Overview

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Predicates and Quantifiers

Definition 1 *A* **predicate (an open statement)** $P(x)$ *is a declarative statement over the variable* x *which specifies a property that the subject variable* x *may have.* ■

- Thus a predicate is a propositional function which has a truth value for a specified value of x.
- A propositional function may be defined over several variables like:

$$P(x_1, x_2, ..., x_n).$$

- When all the variables in a propositional function are assigned values, the resulting statement becomes a proposition, i.e., has a truth value.

Example: *Each of the following is an open statement whose truth or falsity depends on the values assigned to its variables.*

(a) $x + 2 = 7$.

(b) "A is a human being"

(c) $x < y$

(d) "p is a prime number"

- The allowable values for the variables of an open statement are called the *universe of discourse* or *domain.*
- For instance, in the open statements:
  (a) $x + 2 = 7$, x may represent an integer and the universe of discourse will be the set of integers.
  (b) "A is a human being" may make sense if the universe of discourse is the set of all living beings, but
  (c) $x + 2 = 7$ would not make sense for a universe of discourse consisting of the set of all living beings.

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Predicates (cont.)

Example: *Consider the predicate*

$$P(x) : x > 0$$

with the set of integers,

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$$

as its universe of discourse. Then $P(x) : x > 0$ *has no truth value until the variable* x *is bound. It becomes propositions when* x *is assigned a value, for instance,* $P(-3)$ *is false,* $P(0)$ *is false,* $P(3)$ *is true.*

Example: *Let* R *be the three-variable predicate*

$$R(x, y, z) : x + y = z$$

Then $R(2, -1, 5)$ *and* $R(3, 4, 7)$*, are propositions with truth values false and true respectively while* $R(x, 3, y)$ *has no truth value since* x *and* z *are not bound.*

- Propositions can also be created from propositional functions using *quantification*. Two popular quantification types are *existential quantification* and *universal quantification*.
- Quantification is normally associated with a particular domain of reference called the *universe of discourse*.
- The universe of discourse specifies the possible values a quantification variable can take.

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Quantifiers (cont.)

Definition 2 *The* **universal quantification** *of $P(x)$ is the proposition "$P(x)$ is true for all values of x in the universe of discourse". It is written as $\forall x P(x)$.* ■

The expression $\forall x P(x)$ is also rendered: "for all x $P(x)$", or "for every x, $P(x)$".
As an illustration, suppose we are talking about a world (universal discourse) consisting of four named blocks, say a, b, c, and d. Then the sentence $\forall x Cube(x)$ will be true if and only if the following conjunction is true:

$$Cube(a) \land Cube(b) \land Cube(c) \land Cube(d)$$

Definition 3 *The* **existential quantification** *of $P(x)$ is the proposition "There exists an element x in the universe of discourse such that $P(x)$ is true". It is written as $\exists x P(x)$.* ■

The expression $\exists x P(x)$ is also rendered: "there exists an x such that $P(x)$", or "There is at least one x such that $P(x)$" or "For some x, $P(x)$" or "I can find an x such that $P(x)$.
Likewise, $\exists x Cube(x)$ will be true if and only if this disjunction is true:

$$Cube(a) \lor Cube(b) \lor Cube(c) \lor Cube(d)$$

This analogy suggests that the quantifiers may interact with negation in a way similar to conjunction and disjunction. Indeed, the sentence

$$\neg \forall x Cube(x)$$

will be true if and only if the following negation is true:

$$\neg(Cube(a) \land Cube(b) \land Cube(c) \land Cube(d))$$

which, by DeMorgan, can be expressed as:

$$\neg Cube(a) \lor \neg Cube(b) \lor \neg Cube(c) \lor \neg Cube(d)$$
$$\exists x \neg Cube(x)$$

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Negation and Nested Quantifiers

- If $\exists x P(x)$ is false then there is no value of x for which $P(x)$ is true, or put another way, $P(x)$ is always false. Hence
$$\neg[\exists x P(x)] \equiv \forall x \neg P(x).$$
- On the other hand, if $\forall x P(x)$ is false then it is not true that for every x, $P(x)$ holds, hence for some x, $P(x)$ must be false. Thus:
$$\neg[\forall x P(x)] \equiv \exists x \neg P(x).$$
- In open statements with more than one variable it is possible to use several quantifiers at the same time. For instance
$$\forall x \forall y \exists z P(x, y, z),$$
meaning "for all x and all y there is some z such that $P(x, y, z)$."
- Consider the sentence
$$\mathcal{F} : \quad (\forall x)(p(x, y) \ \wedge \ (\exists y) q(y, z))$$
The variable x is " *bound* " by the quantifier $(\forall x)$. The variable z is " *free* " in $\mathcal{F}$. The variable y in $p(x, y)$ is free. The variable y in $q(x, y)$ is bound by the quantifier $(\exists x)$. The variable y is both bound and free in $\mathcal{F}$.
- Note that in general the existential and universal quantifiers cannot be permuted. That is, in general
$$\forall x \exists y P(x, y)$$
means something different from
$$\exists y \forall x P(x, y).$$

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Quantifiers (cont.)

- For instance if x represents human being and y represents fruit and

$$L(x, y) = x\text{likes to eat } y,$$

then

(1) $\forall x \forall y L(x, y)$ means "everybody likes every type of fruit". To verify whether this statement is true, we would have to ask every person in the world if she or he likes every type of fruit;if even one person does not like one type of fruit, then the statement would be false.

(2) $\forall x \exists y L(x, y)$ means "everybody likes at least one type of fruit."

(3) $\exists x \forall y L(x, y)$ means "someone likes every type of fruit."

(4) $\exists y \forall x L(x, y)$ means "there is a type of fruit that all people like."

Example: *Let* $U = \mathbf{R}$, *the real numbers,* $P(x, y) : xy = 0$

$$(\forall x \forall y)P(x, y)$$

$$(\forall x \exists y)P(x, y)$$

$$(\exists x \forall y)P(x, y)$$

$$(\exists x \exists y)P(x, y)$$

The only one that is false is the first one.

- An *interpretation* of a quantified statement is the assignment of a meaning to that statement in some universe of discourse.

- For instance, a possible interpretation for

$$\forall x(x + y = 7)$$

consists of taking the set of integers as universe of discourse, and assigning $y = 3$, i.e.,

$$\forall x(x + 3 = 7)$$

as a statement about integers.

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Quantifiers (cont.)

- Note that the interpretation consists of a choice of definite values for the free variables, and a universe of discourse for the bound variables.
- If the interpretation is true then it is called a *model* and if it is false, it is called a *counter-model*.
- For instance, a possible model for

$$\exists x, (x^2 = y)$$

is the set of integers as universe of discourse, and $y = 9$.

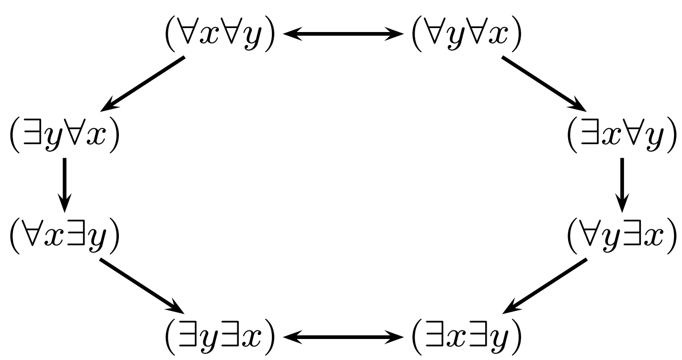
- A counter-model for

$$\exists x, (x^2 = y)$$

is the set of real numbers as universe of discourse and $y = -1$.

The following statements are useful for manipulating logical expressions involving quantifiers.

- Reversal of quantifiers



- Duality of quantifiers

(1.) $\neg(\exists x P(x)) \equiv \forall x \neg P(x)$

(2.) $\neg(\forall x P(x)) \equiv \exists x \neg P(x)$

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Sentences using Quantifiers

When we would translate sentences using quantifier symbols, it is very useful to use **The four Aristotelian forms**:

All P's are Q's.	$\forall x(P(x) \rightarrow Q(x))$
Some P's are Q's.	$\exists x(P(x) \wedge Q(x))$
No P's are Q's.	$\forall x(P(x) \rightarrow \neg Q(x))$
	$\neg\exists(P(x) \wedge Q(x))$
Some P's are not Q's.	$\exists x(P(x) \wedge \neg Q(x))$

Observe the following expressions:
$\forall x(P(x) \wedge Q(x))$, All objects are both P's and Q's.
$\exists x(P(x) \rightarrow Q(x))$, It is true just in case there is an object which is either not a P or else is a Q.

Example: *Let $P(x)$: x is a lion; $Q(x)$: x is fierce; and $R(x)$: x drinks coffee. Then,*

All lions are fierce.	$\forall x(P(x) \rightarrow Q(x))$
Some lions do not drink coffee.	$\exists x(P(x) \wedge \neg R(x))$
Some fierce creatures do not drink coffee	$\exists x(Q(x) \wedge \neg R(x))$

Example: *Let $P(x)$: x is a hummingbird; $Q(x)$: x is large; $R(x)$: x lives on honey; $S(x)$: x is richly colored. Then,*

All hummingbirds are richly colored.	$\forall x(P(x) \rightarrow S(x))$
No large birds live on honey.	$\neg\exists x(Q(x) \wedge R(x))$
Birds that don't live on honey are dull in color	$\forall x(\neg R(x) \rightarrow \neg S(x))$
Hummingbirds are small	$\forall x(P(x) \rightarrow \neg Q(x))$

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Numerical quantification

To say that there are at least two cubes:

$$\exists x \exists y (Cube(x) \wedge Cube(y) \wedge x \neq y)$$

To say that there are at least three cubes:

$$\exists x \exists y \exists z (Cube(x) \wedge Cube(y) \wedge Cube(z) \wedge x \neq y \wedge x \neq z \wedge x \neq z)$$

To say that there are at most two cubes is equivalent to say that there are not at least three cubes.

To express the sentence there are exactly two cubes, we could paraphrase it as follows: *There are at least two cubes and there are at most two cubes.*

$$\exists x \exists y [x \neq y \wedge \forall z (Cube(z) \leftrightarrow (z = x \vee z = y))]$$

Example: *Express the statement "Everyone has exactly one best friend". The domain is all people.*

Solution: *Let $B(x, y)$ be y is the best friend of x. Thus,*

$$\forall x \exists y (B(x, y) \wedge \forall z ((z \neq y) \rightarrow \neg B(x, z)))$$

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