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I Putu Danu Raharja

Proof Strategy

1. Finding proofs can be a challenging business.
2. When you confronted with a statement to prove, you should first replace terms by their definitions and then carefully analyze what the hypotheses and the conclusion mean.
3. After doing so, you can attempt to prove the result using one of the available methods of proof.
4. Whichever method you choose, you need a starting point for your proof.
   (i) Forward reasoning.
   (ii) Backward reasoning.

Forward Reasoning

1. With direct reasoning, you can start with the hypotheses, together with axioms and known theorems, you can construct a proof using a sequence of steps that leads to the conclsion.
2. With indirect reasoning, you can start with the negation of conclusion, and using a sequence of steps, obtain the negation of the hypotheses.

Backward Reasoning

1. To prove a statement q, we find a statement p_1 that we can prove with the property that $p_1 \to q$.
2. In addition, if necessary, we may repeat this process to find $p_2, p_3, \cdots$ that $p_3 \to p_2$, $p_2 \to p_1$.

I Putu Danu Raharja

Example: *Given two positive real numbers a and b, their arithmetic mean is $(a+b)/2$ and their geometric mean is $\sqrt{ab}$. Prove that the arithmetic mean always greater than the geometric mean of pairs of distinct positive real numbers.*

Solution: *To prove it, we can construct a sequence of equivalent inequalities:*

$$\begin{array}{rcl} (a+b)/2 & > & \sqrt{ab} \\ (a+b)^2/4 & > & ab \\ & \vdots & \\ (a-b)^2 & > & 0 \end{array}$$

Now we can construct the proof in a formal way.

Example: *Suppose that two people play a game taking turns removing $1, 2$ or 3 stones at a time from a pile that begins with 15 stones. The person who removes the last stone wins the game. Prove that the first player can win the game no matter what the second player does.*

Solution: *At the last step, the first player can win if this player is left with a pile containing $1, 2$ or 3 stones. The second player will be forced to leave $1, 2$ or 3 stones if this player has to remove from a pile containing 4 stones. Etc.*

Leveraging Proof By Cases

1. When it is not possible to consider all cases of a proof at the same time, a proof by cases should be considered.

Example: *Prove that if n is an integer not divisible by 2 or 3, then $n^2 - 1$ is divisible by 24.*

Solution: *We can view the number n is of the form $6k + j$ for $j = 0, 1, 2, 3, 4, 5$. There are only two cases can be considered.*

I Putu Danu Raharja

Conjectures

1. The present mathematics theorems and their proofs are the ultimate result of a long intellectual process that starts with the exploration of concepts and examples, the formulation of conjectures, and attempt to settle these conjectures either by proof or by counterexample.
2. People formulate conjectures on the basis of many types of possible evidence.
3. At other time, conjecture are made based on intuition or a belief that a result holds.

Example: *Let*

$$T(x) = \begin{cases} x/2 & , x \text{ is even} \\ 3x+1 & , x \text{ is odd} \end{cases}$$

A famous conjecture, sometimes known as the $3x+1$ *conjecture, states that for all positive integers* x*, when we repeatedly apply the transformation* T*, we will eventually reach the integer* 1*.*

Uniqueness Proofs

To prove the existence of a unique element with a particular property, we need to show that:

1. *Existence* : an element x with the desired property exists.
2. *Uniqueness* : if $y \neq x$, then y does not have the desired property.

Example: *Show that if* a *and* b *are real numbers and* $a \neq 0$*, then there is a unique real number* r *such that* $ar + b = 0$*.*

Solution: *First, note that the real number* $r = -b/a$ *is a solution of* $ar + b = 0$*. Consequently, a real number* r *exists for which* $ar + b = 0$*.*
Second, Suppose that s *is a real number such that* $as + b = 0$*. Then* $as + b = ar + b$*, where* $r = -b/a$*. Etc......*

I Putu Danu Raharja

Additional Proof Methods

1. Mathematical Induction.
2. Structural Induction.
3. Cantor diagonalization.
4. Combinatorial proofs.

I Putu Danu Raharja