Overview

1. Set Operations
2. Set Identities
3. Generalized Unions and Intersections
4. Computer Representation of Sets
5. The cardinality of the union of sets
6. Preview: Functions

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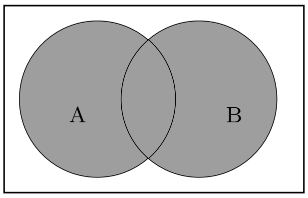
Set Operations

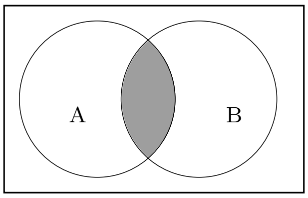
Definition 1 *Let A and B be sets. The* **union** *of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both. That is,*

$$A \cup B = \{x | x \in A \vee x \in B\}.$$ ■



Definition 2 *Let A and B be sets. The* **intersection** *of the sets A and B, denoted by $A \cap B$, is the set that contains those elements in both A and B. That is,*

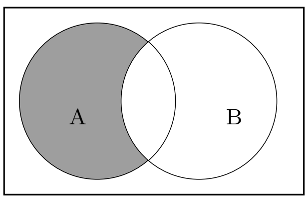
$$A \cap B = \{x | x \in A \wedge x \in B\}.$$ ■



Definition 3 *Two sets are called* **disjoint** *if their intersection is $\emptyset$, the empty set.* ■

Definition 4 *Let A and B be sets. The* **difference** *of the sets A and B, denoted by $A - B$, is the set that contains those elements that are in A but not in B. The difference of A and B is also called the* complement of B with respect to A. *That is,*

$$A - B = \{x | x \in A \wedge x \notin B\} = A \cap \bar{B}.$$ ■



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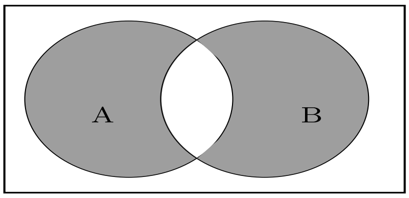
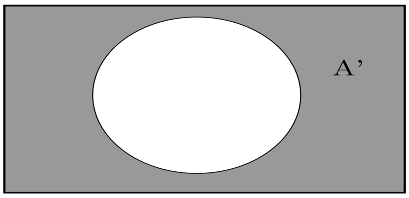
Definition 5 *Let A and B be sets. The* **symmetric difference** *of the sets A and B, denoted by $A \oplus B$, is the set that contains those elements any of the two sets but not in both. That is,*

$$A \oplus B = A \cup B - A \cap B = (A - B) \cup (B - A). \quad \blacksquare$$



Definition 6 *Let U be the universal set. The (absolute)* **complement** *of the set A, denoted by $\bar{A}$, is the complement of A with respect to U. Or, the complement of the set A is $U - A$. That is,*

$$\bar{A} = \{x|\neg(x \in A)\} = \{x \in U|x \notin A\}. \quad \blacksquare$$



Example: *Let*

$$A = \{1, 2, 3, 4, 5\}, B = \{4, 5, 6, 7, 8\}$$

and

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 8, 9, 10\},$$

Then

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. $A - B = \{1, 2, 3\}$.

$A \cap B = \{4, 5\}$. $B - A = \{6, 7, 8\}$.

$\bar{A} = \{0, 6, 7, 8, 9, 10\}$. $A \oplus B = \{1, 2, 3, 6, 7, 8\}$

$\bar{B} = \{0, 1, 2, 3, 9, 10\}$.

Example: *Let $E = \{x \mid (x \notin A \to x \in D) \vee x \in B\}$, where A, B, and D are arbitrary sets. Draw the Venn diagram for E.*

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Set Identities

The following **set identities** correspond to the logical equivalences discussed in Lecture 2.

Identity	**Name**
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotency laws
$A \cup \bar{A} = U$ $A \cap \bar{A} = \emptyset$	Inverse law
$\bar{\bar{A}} = A$	Double complement law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$	DeMorgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption law

Table 1: Algebra of Sets

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Definition 7 *The* **union** *of a collection of sets is the set that contains those elements that are members of at least one set in the collection:* ■

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_1 \cup \ldots \cup A_n = \{x | \exists i (x \in A_i)\}.$$

Definition 8 *The* **intersection** *of a collection of sets is the set that contains those elements that are members of all the sets in the collection:* ■

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_1 \cap \ldots \cap A_n = \{x | \forall i (x \in A_i)\}.$$

These definitions can be applied to infinite collections of sets as well.

Definition 9 *A collection of nonempty sets* $\{A_1, A_2, \ldots, A_n\}$ *is a* ***partition*** *of a set* A *if and only if,*

1 $A = A_1 \cup A_2 \cup \cdots \cup A_n$;

2 $A_1, A_2, \ldots, A_n$ *are mutually disjoint.* ■

Example: *Let* $A = \{1, 2, 3, 4, 5, 6\}$. *Then,* $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, *and* $A_3 = \{5, 6\}$ *is a partition of* A.

Example: *Let* $A = \{1, 2, 3, 4\}$. *How many different partitions of* A *are possibly created?*

Membership Table

Set identities can also be proved using membership tables. We consider each combination of sets that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used.

Example: *Use a membership table to show that*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

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A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Generalized Union and Intersection

Example: *Let*

(i) If $A_i = [i, \infty), 1 \leq i < \infty$, *then:*

$$\bigcup_{i=1}^{n} A_i = [1, \infty).$$

and

$$\bigcup_{i=1}^{n} A_i = [n, \infty).$$

(ii) If $[a, b] = \{x \in R | a \leq x \leq b\}$, *then:*

$$\bigcup_{n=1}^{\infty} [n, n+1] = [1, 2] \cup [2, 3] \cup [3, 4] \cup ... = [1, \infty).$$

Example: *What is*

$$\bigcap_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}]?$$

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Representing Sets On the Computer

- It turns out that sets can be represented using *bit strings*.
- The idea is that if a universal set has n elements, then its subsets etc, are represented as bit strings of length n as well.
- The bit string corresponding to a subset A of U has a 1 in position k if $a_k \in A$, and has a 0 in this position if $a_k \notin A$.
- E.g., the subset

  $$A = \{2, 4, 6, 8, 10\}$$

  of

  $$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

  is represented as a bit string of length 10 as `0101010101` while

  $$\bar{A} = \{1, 3, 5, 7, 9\}$$

  is represented as `1010101010`.
- Furthermore, if

  $$B = \{3, 6, 9\},$$

  represented as `0010010010`, then

  $$A \cap B = \{6\}$$

  is represented as `0000010000` while

  $$B \cup \bar{A} = \{1, 3, 5, 6, 7, 9\}$$

  is represented as `1010111010`.
- To obtain the bit string for the union and intersection of two sets, we perform bitwise Boolean operations on the bit strings representing the two sets.
- Another advantage of this representation for sets is in generating the combinations (subsets) of a set since, for example, the number of subsets of $\{1, 2, ..., n\}$ is equivalent to the number of bit strings of length n (See Rosen, Section 4.6).

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The cardinality of the union of sets

Definition 10 *Let A and B be sets. Then,*

$$|A \cup B| = |A| + |B| - |A \cap B|. \quad \blacksquare$$

Definition 11 *Let A, B and C be sets. Then,*

$$|A \cup B \cup C| = |A| + |B| + |C|$$

$$-|A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \quad \blacksquare$$

Useful exercises

1. Let $S = \{\emptyset, a, \{a\}\}$. Determine whether each of these is an element of S, a subset of S, neither, or both.
   (a) $\{a\}$
   (b) $\{\{a\}\}$
   (c) $\emptyset$
   (d) $\{\{\emptyset\}, a\}$
   (e) $\{\emptyset\}$
   (f) $\{\emptyset, a\}$
2. Prove that $P(A) \cup P(B) \subseteq P(A \cup B)$ is true for all sets A and B. How about its converse?
3. Prove that the following is true for all sets A, B, and C: if $A \cap C \subseteq B \cap C$ and $A \cap \overline{C} \subseteq B \cap \overline{C}$, then $A \subseteq B$.

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