Overview

1. Sets, Equal Sets, Subsets

2. Cardinality of Finite Sets

3. Power Sets

4. Cartesian Products

5. Preview: Set Operations

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Introduction to Sets

Definition 1 *A* **set** *is a well-defined collection of objects in which order and repetition is not relevant. The objects in a set are also called the elements, or members, of the set. A set is said to contain its elements.* ■

- The set is the fundamental discrete structure upon which most other discrete structures like, relations, combinations, graphs, etc. are build.
- Whenever a set is defined there must be an underlying *universal set* U, either specifically stated or understood.

A set can be defined in a variety of ways as exemplified here:

(a) Listing the elements between braces:

(i) $A = \{3, 5\} = \{5, 3\} = \{3, 3, 3, 5, 3\}$.

(ii) $B = \{2, red, a, blue, Ali\}$.

(iii) $C = \{a, b, \{a\}, \{a, b\}\}$.

(b) Using the **set builder** notation, i.e., specification by predicates:

(i) $C = \{x|P(x)\}$.

(ii) $D = \{x|x \text{ } is \text{ } a \text{ } positive \text{ } integer \text{ } < 100\}$.

Note that $\{x|x \text{ } is \text{ } an \text{ } outstanding \text{ } person\}$ may not be a set since 'outstanding' is very subjective.

(c) Using the brace notation with ellipses:

(i) $E = \{..., -3, -2, -1\}$.

(ii) $N = \{0, 1, 2, ...\}$.

(d) Using Venn diagrams.

(e) Describe a set by its characteristic function, defined as

$$\forall x \text{ in } U, \ \mu_A(x) = \begin{cases} 1 & \text{,if } x \in A. \\ 0 & \text{,if } x \notin A. \end{cases}$$

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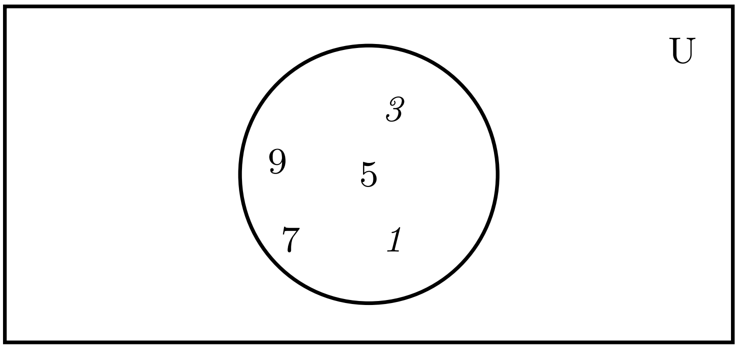
(f) Describe a set by a recursive formula. This is to give one element of the set and a rule by which the rest of the elements of the set may be found.

Example: *The set S of odd positive integers less than 10 can be expressed by* $S = \{1, 3, 5, 7, 9\}$.

- The set S can be written as
  $S = \{\text{x} \mid \text{x is an odd positive integer less than } 10\}$.
- Or can be shown as



- $\mu_A(x) \quad = \begin{cases} 1 \text{ for x } = 1, 3, 5, 7, 9. \\ 0 \text{ otherwise.} \end{cases}$
- $A = \{x_{i+1} = x_i + 2,\ i = 0, 1, \ldots, 4,\ \text{where } x_0 = 1\}$

Notation:
x is a member of S or x is an element of S:

$$x \in S$$

x is not an element of S:

$$x \notin S$$

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Introduction to Sets (cont.)

Some common notations/universal sets are:

(a) $\mathbb{Z}$=the set of integers=$\{0, 1, -1, 2, -2, 3, -3, \ldots\}$.

(b) $\mathbb{N}$=the set of natural numbers $= \{0, 1, 2, 3, \ldots\}$.

(c) $\mathbb{Z}^+$=the set of positive integers.

(d) $\mathbb{Q}$=the set of rational numbers=$\{a/b|a, b \in Z, b \neq 0\}$.

(e) $\mathbb{Q}^+$=the set of positive rational numbers.

(f) $\mathbb{Q}^*$=the set of nonzero rational numbers.

(g) $\mathbb{R}$=the set of real numbers.

(h) $\mathbb{R}^+$=the set of positive real numbers.

(i) $\mathbb{R}^*$=the set of nonzero real numbers.

(j) $\mathbb{C}$=the set of complex numbers.

(k) $\mathbb{C}^*$=the set of nonzero complex numbers.

(l) For any $n \in Z^+$, $Z_n = \{0, 1, 2, 3, ..., n-1\}$.

(m) For real numbers a, b with $a < b$,

  (i) $[a, b] = \{x \in R|a \leq x \leq b\}$. Closed interval.

  (ii) $(a, b) = \{x \in R|a < x < b\}$. Open interval.

  (iii) $[a, b) = \{x \in R|a \leq x < b\}$. Half-open interval.

  (iv) $(a, b] = \{x \in R|a < x \leq b\}$. Half-open interval.

  (v) $(a, \infty) = \{x \in R|a < x\}$.

  (vi) $[a, \infty) = \{x \in R|a \leq x\}$.

  (vii) $(-\infty, b) = \{x \in R|x < b\}$.

  (viii) $(-\infty, b] = \{x \in R|x \leq b\}$.

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Equal Sets, Subsets, The Empty Set

Definition 2 *Two sets are* **equal** *if and only if they have exactly the same elements, i.e.:*

$$A = B \equiv \forall x (x \in A \leftrightarrow x \in B). \qquad \blacksquare$$

Definition 3 *The set A is said to be a subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. Symbolically, we write*

$$A \subseteq B \equiv \forall x (x \in A \to x \in B). \qquad \blacksquare$$

In particular $A \subseteq A$. Note that

$$A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A),$$

so it is possible to prove that two sets A and B are equal by showing that all elements of A are in B, and all elements of B are in A.

- A is a *proper subset* of B, represented $A \subset B$ if $A \subseteq B$ but $A \neq B$, i.e., there is some element in B which is not in A.
- A set with no elements is called the *void set*, or *empty set*, or *null set* and is represented by $\emptyset$ or $\{\}$.
- Note that the assertion $x \in \emptyset$ is **always false**. Hence

  $$\forall x (x \in \emptyset \to x \in B),$$

  is always true (vacuously). Therefore, $\emptyset$ is a subset of every set.

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Power Set, Cardinality of Finite Sets

Note that a set can be both an element as well as a subset of another set. For example, given the set

$$A = \{\emptyset, \{\emptyset\}\},$$

$\emptyset$ is both a member of A and also a subset of A!

Definition 4 *Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.* ∎

In other words, if the cardinality of a set is a natural number, then the set is finite, otherwise the set is infinite.

Example: *If $A = \{a, b, c, d\}$, then*

$$|A| = |\{a, b, c, d\}| = |\{b, c, a, d, d\}| = 4.$$

Example: *If $B = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, then*

$$|B| = 3$$

Note that $|\emptyset| = 0$.

Definition 5 *Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$. In other words,*

$$P(S) = \{A | A \subseteq S\}.$$ ∎

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N-Tuples, Cartesian Products

- For any finite set A with $|A| = n \geq 0$, there are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ subsets of size k.
- Counting the subsets of A according to the number, k, of elements in a subset, we have the combinatorial identity:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n, \ for\ n \geq 0.$$

Example:

(i) $P(\emptyset) = \{\emptyset\}$.

(ii) $P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}$.

(iii) $P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

In general, $|P(S)| = 2^{|S|}$, for an arbitrary set S.

Definition 6 *The* **ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ *is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.* ■

Two tuples are equal if and only if they have the same length and have exactly the same elements in the same order.

$$\begin{aligned} & (a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_m) \\ \Leftrightarrow \quad & n = m \wedge a_1 = b_1 \wedge a_2 = b_2 \wedge \ldots \wedge a_n = b_m. \end{aligned}$$

Definition 7 *Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Hence,*

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}.$$ ■

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N-Tuples, Cartesian Products (cont.)

Example: *Let* $A = \{a, b\}$ *and* $B = \{1, 2, 3\}$*, then*

$$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}.$$

$$B \times A = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}.$$

showing clearly that $A \times B \neq B \times A$ *unless* $A = B$ *or one of the sets is* $\emptyset$*.*

The Cartesian product of any set with $\emptyset$ is $\emptyset$. Why?

Definition 8 *The* **Cartesian product** *of the sets* $A_1, A_2, \ldots, A_n$*, denoted by* $A_1 \times A_2 \times \ldots \times A_n$*, is the set of ordered* n*-tuples* $(a_1, a_2, \ldots,$ $a_n)$ *where* $a_i \in A_i$ *for* $i = 1, 2, \ldots, n$*.* ■

In other words,

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i,\ for\ i = 1, 2, \ldots, n\}.$$

From Example above, we can write

$$\begin{aligned} A \times B \times A = \{ \quad & (a,1,a),(b,1,a),(a,1,b),(b,1,b),\\ & (a,2,a),(b,2,a),(a,2,b),(b,2,b),\\ & (a,3,a),(b,3,a),(a,3,b),(b,3,b)\}. \end{aligned}$$

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