Overview

1. Functions
2. Types of Functions
3. Inverse and Composition Functions
4. Preview: Sequences and Summations

I Putu Danu Raharja

Introduction to Functions

Definition 1 *Let A and B be sets. A* **function** *from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.* ∎

Formally, f is a function from A to B ($f : A \to B$) if and only if the following hold:

(1) $f \subseteq A \times B$: f is a set of ordered pairs whose first components are from A and whose second components are from B, and,

(2) $\forall a \in A \exists b \in B((a, b) \in f)$: every element from A is mapped to some element in B, and,

(3) $\forall a, b, c(((a, b) \in f \wedge (a, c) \in f \to b = c)$: every element from A is assigned at most one element of B.

Definition 2 *If f is a function from A to B, we say that A is the* **domain** *of f and B is the* **codomain** *of f. If $f(a) = b$, we say that b is the* **image** *of a and a is a* **pre-image** *of b. The* **range** *of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f* **maps** *A to B.* ∎

If f is a function from A to B and the domain and range of f are denoted by $dom(f)$ and $range(f)$ respectively, then

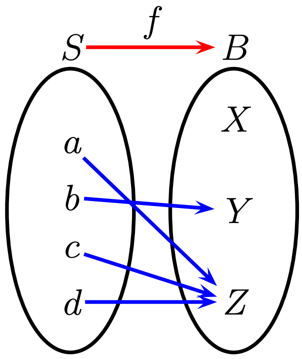
$$\begin{aligned} dom(f) &= \{a \mid \exists b((a, b) \in f)\} = A. \\ range(f) &= f(A) = \{f(a) \mid a \in A\} = \{b \mid \exists a((a, b) \in f)\}. \end{aligned}$$

Definition 3 *Let f be a function from the set A to the set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so that, $f(S) = \{f(s) \mid s \in S\}$.* ∎

I Putu Danu Raharja

Introduction to Functions (cont.)

Example: *Given the sets* $A = \{a, b, c, d\}$ *and* $B = \{X, Y, Z\}$ *and the figure,*



then,

(a) $f(a) = Z$

(b) *The image of* d *is* Z

(c) *The domain of* f *is* $A = \{a, b, c, d\}$

(d) *The codomain is* $B = \{X, Y, Z\}$

(e) $f(A) = \{Y, Z\}$

(f) *The pre-image of* Y *is* b

(g) *The pre-images of* Z *are* a, c *and* d

(h) $f(\{c, d\}) = \{Z\}$

Example: *Let ICS253 be the set of students in this class. Define*

$$d : ICS253 \rightarrow \mathbb{N}$$

by "if the last name of $s \in ICS253$ *begins with letters A through H, then* $d(s) = 2$*, else,* $d(s) = 1$*".*

(a) *What is the image of "Hamad Ali"?*

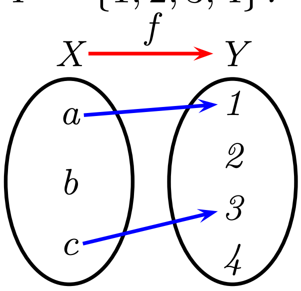
(b) *What is the the pre-image of 1?*

(c) *What is the codomain of d?*

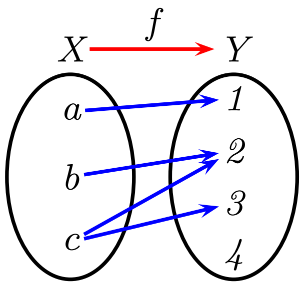
(d) *What is the range of d?*

I Putu Danu Raharja

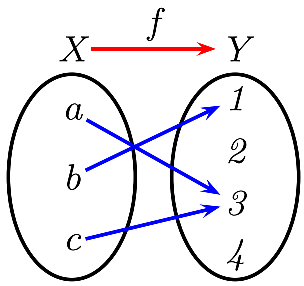
Example: *Which of the following define functions from $X = \{a, b, c\}$ to $Y = \{1, 2, 3, 4\}$?*



(a)



(b)



(c)

Only (c) defines a function.

Example: *Someone tries to define a function $f : \mathbb{Q} \to \mathbb{Z}$ by the formula:*

$$f\left(\frac{m}{n}\right) = m$$

That is, the integer associated by f to the number m/n is m. Is f a function?

Solution: *Fractions have more than one representation as quotients of integers. For instance, $\frac{1}{2} = \frac{4}{8}$. Now if f were a function then $f(\frac{1}{2}) = f(\frac{4}{8})$ since $\frac{1}{2} = \frac{4}{8}$. But, we get $f(\frac{1}{2}) = 1$ and $f(\frac{4}{8}) = 4$. This contradiction shows that the relation f is not a function.*

Definition 4 *Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then f_1+f_2 and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined by*

$$(f_1 + f_2)(x) = f_1(x) + f_2(x).$$

$$(f_1.f_2)(x) = f_1(x).f_2(x).$$

■

I Putu Danu Raharja

Functions Equality

Definition 5 *Two functions $f : A \to B$ and $g : C \to D$ are equal, $(f = g)$ if $A = C$, $B = D$, and for every $a \in A$, $f(a) = g(a)$.* ■

Note that in order to be equal f and g must have the same domain and the same codomain.

Example: *Consider the functions:*

$$f : \mathbf{R} - \{0\} \to \mathbf{R}$$

defined by $f(x) = 3x/x$, and

$$g : \mathbf{R} \to \mathbf{R}$$

defined by $g(x) = 3$. Then,

$$f(x) = g(x)$$

for every x in the domain of f, however, $f \neq g$, because g is defined on a larger domain.

Example: *Define $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ by the following formulas:*

$$\begin{aligned} f(x) &= |x| & \forall x \in \mathbf{R} \\ g(x) &= \sqrt{x^2} & \forall x \in \mathbf{R} \end{aligned}$$

Does $f = g$?

Solution: *Yes. Since the absolute value of a number equals the square root of its square.*

$$|x| = \sqrt{x^2} \ , \forall x \in \mathbf{R}$$

Hence $f = g$.

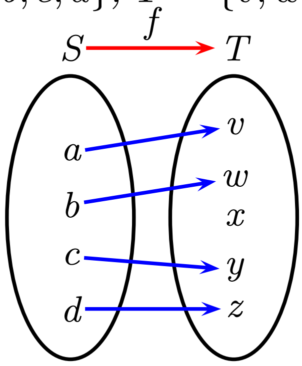
I Putu Danu Raharja

Types of Functions

Definition 6 *A function f is said to be* one-to-one*, or* injective*, iff if $f(x) = f(y)$ implies that $x = y$ for all x and y in the domain of f. A function is said to be an injection if it is one-to-one.*

$$F : X \longrightarrow Y \text{ is one-to-one } \Leftrightarrow \forall a \forall b \in X(F(a) = F(b) \rightarrow a = b) \quad \blacksquare$$

Example: *Let $S = \{a, b, c, d\}$, $T = \{v, w, x, y, z\}$ and $f : S \rightarrow T$.*



An Injective Function that is not Surjective.

Example: *The function $f : \mathbb{Z} \rightarrow \mathbb{Z}$, defined as $f(x) = 2x$ is injective.*

Example: *If the function $g : \mathbb{Z} \rightarrow \mathbb{Z}$ is defined by the formula $g(n) = n^2, \forall n \in \mathbb{Z}$, then g is not one-to-one.*
Counterexample: $g(2) = g(-2) = 4$ but $2 \neq -2$.

Definition 7 *A function f from A to B is called* onto*, or* surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a surjection if it is onto.*

$$f : A \rightarrow B \text{ is onto } \Leftrightarrow \forall y \in B, \exists x \in A(f(x) = y) \quad \blacksquare$$

Definition 8 *A function f is a* one-to-one correspondence*, or a* bijection*, if it is both one-to-one and onto.* $\blacksquare$

Example: *The function $f : R \rightarrow R^+ \cup \{0\}$, defined as $f(x) = x^2$ is surjective.*

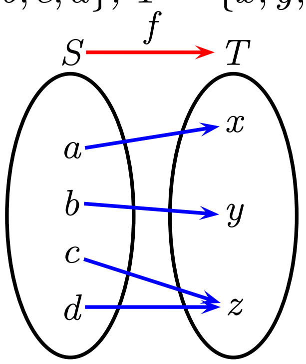
Example: *If $f : \mathbf{R} \rightarrow \mathbf{R}$ is the function defined by the rule $f(x) = 4x - 1, \forall x \in \mathbf{R}$, then f is onto.*

I Putu Danu Raharja

Solution: *Let* $y \in \mathbf{R}$. *We must show that* $\exists x \in \mathbf{R}$ *such that* $f(x) = y$. *Let* $x = \frac{y+1}{4}$. *Then* x *is a real number since sums and quotients (other than 0) of real numbers are real numbers. It follows that*

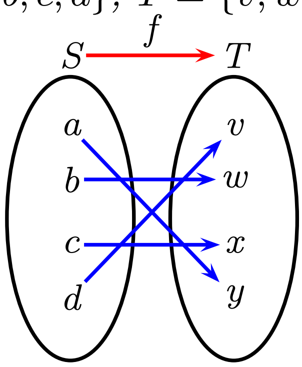
$$\begin{aligned} f(x) &= f\left(\frac{y+1}{4}\right) \\ &= 4.\left(\frac{y+1}{4}\right) - 1 = y \end{aligned}$$

Example: *Let* $S = \{a, b, c, d\}$, $T = \{x, y, z\}$ *and* $f : S \to T$.



An Surjective Function that is not Injective.

Example: *Let* $S = \{a, b, c, d\}$, $T = \{v, w, x, y\}$ *and* $f : S \to T$.



A one-to-one Correspondence (Bijective Function).

Example:

- *The function* $f : \mathbb{Z} \to \mathbb{Z}$, *defined as* $f(x) = x + 3$ *is bijective.*
- $h : \emptyset \to \emptyset$ *is bijective.*
- $d : ICS253 \to \mathbb{N}$ *is not 1-1 because more than one student is assigned to 1. It is not onto because no student is assigned to 3.*
- $d : ICS253 \to \{1, 2\}$ *is not 1-1 but it is onto.*

I Putu Danu Raharja

Identity and Inverse Functions

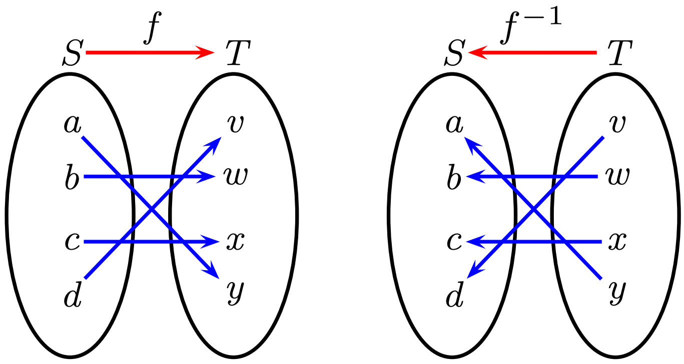
Let A be any set. Define the *identity function on* A ($\iota_A : A \to A$) by, $\iota(a) = a$, i.e.,

$$\iota_A = \{(a, a) | a \in A\}.$$

Observe that ι_A is a 1-1 correspondence.

Definition 9 *Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence $f^{-1}(b) = a$ when $f(a) = b$.* ■

Example: *The following figure show the function $f : S \to T$ and its inverse.*



Example: *Here are some other examples:*

(a) *If $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = x + 3$, then its inverse is $f^{-1}(x) = x - 3$.*

(b) *If $f : \mathbf{R} \to \mathbf{R}^+ \cup 0$ is defined by $f(x) = x^2$, one may think that its inverse is $g(x) = \sqrt{x}$, but that is incorrect.*

Definition 10 *A function f whose domain and codomain are subsets of $\mathbf{R}$ is called* strictly increasing *if $f(x) < f(y)$ whenever $x < y$ and x and y are in the domain of f.*

f is strictly increasing: $\forall x \forall y((x < y) \to (f(x) < f(y))$ ■

Definition 11 *A function f is called* strictly decreasing *if $f(x) > f(y)$ whenever $x < y$ and x and y are in the domain of f.*

f is strictly decreasing: $\forall x \forall y((x < y) \to (f(x) > f(y))$ ■

I Putu Danu Raharja

Function Composition

A function that has an inverse is called *invertible*. The necessary and sufficient condition for a function to be invertible is to be a 1-1 correspondence.

Definition 12 *Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.* ■

Example: *Diagrammatic view of functions f and g and their composition $g \circ f$.*

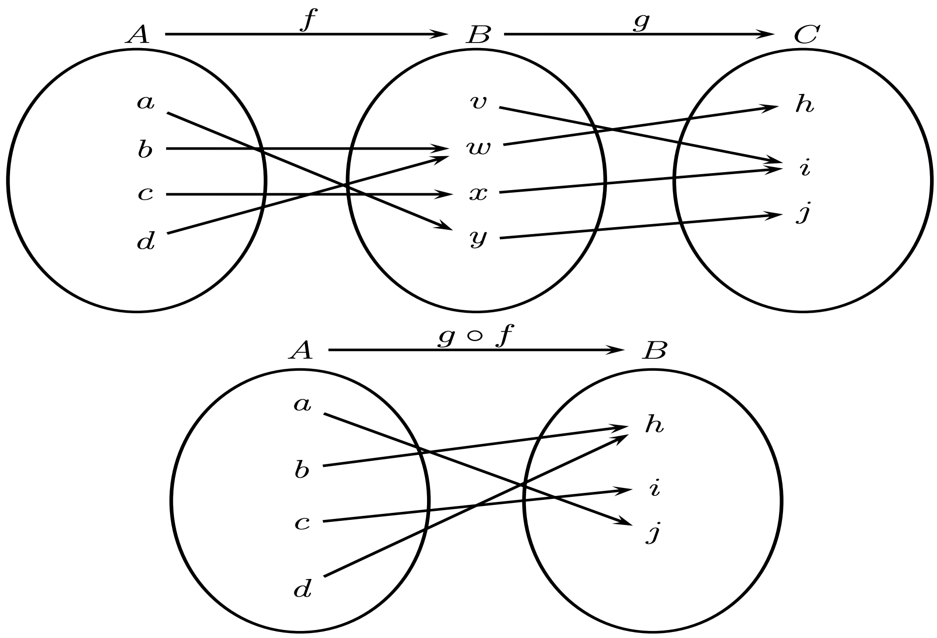


Figure 1: Function Composition.

Example: *As another example, if $A = B = C = Z$, $f(x) = x+1, g(x) = x^2$, then $(g \circ f)(x) = f(x)^2 = (x+1)^2$. Also $(f \circ g)(x) = g(x) + 1 = x^2 + 1$.*

This demonstrates that, in general, function composition is not commutative.

I Putu Danu Raharja

Function Composition (cont.)

Some properties of function composition:

(1.) If $f : A \to B$ is a function from A to B, we have that

$$f \circ \iota_A = \iota_B o f = f.$$

(2.) Given two functions, $f : A \to B$ and $g : B \to C$, we have:

- (a) If f and g are one-to-one, then $g \circ f$ is one-to-one.
- (b) If f and g are onto, then $g \circ f$ is onto.
- (c) If $g \circ f$ is one-to-one then f is one-to-one.
- (d) If $g \circ f$ is onto then g is onto.

(3.) Function composition is associative, i.e., given three functions

$$f : A \to B, g : B \to C, h : C \to D,$$

we have that $h \circ (g \circ f) = (h \circ g) \circ f$.

Definition 13 *Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a, b) | a \in A \wedge f(a) = b\}$.* ■

Definition 14 *The floor function assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. The value of the floor function at x is denoted by $\lceil x \rceil$.* ■

Example:

(a) $\lfloor 2 \rfloor = 2, \lfloor 2.3 \rfloor = 2, \lfloor \pi \rfloor = 3, \rfloor -2.5 \rfloor = -3.$

(b) $\lceil 2 \rceil = 2, \lceil 2.3 \rceil = 3, \lceil \pi \rceil = 4, \lceil -2.5 \rceil = -2.$

See Rosen p. 107 for some useful properties of the Floor and Ceiling functions.

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1 \ , \forall x \in \mathbf{R}$$

I Putu Danu Raharja

Example: *Prove or disprove the assertion*

$$\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor, \; \textit{real } x \geq 0$$

Solution: *Let us try to prove it. Let* $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$. *Then,*

$$\begin{aligned}
& m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor \\
& m \leq \sqrt{\lfloor x \rfloor} < m + 1 \\
& m^2 \leq \lfloor x \rfloor < (m+1)^2 \\
& m^2 \leq x < (m+1)^2 \\
& m \leq \sqrt{x} < m + 1 \\
& m = \lfloor \sqrt{x} \rfloor
\end{aligned}$$

Warmups Exercises

(1) Prove or disprove the assertion

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \;, \forall x \in \mathbf{R} \forall y \in \mathbf{R}$$

(2) Consider $f(n) : \mathbb{N} \to \mathbb{N}$ is defined as:

$$f(1) = 1, f(n+1) = \begin{cases} \frac{1}{2} f(n) & \text{if } f(n) \text{ is even;} \\ 5f(n) + 1 & \text{otherwise.} \end{cases}$$

  (a) Is f a function?

  (b) Is f injective, surjective, bijective?

(3) Prove that

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor,$$

for all integers n and all positive integers m.

I Putu Danu Raharja