Introduction

1. Mathematical and scientific discovery often arises from the recognition of a pattern.
2. There are two main aspects of inquiry in mathematics and science whereby new results can be discovered
   - deduction, and
   - induction.
3. Deduction, accepting certain statements as premises and axioms, we can deduce other statements on the basis of valid inferences.
4. Induction,
   - the process of discovering general laws by observation and experimentation.
   - arriving at a conjecture for a general rule by inductive reasoning.
   - proof technique for verifying conjectures about positive integers.

The Well-Ordering Property

The validity of mathematical induction follows from the following fundamental axiom about the set of integers.

Definition 1 *A set S is well ordered if every subset has a least element.* ■

Note: $(0, 1]$ is not well ordered since $(0, 1]$ does not have a least element.

Example: $\mathbb{N}$ *is well ordered (under the $\leq$ relation)*

Example: *Any countably infinite set can be well ordered.*

Example: *The set of all positive real numbers has no least element. For if x is any positive real number, then $x/2$ is a positive real number that is less than x. However, no violation of the well-*

I Putu Danu Raharja

ordering principle occurs because the well-ordering principle refers only to the sets of integers and this set is not a set of integers.

Example: *The set of all nonnegative integers n such that $n^2 < n$ has no least nonnegative integer n such that $n^2 < n$ because there is no nonnegative integer that satisfies this inequality. No violation of the well-ordering principle occurs because the well-ordering principle refers only to the sets that contain at least one or more elements.*

Let $P(x)$ be a predicate over a well ordered set S. The problem is to prove $\forall x P(x)$.

The rule of inference called The **(first) principle of Mathematical Induction** can sometimes be used to establish the universally quantified assertion.

In the case that $S = \mathbb{Z}^+$, the positive integers, the principle has the following form.

$$P(1)$$
$$P(n) \to P(n+1)$$
$$\therefore \forall x P(x)$$

The hypotheses are $H_1 : P(1)$ and $H_2 : P(n) \to P(n+1)$ for n arbitrary. H_1 is called The Basis Step. H_2 is called The Induction (Inductive) Step.

The Principle of Mathematical Induction

- Let $P(n)$ be a statement which, for each positive integer n, may either be true or false.
- To prove $P(n)$ is true for all integers $n \geq 1$, it suffices to prove
  1. $P(1)$ is true.
  2. For all $k \geq 1$, $P(k)$ implies $P(k+1)$.

Then,

- knowing it is true for the first element means it must be true for the element following the first or the second element.
- knowing it is true for the second element implies it is true for the third and so forth.

I Putu Danu Raharja

Therefore,induction is equivalent to modus ponens applied a countable number of times!!

General Case

- For all n that belong to $\mathbb{Z}$, $P(n)$ is true for all $n \geq n_0$ if
  1. $P(n_0)$ is true.
  2. For all $k \geq n_0$, $P(k)$ implies $P(k+1)$.
- n_0 is called the basis of induction and it may be any integer.

Three Steps to a Proof using Induction

1. Basis of Induction
   Show that $P(n_0)$ is true.
2. Inductive Hypothesis
   Assume $P(k)$ is true for $k \geq n_0$.
3. Inductive Step
   Show that $P(k+1)$ is true on the basis of the inductive hypothesis.

Example: *To determine a formula for the sum of the first n positive integers.*

Let $S(n) = \sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n.$

Examine a few values for $S(n)$,

n	1	2	3	4	5	6	7
$S(n)$	1	3	6	10	15	21	28

Observe the following pattern

$$
\begin{aligned}
2S(1) &= 2 &&= 1 \times 2 \\
2S(2) &= 6 &&= 2 \times 3 \\
2S(3) &= 12 &&= 3 \times 4
\end{aligned}
$$

I Putu Danu Raharja

$$\begin{aligned} 2S(4) &= 20 &= 4 \times 5 \\ 2S(5) &= 30 &= 5 \times 6 \\ 2S(6) &= 42 &= 6 \times 7 \end{aligned}$$

Conjecture that $2S(n) = n(n+1)$ or $S(n) = n(n+1)/2$.
Prove by mathematical induction
Statement: Let P(n) be the statement: the sum $S(n)$ of the first n positive integers is equal to $n(n+1)/2$.

1. Basis Step
   Since $S(1) = 1 = 1(1+1)/2$, the formula is true for $n = 1$.
2. Inductive Hypothesis
   Assume that $P(n)$ is true for $n = k$, that is $S(k) = 1 + 2 + \ldots + k = k(k+1)/2$.
3. Inductive Step
   - Now show that the formula is true for $n = k + 1$.
   - Observe that $S(k+1) = 1+2+\ldots+k+(k+1) = S(k)+(k+1)$.
   - Since $S(k) = k(k+1)/2$ by the inductive hypothesis, then

   $$\begin{aligned} S(k+1) &= S(k) + (k+1) \\ &= (k/2)(k+1) + (k+1) \\ &= (k+1)(k/2+1) \\ &= ((k+1)(k+2))/2 \end{aligned}$$

   and the formula holds for k + 1. QED

In general, to prove by mathematical induction that the summation formula

$$\sum_{k=1}^{n} f(k) = F(n)$$

is true for every natural number n, we simply have to check that:

(a) **Basis step**: $f(1) = F(1)$,

(b) **Inductive step**: $F(n+1) - F(n) = f(n+1)$, holding for arbitrary n.

I Putu Danu Raharja

Example: *Let m be a nonnegative integer. Then*

$$\sum_{k=1}^{n} k(k+1)\cdots(k+m) = \frac{n(n+1)\cdots(n+m+1)}{m+2}$$

For $n = 1$, both sides are equal to $(m+1)!$ so (a) holds. To check (b), we evaluate $F(n+1) - F(n)$, where $F(n)$ is the expression $\frac{n(n+1)\cdots(n+m+1)}{m+2}$. Simplifying the expression

$$\frac{(n+1)(n+2)\cdots(n+m+2)}{m+2} - \frac{n(n+1)\cdots(n+m+1)}{m+2}$$

we find

$$F(n+1) - F(n) = (n+1)(n+2)\cdots(n+m+1) = f(n+1)$$

Thus the formula holds in all cases. Using this formula, we can derive other sums. For example, since

$$k^3 = k(k+1)(k+2) - 3k(k+1) + k,$$

then, we obtain

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example: *Prove that if $n > 1$, then*

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}.$$

Basis Step: If $n = 1$, we have

$$\frac{1}{2} = \frac{1}{\sqrt{3\cdot 1+1}}.$$

Inductive Step: Assume now that for some n,

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}.$$

If both sides of this inequality are multiplied by $\dfrac{2n+1}{2n+2}$, it becomes

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n+2} \leq \frac{2n+1}{(2n+2)\sqrt{3n+1}}.$$

I Putu Danu Raharja

Now,

$$\begin{aligned}\left[\frac{2n+1}{(2n+2)\sqrt{3n+1}}\right]^2 &= \frac{(2n+1)^2}{12n^3+28n^2+20n+4}\\ &= \frac{(2n+1)^2}{(12n^3+28n^2+19n+4)+n}\\ &= \frac{(2n+1)^2}{(2n+1)^2(3n+4)+n} < \frac{1}{3n+4},\end{aligned}$$

and it follows that

$$\frac{2n+1}{(2n+2)\sqrt{3n+1}} < \frac{1}{\sqrt{3n+4}}.$$

Thus, we obtain

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n+2} < \frac{1}{\sqrt{3(n+1)+1}}.$$

Example: *Use the Principle of Mathematical Induction to prove that* $4 \mid (9^n - 5^n), \forall n \geq 0$.

$P(0) : 4 \mid 1-1$ is true since $4 \mid 0$. $P(k) \to P(k+1) : 9^{k+1} - 5^{k+1} = 9(9^k - 5^k) + 5^k(9-5)$. Each term is divisible by $4 : 4 \mid 9^k - 5^k$ (by $P(k)$) and $4 \mid 9-5$.

Example: *Use the Principle of Mathematical Induction to prove that* $3 \mid (n^3 + 3n^2 + 2n), \forall n \geq 1$.

Recursive Definitions

Recursive or inductive definitions of sets and functions on recursively defined sets are similar.

1. Basis step:
   - (i) For sets: State the basic building blocks (BBB's) of the set.
   - (ii) For functions: State the values of the function on the BBBs.
2. Inductive or recursive step:
   - (i) For sets: Show how to build new things from old with some construction rules.
   - (ii) For functions: Show how to compute the value of a function

I Putu Danu Raharja

on the new things that can be built knowing the value on the old things.

3. Extremal clause:
   (i) For sets: If you can't build it with a finite number of applications of steps 1.and 2. then it isn't in the set.
   (ii) For functions: A function defined on a recursively defined set does not require an extremal clause.

NOTE:

- To prove something is in the set you must show how to construct it with a finite number of applications of the basis and inductive steps.
- To prove something is not in the set is often more difficult.

Example: *A recursive definition of* $\mathbb{N}$.

1. **Basis Step**: 0 is in $\mathbb{N}$ (0 is the BBB).
2. **Recursive Step**: if n is in $\mathbb{N}$ then so is $n+1$ (how to build new objects from old: add one to an old object to get a new one).

Example: *A recursive definition of* $F(n) = n!$

- *Basis Step:* $F(0) = 0$
- *Recursive Step:* $F(n+1) = (n+1) \cdot F(n)$

Example: *Give an inductive definition of* a^n *where* a *is a nonzero real number and* n *is a nonnegative integer.*

- Basis Step: $a^0 = 1$
- Recursive Step: $a^{n+1} = a^n \cdot a$

Example: *A recursive definition of the Fibonacci sequence*

1. Basis Step: $f(0) = f(1) = 1$ (two initial conditions)
2. Recursive Step: $f(n+1) = f(n) + f(n-1)$ (the recurrence equation).

Example: *Prove the remarkable fact that for* $n \geq 1$,

$$\phi^{n-2} \leq f(n) \leq \phi^{n-1}$$

I Putu Danu Raharja

where $\phi = (1+\sqrt{5})/2$. ϕ *is called the golden ratio.*

To prove it we will use this Lemma: $\phi^2 = \phi + 1$. The proof consists of two parts:

- $P(n) : f(n) \leq \phi^{n-1}$
  1. Base case: $P(1)$. Since both sides of $P(1)$ reduce to 1, $P(1)$ holds.
  2. base case: $P(2)$. Since $1 \leq (1+\sqrt{5})/2$, which is true. Hence, $P(2)$ holds.
  3. Inductive case: For arbitrary $n \geq 2$, we assume inductive hypothesis $P(i)$ for $1 \leq i \leq n$ and prove $P(n+1)$:

$$\begin{aligned} f(n+1) &= f(n) + f(n-1) \\ &\leq \phi^{n-1} + \phi^{n-2} \\ &= \phi^{n-2} \times (\phi + 1) \\ &= \phi^n \end{aligned}$$

- $P(n) : \phi^{n-2} \leq f(n)$: This is an exercise for the students.

Structural Induction

A proof by structural induction consists of two parts.

- Basis Step: Show that the result holds for all elements specified in the basis step of the recursive definition to be in the set.
- Recursive Step: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

Example: *Give a recursive definition of the function* $m(s)$*, which equals the smallest digit in a nonempty string of decimal digits. Prove that* $m(st) = min(m(s), m(t))$.

- Basis step: If $x \in D = \{0,1,2,3,4,5,6,7,8,9\}$, then $m(x) = x$;
- Recursive step: if $s = tx$, where $t \in D^*$ and $x \in D$, then $m(s) = min(m(t), x)$.

To prove that $m(st) = min(m(s), m(t))$ using structural induction, let $t = wx$, where $t \in D^*$ and $x \in D$. If $w = \lambda$, then

I Putu Danu Raharja

$m(st) = m(sx) = min(m(s), x) = min(m(s), m(x))$ by the recursive step and the basis step of the definition of m. Otherwise, $m(st) = m((sw)x) = min(m(sw), m(x))$ by the definition of m. Now, $m(sw) = min(m(s), m(w))$, by the inductive hypothesis so $m(st) = min(min(m(s), m(w)), x) = min(m(s), min(m(w), x))$. But, $\min(m(w), x) = min(wx) = m(t)$ by the recursive step. Thus, $m(st) = min(m(s), m(t))$.

I Putu Danu Raharja