Overview

1. Sequences

2. Summations

3. Cardinality of Infinite Sets

4. Preview: Inductions and Recursions

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Introduction to Sequences

Definition 1 *A* **sequence** *is a function from a subset of the of integers (usually either the set {0, 1, 2, . . .} or the set {1, 2, . . .}) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a* **term** *of the sequence.* ■

That is, if f is a function from $\{0, 1, 2, ...\}$ to S we usually denote $f(i)$ by a_i and we write

$$\{a_0, a_1, a_2, a_3, \ldots\} = \{a_i\}_{i=0}^{k} = \{a_i\}_0^k$$

where k is the upper limit (usually ∞).

Note: the sets $\{0, 1, 2, 3, \ldots, k\}$ and $\{1, 2, 3, 4, \ldots, k\}$ are called initial segments of $\mathbb{N}$.

Example: *Using zero-origin indexing, if $f(i) = \frac{1}{(i+1)}$, then the sequence*

$$f = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} = \{a_0, a_1, a_2, a_3, \ldots\}$$

Example: *Using one-origin indexing the sequence f becomes*

$$\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} = \{a_1, a_2, a_3, \ldots\}$$

Definition 2 *A* geometric progression *is a sequence of the form*

$$a, ar, ar^2, ar^3, \ldots, ar^n$$

,where the initial term a and the common ratio r are $\in$ **R***.* ■

Definition 3 *An* arithmetic progression *is a sequence of the form*

$$a, a + d, a + 2d, \ldots, a + nd$$

,where the initial term a and the common difference d are $\in$ **R***.* ■

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Summation Notation

Given a sequence $\{a_i\}_0^k$ we can add together a subset of the sequence by using the summation and function notation

$$a_{g(m)} + a_{g(m+1)} + \ldots + a_{g(n)} = \sum_{j=m}^{n} a_{g(j)}$$

The above summation can be written in a more general form as,

$$\sum_{m \le j \le n} a_{g(j)} = \sum_{j \in S} a_{g(j)}$$

Example:

$$r^0 + r^1 + r^2 + r^3 + \ldots + r^n = \sum_{j=0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \sum_{i=1}^{\infty} \frac{1}{i}$$

$$a_{2m} + a_{2(m+1)} + \ldots + a_{2(n)} = \sum_{j=m}^{n} a_{2j}$$

If $S = \{2, 5, 7, 10\}$ then

$$\sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Similarly for the product notation:

$$\prod_{j=m}^{n} a_j = a_m a_{m+1} \cdots a_n$$

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Let K be any finite set of integers. Sum over the elements of K can be transformed by using the following rules:

(a) $\sum_{k \in K} ca_k = c \sum_{k \in K} a_k$, (distributive law)

(b) $\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{\in K} b_k$,(associative law)

(c) $\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}$,(commutative law)

(d) $\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \sum_{k \in K \cap K'} a_k + \sum_{k \in K \cup K'} a_k$,

(e) $\sum_{j} \sum_{k} a_{j,k} = \sum_{P(j,k)} a_{j,k} = \sum_{k} \sum_{j} a_{j,k}$, (interchanging the order of summation).

Example:

$$\sum_{k=1}^{m} a_k + \sum_{k=m}^{n} a_k = a_m + \sum_{k=1}^{n} a_k, 1 \leq m \leq n;$$

Example:

$$\sum_{k=0}^{n} a_k = a_0 + \sum_{k=1}^{n} a_k, n \geq 0.$$

Example:

$$\sum_{j=1}^{n} \sum_{k=j}^{n} a_{j,k} = \sum_{1 \leq j \leq k \leq n} a_{j,k} = \sum_{k=1}^{n} \sum_{j=1}^{k} a_{j,k}$$

Example: *Let* $S_n = \sum_{k=0}^{n} a_k$.

$$\begin{aligned} S_n + a_{n+1} = \sum_{0 \leq k \leq n+1} a_k &= a_0 + \sum_{1 \leq k \leq n+1} a_k \\ &= a_0 + \sum_{1 \leq k+1 \leq n+1} a_{k+1} \\ &= a_0 + \sum_{0 \leq k \leq n} a_{k+1} \end{aligned}$$

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Example: *Find the sum of a geometric progression,* $S_n = \sum_{k=0}^{n} ax^k$.

$$\begin{aligned} S_n + ax^{n+1} &= ax^0 + \sum_{0\le k\le n} ax^{k+1} \\ &= a + xS_n \\ S_n &= \frac{ax^{n+1}-a}{x-1}, x \neq 1. \end{aligned}$$

Example: *Let* x *be a real number with* $|x| < 1$. *Find* $\sum_{k=0}^{\infty} x^k$.

Solution: *By prior example, with* $a = 1$ *we see that* $\sum_{k=0}^{n} x^k = \frac{x^{k+1}-1}{x-1}$. *Because* $|x| < 1$, x^{k+1} *approaches 0 as* k *approaches* ∞. *It follows that*

$$\sum_{k=0}^{\infty} x^k = \lim_{x\to\infty} \frac{x^{k+1}-1}{x-1} = \frac{1}{1-x}$$

Example:

- *Show that the sum to* n *terms of the series*

$$\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \cdots$$

  is equal to $n - 1 + 2^{-n}$.

- *Show that*

$$\sum_{k=0}^{n} kx^k = \frac{x - (n+1)x^{n+1} + xn^{x+2}}{(1-x)^2}, \text{ for } x \neq 1$$

- *Find the sum of the series*

$$S = 1^2 - 2^2 + 3^2 - 4^2 + \cdots - 2002^2 + 2003^2.$$

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Cardinality of Infinite Sets

Definition 4 *The sets A and B have the same* **cardinality** *if and only if there is a* one-to-one correspondence *from A to B (a bijective map $f : A \to B$).* ∎

With infinite sets proper subsets can have the same cardinality. This cannot happen with finite sets.

By definition, $|A| \leq |B|$ if there is an injection from A to B.

Theorem 1 *If $A \subseteq B$, then $|A| \leq |B|$. This can be seen by defining a function $f(x) = x$ which is clearly an injection from A to B.* ∎

Example:

$$|\{1, 2, 5\}| \leq \aleph_0.$$

The injection $f : \{1, 2, 5\} \to \mathbb{N}$ defined by $f(x) = x$ is shown in the figure below.

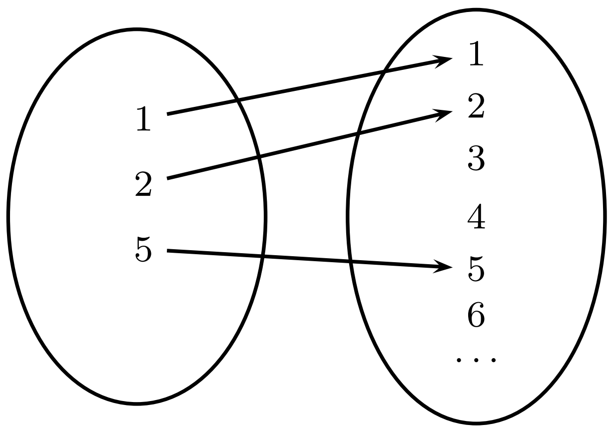

Figure 1: Injection from $\{1, 2, 5\}$ to $\mathbb{N}$.

Lemma 1 *Let A, B, and C be sets.*

(i) A and A have the same cardinality.

(ii) if $|A| = |B|$, then $|B| = |A|$.

(iii) if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

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Definition 5 *Let $n \in \mathbb{Z}$. We let $\mathbb{N}[1,n]$ denote the set $\mathbb{N}[1,n] = \{x \in \mathbb{N} \mid 1 \leq x \leq n\}$.* ■

Definition 6

(1) A set is **finite** *if it is either the empty set or it has the same cardinality as $\mathbb{N}[1,n]$ for some $n \in \mathbb{N}$.*

(2) A set is **infinite** *if it is not finite.*

(3) A set is **countably infinite** *if it has the same cardinality as $\mathbb{N}$ or $\mathbb{Z}^+$.*

(4) A set is **countable** *if it is finite or countably infinite.*

(5) A set is **uncountable** *if it is not countable.* ■

Example: *The set of squares $S = \{1, 4, 9, 16, \ldots\}$ and the set of natural numbers $\mathbb{N}$ have the same cardinality.*

There exists the map $h : \mathbb{N} \to S$ given by $h(n) = (n+1)^2$ for all $n \in \mathbb{N}$ that is a bijective map. That h is bijective follows from the fact that $k : S \to \mathbb{N}$ given by $k(n) = \sqrt{n} - 1$ for all $n \in S$ is an inverse map for h.

Example: *The set of natural numbers $\mathbb{N}$ and the set of integers $\mathbb{Z}$ have the same cardinality.*

Let $f(n)$ be a bijective map $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Theorem 2

(1) The set $\mathbb{N}$ is infinite.

(2) A countably infinite set is infinite.

(3) Let A be a set. Then A is infinite iff it it contains an infinite subset. ■

Example: *The set of (finite length) strings S over a finite alphabet A is countably infinite.*

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Solution: *To show this we assume that*

(a) $A \neq \emptyset$.

(b) There is an "alphabetical" ordering of the symbols in A.

Continuing the proof, let us list the strings in lexicographic order:

(a) all the strings of zero length,

(b) then all the strings of length 1 in alphabetical order,

(c) then all the strings of length 2 in alphabetical order,

(d) etc.

This implies a bijection from $\mathbb{N}$ *to the list of strings and hence it is a countably infinite set.*

Example: *Let us demonstrate the above example with the set*

$$A = \{a, b, c\}.$$

The lexicographic ordering of A *is*

$$\begin{aligned} \{ \quad & (\lambda), (a, b, c), (aa, ab, ac, ba, bc, ca, cb, cc), \\ & (aaa, aab, aac, aba, \ldots), \ldots\} \\ & = \{f(0), f(1), f(2), f(3), f(4), \ldots\} \end{aligned}$$

Example: *The set of all Java programs is countable.*

Solution: *Let* S *be the set of legitimate characters which can appear in a Java program. A Java compiler will determine if an input program is a syntactically correct Java program (the program doesn't have to do anything useful).*

Use the lexicographic ordering of S *and feed the strings into the compiler.*

If the compiler says YES, this is a syntactically correct Java program, we add the program to the list. Else we move on to the next string.

In this way we construct a list or an implied bijection from N *to the set of Java programs. Hence, the set of Java programs is countable.*

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Theorem 3 *The set of real numbers between 0 and 1 is uncountable.* ■

Solution: *We assume that it is countable and derive a contradiction. If it is countable we can list them (i.e., there is a bijection from a subset of* $\mathbb{N}$ *to the set).*

We show that no matter what list you produce we can construct a real number between 0 and 1 which is not in the list. Hence, there cannot exist a list and therefore the set is not countable.

It's actually much bigger than countable. It is said to have the **cardinality of the continuum***, c.*

Represent each real number in the list using its decimal expansion, e.g

$$\begin{aligned} \tfrac{1}{3} &= .3333333\ldots \\ \tfrac{1}{2} &= .5000000\ldots \\ &= .4999999\ldots \end{aligned}$$

If there is more than one expansion for a number, it doesn't matter as long as our construction takes this into account.

$$\begin{array}{ll} r_1 = & .d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\cdots \\ r_2 = & .d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\cdots \\ r_3 = & .d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\cdots \\ \vdots & \vdots \end{array}$$

Now construct the number $x = .x_1x_2x_3x_4x_5x_6x_7....$ *,where*

$$x_i = \begin{cases} 3 & \textit{if } d_{ii} \neq 3 \\ 4 & \textit{if } d_{ii} = 3 \end{cases}$$

(Note: choosing 0 and 9 is not a good idea because of the non uniqueness of decimal expansions.) Then x is not equal to any number in the list.

Hence, no such list can exist and hence the interval $(0, 1)$ *is uncountable.*

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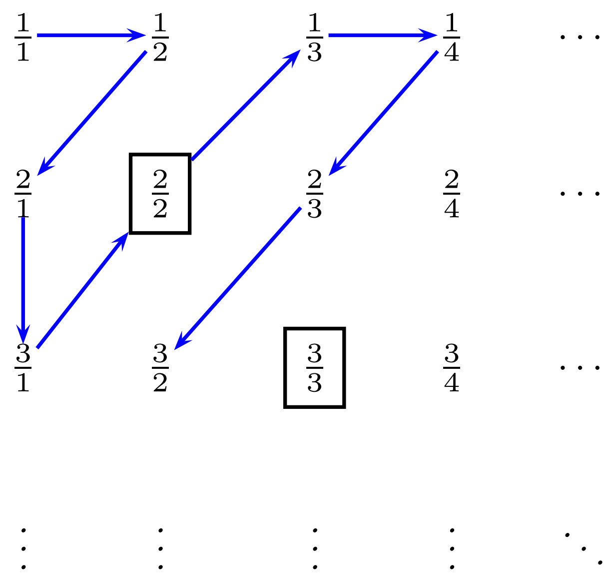
Example: *Show that the set $\mathbb{Q}^+$ is countable.*

Solution: *Display the elements of the set $\mathbb{Q}^+$ in a grid as shown below.*



Define a function $F : \mathbb{Z}^+ \to \mathbb{Q}^+$ by starting to count at $\frac{1}{1}$ and following the arrows as indicated, skipping over any number that has already been counted.

To be specific: Set $F(1) = \frac{1}{1}$, $F(2) = \frac{1}{2}$, $F(3) = \frac{2}{1}$. Skip $\frac{2}{2}$ since $\frac{2}{2} = \frac{1}{1}$, which was counted first. Continue in this way, defining $F(n)$ for each positive integer n.

Note that every positive rational number appears somewhere in the grid, and the counting procedure is set up so that every point in the grid is reached eventually. Thus the function F is onto. Also by skipping numbers that have already been counted, no number is counted twice. So F is one-to-one. Consequently, F is a function from $\mathbb{Z}^+$ to $\mathbb{Q}^+$ that is 1-to-1 and onto, and so $\mathbb{Q}^+$ is countably infinite and hence countable.

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