Overview

1. Introduction to Propositional Logic
2. Operations on Propositions
3. Truth Tables
4. Translating Sentences into Logical Expressions
5. Preview: Propositional Equivalence

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An Introduction

1. Logic is the framework upon which rigorous reasoning are built. Without some basic logical concepts, it would not be possible to reason properly, especially in structuring mathematical proofs.
2. The rules of logic are used to specify precise meaning to mathematical statements and to distinguish between logically valid and logically invalid statements. Logic, at its most basic, is concerned with the construction of well-formed statements and valid arguments.
3. Clearly, logic has wide practical applicability in, for example,
   - the design of computing machines
   - artificial intelligence
   - programming languages
   - a court of law
   - etc.
4. There are many different kinds of logic such as propositional, predicate, temporal and fuzzy logics.
5. In propositional logic, we use two assumptions when dealing with statements:
   (a) Every statement is either true or false. (The Law of the excluded middle or ***bivalence***).
   (b) No statement is both true and false.
6. One of the consequences of those assumptions is that if a statement is not false, then it must be true. Hence, to prove that something is true, it would suffice to prove that it is not false.
7. In propositional logic, we deal with *propositions*. A *proposition* is a declarative sentence that is either true or false (but not both). The following are some propositions:

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(a) Amr Ali is a student of KFUPM.

(b) The largest Airport in the Eastern Province is in Dammam.

(c) $\sqrt{9} = 4$.

(d) ICS102 is a pre-requisite of ICS253.

(e) $5 - 2 = 3$.

(f) There are monkeys on Jupiter or there are no monkeys on Jupiter.

8. Note that not all sentences are propositions. Here are some examples:

   (a) You should study hard to get the grade you want.

   (b) What is the instructor's name?

   (c) What a brilliant student!

   (d) $x + 2 \geq 5$

9. The ***truth value*** of a proposition is true, `T`, if it is a true proposition and is false, `F`, if it is a false proposition.

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Propositions and Logical Connectives

Definition 1 (propositions) *The sentences of propositional logic are made up of the following symbols, called* propositions*:*

- *The* truth symbols
  true (T) or false (F)
- *The* propositional symbols
  $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}$, ... ■

Definition 2 (sentences) *The* sentences *of propositional logic are built up from the propositions by application of the* propositional connectives *such as negation* ($\neg$)*, conjunction* ($\wedge$)*, disjunction* ($\vee$)*, exclusive-or* ($\oplus$)*, implication* ($\rightarrow$)*, biconditional* ($\leftrightarrow$)*.* ■

Definition 3 (not rule) *The* **negation** ($\neg\mathcal{F}$) *is true when* $\mathcal{F}$ *is false, and false when* $\mathcal{F}$ *is true.* ■

Definition 4 (and rule) *The* **conjunction** ($\mathcal{F} \wedge \mathcal{G}$) *is true when* $\mathcal{F}$ *and* $\mathcal{G}$ *are both true, and is false otherwise.* ■

Definition 5 (or rule) *The* **inclusive disjunction** ($\mathcal{F} \vee \mathcal{G}$) *is false when* $\mathcal{F}$ *and* $\mathcal{G}$ *are both false, and true otherwise.* ■

Definition 6 *The* **exclusive disjunction** ($\mathcal{F} \oplus \mathcal{G}$) *is true when* $\mathcal{F}$ *is true or when* $\mathcal{G}$ *is true, but not both, and false otherwise.* ■

Definition 7 *The* **implication** ($\mathcal{F} \rightarrow \mathcal{G}$) *is true when* $\mathcal{F}$ *is false or when* $\mathcal{G}$ *is true, and false otherwise.* ■

if $\mathcal{F}$, then $\mathcal{G}$	$\mathcal{F}$ is sufficient for $\mathcal{G}$
if $\mathcal{F}$, $\mathcal{G}$	$\mathcal{G}$ if $\mathcal{F}$
$\mathcal{F}$ implies $\mathcal{G}$	$\mathcal{G}$ whenever $\mathcal{F}$
$\mathcal{F}$ only if $\mathcal{G}$	$\mathcal{G}$ is necessary for $\mathcal{F}$

In other words, to say "$\mathcal{F}$ is a sufficient condition for $\mathcal{G}$" means that the occurrence of $\mathcal{F}$ is *sufficient* to guarantee the occurrence of $\mathcal{G}$. On the other hand, to say "$\mathcal{F}$ is a necessary condition for $\mathcal{G}$" means that if $\mathcal{F}$ does not occur, then $\mathcal{G}$ cannot occur either.

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Definition 8 *The* **biconditional** *$(\mathcal{F} \leftrightarrow \mathcal{G})$ is true when $\mathcal{F}$ and $\mathcal{G}$ have the same truth values, and is false otherwise.* ∎

"$\mathcal{F}$ if and only if $\mathcal{G}$", or "$\mathcal{F}$ is necessary and sufficient for $\mathcal{G}$", or "if $\mathcal{F}$ then $\mathcal{G}$, and conversely."

Contrapositive, converse, inverse

Definition 9 *Suppose a conditional statement of the form $p \rightarrow q$ is given.*

- *The* ***contrapositive*** *is $\neg q \rightarrow \neg p$*
- *The* ***converse*** *is $q \rightarrow p$*
- *The* ***inverse*** *is $\neg p \rightarrow \neg q$* ∎

Example: *Let P be the statement "I am a man." Let Q be the statement "I am not a mother.". Then,*

Name	English Translation
implication	If I am a man, then I am not a mother.
contrapositive	If I am a mother,then I am not a man.
converse	If I am not a mother, then I am a man.
inverse	If I am not a man, then I am a mother.

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Truth Tables for Propositions

A `truth table` displays the relationships between the truth values of propositions.

P	Q	$P \wedge Q$	$P \vee Q$	$P \oplus Q$	$P \rightarrow Q$	$P \leftrightarrow Q$	$\neg(P \leftrightarrow Q)$
T	T	T	T	F	T	T	F
T	F	F	T	T	F	F	T
F	T	F	T	T	T	F	T
F	F	F	F	F	T	T	F

Table 1: Truth Table Showing the Basic Logical Connectives.

Example: *The truth table of the statement* $P \vee (Q \rightarrow \neg R)$ *is*

P	$\vee$	$(Q$	$\rightarrow$	$\neg R)$
T	T	T	F	F
T	T	T	T	T
T	T	F	T	F
T	T	F	T	T
F	F	T	F	F
F	T	T	T	T
F	T	F	T	F
F	T	F	T	T
1	*5*	*2*	*4*	*3*

In the truth table shown above, we first write columns 1, 2, and 3; we then write column 4 which is formed from columns 2 and 3, and column 5 is formed from columns 1 and 4. The final result (in column 5) refers to the compound statement in which we are interested.

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1. Note that *bits* (binary digits) consisting of 0 (zero) and 1 (one) are sometimes used to represent truth values. When bits are used, the bit 1 is customarily used to represent true while the bit 0 is used to represent false.
2. Some programming languages such as C and Java allow operations on bit strings which correspond to logical operations using connectives.
3. The hierarchy for evaluation of connectives (from "do first" to "do last") is: $\neg$ takes precedence over the other operations, followed by $\wedge, \vee, \oplus, \rightarrow, \leftrightarrow$.
4. A string of the same operations is evaluated from left to right. For instance, $\neg P \rightarrow (P \wedge Q \wedge S \vee R) \leftrightarrow Q$ means $((\neg P) \rightarrow (((P \wedge Q) \wedge S) \vee R)) \leftrightarrow Q$.

Example: *Evaluate each of the following expressions.*

(a) $11000 \wedge (01011 \vee 11011)$

(b) $(01010 \oplus 11011) \oplus 01000$

Solution:

Example: *It is known that Hassan was a good friend of Ali and Jafar. The following facts are true about them:*

(i) Either Ali or Jafar is the oldest of the three.

(ii) Either Hassan is the oldest or Ali is the youngest.

Who is the oldest and who is the youngest?

Solution: *There are nine possible statements:* $H1$, $H2$, $H3$, $A1$, $A2$, $A3$, $J1$, $J2$, $J3$. *Also, the proposition* $(A1 \vee J1) \wedge (H1 \vee A3)$ *must be true.*

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From Logical Expressions to English Sentences

Example: *Let $\mathcal{F}$ and $\mathcal{G}$ be the propositions:*

$\mathcal{F}$: *I bought a new car last year.*

$\mathcal{G}$: *I won the neatest car award on Friday.*

Express each of the following as an English sentence.

(a) $\neg\mathcal{F}$ (b) $\mathcal{F} \vee \mathcal{G}$ (c) $\mathcal{F} \rightarrow \mathcal{G}$

(d) $\mathcal{F} \wedge \mathcal{G}$ (e) $\mathcal{F} \leftrightarrow \mathcal{G}$ (f) $\neg\mathcal{F} \rightarrow \neg\mathcal{G}$

(g) $\neg\mathcal{F} \wedge \neg\mathcal{G}$ (h) $\neg\mathcal{F} \vee (\mathcal{F} \wedge \mathcal{G})$

Solution:

(a) I didn't buy a new car last year.

(b) Either I bought a new car last year or (in the inclusive sense) I won the neatest car award on Friday.

(c) If I bought a new car last year, then I won the neatest car award on Friday.

(d) I bought a new car last year and I won the neatest car award on Friday.

(e) I bought a new car last year if and only if I won the neatest car award on Friday.

(f) If I did not buy a new car last year, then I did not win the neatest car award on Friday.

(g) I didn't buy a new car last year and I didn't win the neatest car award on Friday.

(h) Either I did not buy a new car last year, or else I did buy one and won the neatest car award on Friday.

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Logical Expressions to English Sentences (cont.)

Example: *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and $\mathcal{I}$ be the propositions:*

$\mathcal{F}$: *You have the flu.*

$\mathcal{G}$: *You miss the final examination.*

$\mathcal{H}$: *You pass the course.*

Express each of the following as an English sentence.

(a) $\mathcal{F} \rightarrow \mathcal{G}$ (b) $\neg\mathcal{G} \leftrightarrow \mathcal{H}$ (c) $\mathcal{F} \vee \mathcal{G} \vee \mathcal{H}$

(d) $(\mathcal{F} \rightarrow \neg\mathcal{G}) \vee (\mathcal{G} \rightarrow \neg\mathcal{H})$ (e) $(\mathcal{F} \wedge \mathcal{G}) \vee (\neg\mathcal{G} \wedge \mathcal{H})$

Solution:

(a) If you have the flu, then you miss the final exam.

(b) You do not miss the final exam if and only if you pass the course.

(c) You have the flu, or you miss the final exam, or you pass the course.

(d) It is either the case that if you have the flu then you do not pass the course or the case that if you miss the final exam then you do not pass the course (or both, it is understood).

(e) Either you have the flu and you miss the final exam, or you do no miss the final exam and do pass the course.

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From English Sentences to Logical Expressions

Example: *Let $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ be the propositions:*

$\mathcal{F}$: *You get an A on the final exam.*

$\mathcal{G}$: *You do every exercise in this book.*

$\mathcal{H}$: *You get an A in this class.*

Write the following propositions using $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ and logical connectives.

(a) You get an A in this class, but you do not do every exercise in this book.

(b) You get an A on the final, you do every exercise in this book, and you get an A in this class.

(c) To get an A in this class, it is necessary for you to get an A on the final.

(d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

(e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

Solution:

(a) $\mathcal{F} \wedge \neg\mathcal{G}$ *(b)* $\mathcal{F} \wedge \mathcal{G} \wedge \mathcal{H}$ *(c)* $\mathcal{H} \rightarrow \mathcal{F}$

(d) $\mathcal{F} \wedge \neg\mathcal{G} \wedge \mathcal{H}$ *(e)* $(\mathcal{F} \wedge \mathcal{G}) \rightarrow \mathcal{H}$

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