
```

  - The number of iterations is: $\lceil \mathbf{(Log_m (n / k))} \rceil$

- In the following for-loop: (with <=)

```
for (int i = k; i <= n; i = i * m){
        statement1;
        statement2;
 }
```

  - The number of iterations is: $\lfloor \mathbf{(Log_m (n / k) + 1)} \rfloor$