Modelling by a Rational Spline with Interval Shape Control

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Abstract

Various models have been developed, for the designing of distinct objects, for applications like font designing, Computer Aided Design (CAD), Computer Aided Engineering (CAE}, etc. Some methods are better suited for controlling the shape of the curve on an interval, while others are better suitedfor controlling the shape at the individual control points. This work reviews C^2 rational splines with interval tension [3,5] and extends this work for the modelling of interpolatory curves and surfaces through B-spline formulation.

1 Introduction

Modelling, for interactively designing 2D or 3D objects using splines, is a significant area of Computer Graphics. Splinning gives rise to a lot of features including reasonable smoothness, economical computation, ideal storage facility, extra degrees of freedom in the form of shape parameters, useful geometric transformations like translation, scaling, rotation etc. Each of the above features justify the use of splines in computer graphics in their own right.

A rational cubic spline with interval tension was described and analysed in [1]. It provides a C^2 computationally simpler alternative to the exponential spline under tension [2] and an alternative to C' and GC^2 spline methods like the weighted v-spline [3] and y-spline [4]. The rational cubic spline maintains the C^2 parametric continuity of the curve, rather than the more general geometric GC^2 arc length continuity achieved by the v-spline and β-spline. Regarding shape characteristics, it has a shape control parameter associated with each interval which can be used to flatten or tighten the curve both locally and globally. Since the spline is defined on a non-uniform knot partition, the partition itself provides additional degrees of freedom on the curve. However the parameterization is normally expected to be defined on a uniform knot partition, or by cumulative chord length, or by some other appropriate means.

This paper presents a description and analysis of a rational cubic interpolatory spline which has a shape parameter associated with each interval. The spline can be used in CAGD, to represent the parametric curves and surfaces in interpolatory form. The rational spline is based on a rational cubic Hermite interpolant. Section(4) describes the freeform rational spline and analyses its behavior with respect to shape parameter in each interval. Section(S) describes the interpolatory rational spline, with examples which illustrate the interval tension property of the rational spline. Section(6) and section(7) describe freeform and interpolatory surfaces respectively.

2 C^1 Rational Cubic Hermite Interpolant

A piecewise rational cubic Hermite parametric function $p \in C^1[t_0,\ t,]$, with parameters $r_i, i = 0,\ldots, n-1$ and data points $F_i \in R^N, N \geq 2$, is defined for $t \in [t_i, t_{i+1}]$, $i = 0, \ldots, n-1$, by

$p(t) = p_i(t;r_i) =$

$$\left(\frac{(1-\theta)^3 F_i + \theta(1-\theta)^2 (r_i F_i + h_i D_i) + \theta^2 (1-\theta)(r_i F_{i+1} - h_i D_{i+1}) + \theta^3 F_{i+1}}{1 + (r_i - 3)\theta(1-\theta)}\right) \quad (1)$$

Where $\theta = (F_{i+1} - F_i)/h_i$, and $h_i = t_{i+1} - t_i$. The D_i's here denote the first derivatives at t_i's. The $r_i \geq 0$, will be used as tension parameters to control the shape of the curve. The case $r_i = 3$, $i = 0,\ldots, n-1$, is that of cubic Hermite interpolation and the restriction $r_i > -1$ ensures a positive denominator in equation(1).

The function p(t) has the Hermite interpolation properties :

$$p(t_i) = F_i \text{ and } p^{(1)}(t_i) = D_i, i = 0, \ldots, n.$$

For $r_i \neq 0$, equation(1) can be written in the form:

$$(p_i(t;r_i) = R_0(\theta;r_i)F_i + R_1(\theta;r_i)V_i + R_2(\theta;r_i)W_i + R_3(\theta;r_i)F_{i+1}), \quad (2)$$

where

$$V_i = F_i + h_i D_i/r_i, W_i = F_{i+1} - h_i D_{i+1}/r_i,$$

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and $R_j(6'; r_i)$, $j = 0, 1, 2, 3$, are appropriately defined rational functions with

$$\sum_{j=0}^{3} R_j(\theta; r_i) = 1. \quad (3)$$

Moreover, these functions are rational Bernstein-Bezier weight functions which are non-negative for $r_i > 0$. Thus in R^N, $N > 1$ and for $r_i > 0$ the ***convex hullproperty*** holds ***i.e*** the curve segment P_i lies in the convex hull of the control points $\{F_i, V_i, W_i, F_{i+1}\}$. Moreover, the ***variation diminishing property*** also holds of the rational cubic ***i.e*** the curve segment p_i crosses any (hyper) plane of dimensions N-l no more times than it crosses the control polygon joining F_i, V_i, W_i, F_{i+1}.
The rational cubic of equation(1) can be expressed as:

$$p_i(t; r_i) = l_i(t) + e_i(t; r_i), \text{ where}$$
$$l_i(t) = (1-\theta)F_i + \theta F_{i+1}$$
$$e_i(t; r_i) = \frac{h_i\theta(1-\theta)[(\Delta_i - D_i)(\theta-1) + (\Delta_i - D_{i+1})\theta]}{1+(r_i-3)\theta(1-\theta)}$$

This immediately leads to ***Interval tension property*** i.e. for given fixed (or bounded) D_i, D_{i+1}, the rational cubic Hermite interpolant of equation(1) converges uniformly to the linear interpolant on $[t_i, t_{i+1}]$ as $r_i \to \infty$.
In the following section, a C^2 rational spline interpolant is constructed. This requires knowledge of the 2^{nd} derivative of equation(1) which, after some simplification, is given by

$$(p_i^{(2)}(t; r_i) = \frac{2\{\alpha_i\theta^3 + \beta_i\theta^2(1-\theta) + \gamma_i\theta(1-\theta)^2 + \delta_i(1-\theta)^3\}}{h_i\{1+(r_i-3)\theta(1-\theta)\}^3}) \quad (4)$$

where

$$\alpha_i = r_i(D_{i+1} - \Delta_i) - D_{i+1} + D_i$$
$$\beta_i = 3(D_{i+1} - \Delta_i)$$
$$\gamma_i = 3(\Delta_i - D_i)$$
$$\delta_i = r_i(\Delta_i - D_i) - D_{i+1} + D_i$$

3 C^2 Rational Cubic Spline Interpolant

We now follow the familiar procedure of allowing the derivative parameters $D_i, i = 0, \ldots, n$ to be degrees of freedom which are constrained by the imposition of the C^2 continuity conditions

$$p^{(2)}(t_{i+}) = p^{(2)}(t_{i-}), i = 1, \ldots, n-1. \quad (5)$$

These C^2 conditions give, from equation(4), the linear system of consistency equations,

$$(h_i D_{i-1} + \{h_i(r_{i-1}-1) + h_{i-1}(r_i - 1)\}D_i + h_{i-1}D_{i+1}) \quad (6)$$
$$= (h_i r_{i-1}\Delta_{i-1} + h_{i-1} r_i \Delta_i, i = 1, \ldots, n-1)$$

For simplicity, assume that D_0 and D_n ***are*** given as end conditions (clearly other end conditions are also appropriate). Assume also that

$$r_i \geq r > 2, \ i = 0, \ldots, n-1$$

Then the equation(6) defines a diagonally dominant, tri-diagonal linear system in the unknowns $D_i, i = 1, \ldots, n-1$. Hence there exists a unique solution which can be easily calculated by use of the tri-diagonal LU decomposition algorithm. Thus a rational cubic spline interpolant can be constructed with tension parameters $r_i, i = 0, \ldots, n-1$, where the special case $r_i = 3, i = 0, \ldots, n-1$, is that of cubic spline interpolation.

4 Rational Cubic Spline with Interval Tension

This section reviews the rational spline with tension (B-spline representation) method [1]. For the purpose of the analysis, let additional knots be introduced outside the interval $[t_0, t_n]$, defined by $t_{-3} < t_{-2} < t_{-1} < t_0$ and $t_n < t_{n+1} < t_{n+2} < t_{n+3}$. Let

$$r_i \geq r > 2, i = -3, \ldots, n+2 \quad (7)$$

be shape parameters defined on this extended partition. Rational cubic spline functions $\psi_j, j = -1, \ldots, n+3$, have been constructed, see Figure l(a), such that

$$\psi_j(t) = 0, \text{ for } t < t_{j-2} \quad (8)$$
$$\psi_j(t) = 1, \text{ for } t \geq t_{j+1}$$

The local support rational cubic B-spline basis, see Figure 1 (c) is now defined by the difference functions:
$B_j(t) = \psi_j(t) - \psi_{j+1}(t)$, $j = -1, \ldots, n+1$ Let $R_k(\theta; r_i)$, $k = 0, 1, 2, 3$ be defined as :

$$R_0(\theta; r_i) = (1-\theta)^3 / Q_0(\theta; r_i),$$
$$R_1(\theta; r_i) = r_i\theta(1-\theta)^2 / Q_0(\theta; r_i),$$
$$R_2(\theta; r_i) = r_i\theta^2(1-\theta) / Q_0(\theta; r_i),$$
$$R_3(\theta; r_i) = \theta^3 / Q_0(\theta; r_i)$$

where $Q_0(\theta; r_i) = 1 + (r_i - 3)\theta(1 - 6')$.
A local support rational cubic B-spline basis $B_j(t), j = -1, \ldots, n+1$ was constructed and an explicit representation was given as:

$$B_j(t) = (R_0(\theta; r_i)B_j(t_i) + R_1(\theta; r_i)(B_j(t_i) + h_i B_j^{(1)}(t_i)/r_i)) + \quad (9)$$
$$(R_2(\theta; r_i)(B_j(t_{i+1}) - h_i B_j^{(1)}(t_{i+1})/r_i) + R_3(\theta; r_i)B_j(t_{i+1})),$$

where

$$B_j(t_i) = B_j^{(1)}(t_i) = 0, \text{ for } i \neq j-1, j, j+1 \quad (10)$$

and

$$B_j(t_{j-1}) = \mu_{j-1}, \quad B_j^{(1)}(t_{j-1}) = \widehat{\mu}_{j-1}, \quad (11)$$
$$B_j(t_j) = 1 - \lambda_j - \mu_j, \quad B_j^{(1)}(t_j) = \widehat{\lambda}_j - \widehat{\mu}_j,$$
$$B_j(t_{j+1}) = \lambda_{j+1}, \quad B_j^{(1)}(t_{j+1}) = -\widehat{\lambda}_{j+1}$$

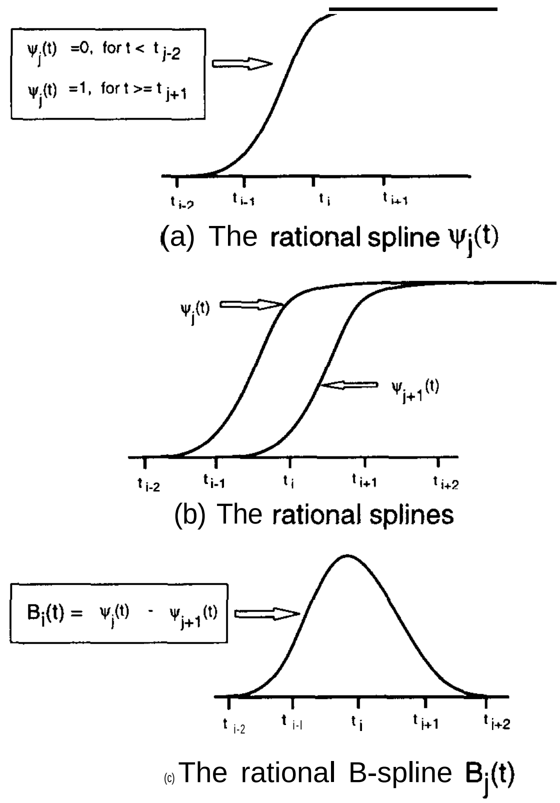


Figure 1. The rational spline forms

with

$$(\lambda_j = h_j \widehat{\lambda_j}/r_j, \quad \mu_j = h_{j-1}\widehat{\mu_j}/r_{j-1}),$$
$$(\lambda_j = h_j d_{j-1}/c_j, \quad \widehat{\mu}_j = h_{j-1} d_{j+1}/c_{j+1}),$$
$$(c_j = h_{j-2} d_j(\frac{h_{j-2}}{r_{j-2}} + \frac{h_{j-1}}{r_{j-1}}) + h_j d_{j-1}(\frac{h_{j-1}}{r_{j-1}} + \frac{h_j}{r_j}) + \frac{h_{j-1} d_{j-1} d_j}{r_{j-1}}),$$
$$d_j = h_j(r_{j-1} - 2) + h_{j-1}(r_j - 2)$$

These rational spline functions, see Figure 1, are such that

1. (local support) $B_j(t) = 0$, fort $\in (t_{j-2}, t_{j+2})$,
2. (Partition of unity) $\sum_{j=-1}^{n+1} B_j(t) = 1$, for $t \in$ [to, t_n]
3. (Positivity) $B_j(t) \geq 0$, for all t,

and hence enjoy all the B-spline properties
The design curve is given by:

$$p(t) = \sum_{j=-1}^{n+1} P_j B_j(t), \; t \in [t_0, t_n] \tag{12}$$

where $Pj \in R^N$ define the control points, was transformed to the piecewise defined rational Bernstein-Bezier representation, see Figure 2, as

$$(p(t) = R_0(\theta; r_i)F_i + R_1(\theta; r_i)V_i + R_2(\theta; r_i)W_i + R_3(\theta; r_i)F_{i+1}), \tag{13}$$

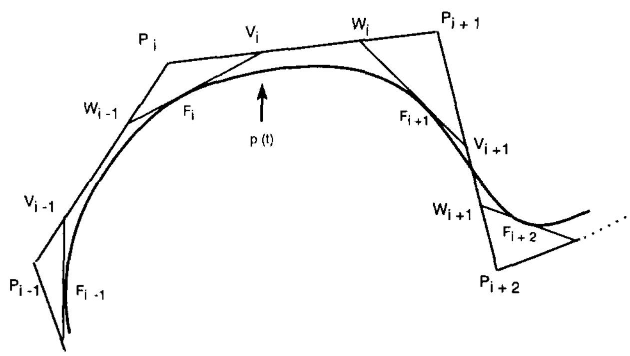


Figure 2. Rational Bernstein-Bezier representation

where

$$\begin{aligned} P_{i-1}\lambda_i + P_i(1 - \lambda_i - \mu_i) + P_{i+1}\mu_i = F_i \\ (1 - \alpha_i)P_i + \alpha_i P_{i+1} = V_i \\ \beta_i P_i + (1 - \beta_i)P_{i+1} = W_i \end{aligned} \tag{14}$$

with

$$\begin{aligned} \alpha_i = \mu_i + h_i\widehat{\mu_i}/r_i \\ \beta_i = \lambda_{i+1} + h_i\widehat{\lambda}_{i+1}/r_i \end{aligned} \tag{15}$$

Let

$$X_i = [F_i \; V_i \; W_i \; F_{i+1}]^T, Z_i = [P_{i-1} \;\; Pi \;\; P_{i+1} \; P_{i+2}]^T$$

and

$$Y_i = \begin{bmatrix} \lambda_i & 1 - \lambda_i - \mu_i & \mu_i & \\ & 1 - \alpha_i & \alpha_i & \\ & \beta_i & 1 - \beta_i & \\ & \lambda_{i+1} & 1 - \lambda_{i+1} - \mu_{i+1} & \mu_{i+1} \end{bmatrix} \tag{16}$$

Then the transformation of equation(14) can also be represented in matrix notation as

$$X_i = Y_i . Z_i \tag{17}$$

The transformation to rational Bernstein-Bezier form is very convenient for computational purposes and also leads to:

1. **Variation diminishing property:** The rational B-spline curve $p(t), t \in [t_0, t_n]$, defined by equation(12), crosses any (hyper) plane of dimension N-l no more times than it crosses the control polygon P joining the control points $\{P_j\}_{=-1}^{n+1}$, see Figure 3.

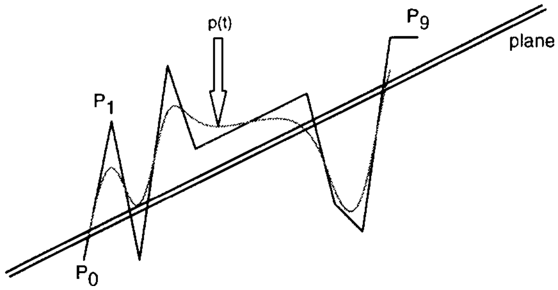


Figure 3. Variation diminishing property

2. **Global tension property:** Let $r_i \geq r > 2$, $i =$ -2,. ., $n + 1$, and let P denote the rational B-spline control polygon, defined explicitly on $[t_i, t_{i+1}], i = -1, \ldots, n$ by

$$P(t) = (1 - \theta)P_i + \theta P_{i+1}, \qquad (18)$$

then the rational B-sphne representation (12) converges uniformly on $[t_{-1}, t_{n+1}]$ as $r \rightarrow \infty$, see Figure 4(a)

3. **Interval tension property:** Consider an interval $[t_k, t_{k+1}]$ for a fixed $k \in \{0, \ldots, n-1\}$ and let

$$Q_k = (1 - \mu)P_k + \mu P_{k+1}.$$
$$Q_{k+1} = \lambda P_k + (1 - \lambda)P_{k+1}$$

denote two distinct points on the line segment of the control polygon joining P_k, P_{k+1}, where

$$\lambda = \frac{h_{k+1}/r_{k+1}}{(h_{k-1}/r_{k-1} + h_{k+1}/r_{k+1} + h_k)}$$
$$\mu = \frac{h_{k-1}/r_{k-1}}{(h_{k-1}/r_{k-1} + h_{k+1}/r_{k+1} + h_k)}$$

Then the rational B-spline representation of equation(12) converges uniformly to Q, see Figure 4(b) on $[t_k, t_{k+1}]$ as $r_k \rightarrow$ co, where

$$Q(t) = (1 - \theta)Q_k + \theta Q_{k+1} \qquad (19)$$

Figure 4(b) illustrates the interval tension behavior of the curves. As the value of r_k and r_{k+1} increases the resulting curve segment approaches the line segment P_k, P_{k+1}.

For the proof of the above properties the reader is referred to [5].

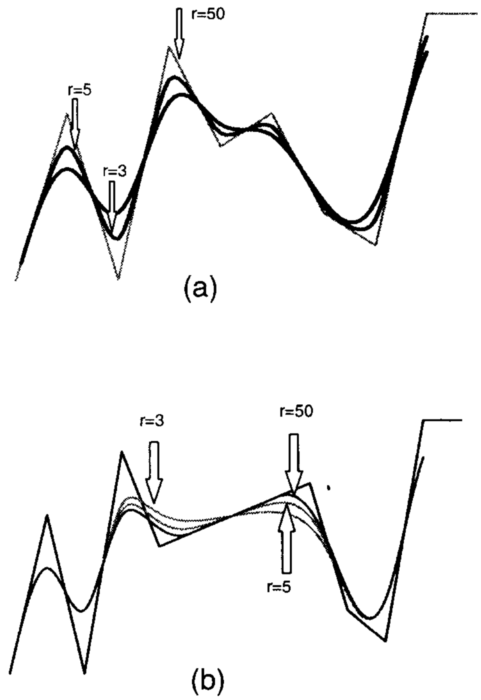


Figure 4. Global and local tension properties

5 Interpolatory Rational Spline with Interval Tension

In interpolatory case we are given a set of data points $F_0, F_1, \ldots, F_n$. We require a cubic B-spline curve p determined by unknown control vertices $P_{-1}, P_0, \ldots, P_{n+1}$, such that $p(t_i) = F_i$, in other words, p interpolates to the data points. The process of obtaining the interpolatory rational cubic B-spline with interval shape control is accomplished through

$$\sum_{j=-1}^{n+1} P_j B_j(t) = F_i, \quad \forall i \qquad (20)$$

where the matrix $B_j(t)$ is the tridiagonal matrix. From equation(14) F_i's, i = 0, $\ldots, n$ can be written as

$$F_i = P_{i-1}\lambda_i + P_i(1 - \lambda_i - \mu_i) + P_{i+1}\mu_i \qquad (21)$$

$$T.P = F \qquad (22)$$

The above set of equations for $F_i, i = 0, \ldots, n$, *i.e* the given set of data points, through which the resulting curve must pass, and the control points P's can be written as in equation(22). As such the above system is underdefined and for a unique solution we need to specify two further conditions, one at the start and one at the end of the curve. We shall repeat the two end control points, although it is not the only end condition available. The above system of equations is tridiagonal- only the diagonal elements and the two neighbors are nonzero. By exploiting the structure of the tridiagonal matrix we can solve the resulting system of equations more efficiently and bypass the standard elimination techniques.
From equation(22) we get

$$P = T^{-1}.F \qquad (23)$$

That is the control vertices of the curve which passes through the given data points F_i's are given by the P_i's as in equation(23). When these values of P_i's are substituted in equation(12), we get the required C^2 interpolatory B-spline curve with interval tension.

Examples The shape behavior of the interpolatory rational splines with interval tension are illustrated by the following examples for the data set in R^2. The global tension behavior is shown in Figure 5, where all shape parameters are progressively increased with values 3, 7, and 50. The effect of the high interval tension is clearly seen in that the resulting spline curve in Figure 6(c) approaches the control polygon. Figure 6(b) and Figure 6(c) display the interval tension behavior applied to the curve of Figure 6(a).

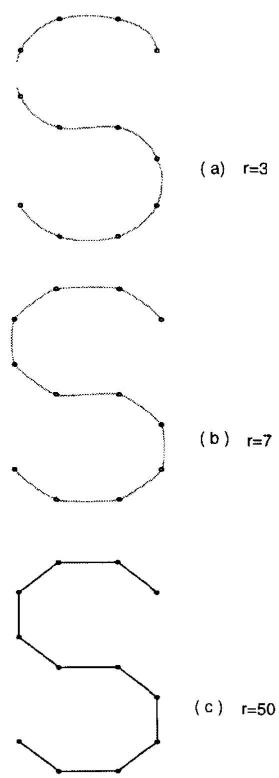


Figure 5. Interpolatoty curves with global tension

6 Freeform Rational B-Spline Surfaces

In **this section we generalize the idea of** section(4) **to surfaces.**

6.1 Rational B-Splines and the Design Surface

Suppose that we are given points

$$P_{i,j} \in R^3, i = -1, \ldots, m+1,\ j = -1, \ldots, n+1 \quad \textbf{(24)}$$

and knot sequences

$$\widetilde{t}_o < \widetilde{t}_1 < \ldots < \widetilde{t}_m$$
$$t_o < t_1 < . < t_n$$

with appropriate additional knots

$$\widetilde{t}_{-3} < \widetilde{t}_{-2} < \widetilde{t}_{-1} < \widetilde{t}_0; \quad \widetilde{t}_m < \widetilde{t}_{m+1} < \widetilde{t}_{m+2} < \widetilde{t}_{m+3},$$
$$t_{-3} < t_{-2} < t_{-1} < t_0; \quad t_n < t_{n+1} < t_{n+2} < t_{n+3}$$

We need to find a parametric rational B-spline surface $p(\widetilde{t}, t)$ in such a way that $p(\widetilde{t}, t_j)$ and $p(\widetilde{t}_i, t)$ are freeform rational cubic splines with tension in $\widetilde{t}-$ and t-directions for all i and j respectively.
Suppose we are given tension parameters

$$(\widetilde{r}_{i,j} > 2 \text{ and } r_{i,j} > 2, i = -3, \ldots, m+2, j = -3, \ldots, n+2), \qquad (25)$$

with $\widetilde{B}_k(\widetilde{t}, t)$ and $B_l(t, \widetilde{t})$ the corresponding rational B-spline basis functions, as in section(4) but with variable cubic B-spline tensions $\widetilde{r}_i(t)$ and $r_j(\widetilde{t})$ defined as:

$$\widetilde{r}_i(t) = \sum \widetilde{r}_{i,j} N_j(t), \qquad (26)$$

$$r_j(\widetilde{t}) = \sum r_{i,j} N_i(\widetilde{t})$$

where $N_j(t)$'s are cubic B-splines. These can be computed as a special case of the rational cubic splines of section(4).
Remark The convex hull property of N_j and equation(25) show that $\widetilde{r}_i(t), r_j(t) >$ **2, Vi,** j, t, and $\widetilde{t}$. Also, for any j

$$\widetilde{r}_i(t) = 0,\ t \notin (t_{j-2}, t_{j+2})$$

and for any i

$$r_j(\widetilde{t}) = 0,\ \widetilde{t} \notin (t_{i-2}, t_{i+2})$$

Furthermore, a large value of the shape parameter $\widetilde{r}_{i,j}$ for any j results in large values of $\widetilde{r}_i(t)$ in the intervals $[t_{j-1}, t_j]$, $[t_j, t_{j+1}]$ and $[t_j, t_{j+1}]$. A similar characteristic is possessed by $r_j(\widetilde{t})$. Thus a sufficiently large value of any of the shape parameters in equation(25) (for $i = 0, \ldots,$ m -1, $j = 0,\ ,\ n-$ 1) results in a sufficiently large value of the

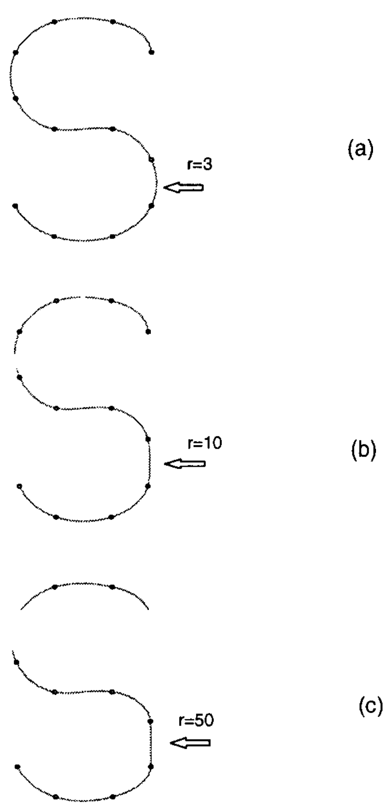


Figure 6. Interpolatoty curves with local tension

variable weight in the corresponding interval. In particular, if any of $\widetilde{r}_{i,j}$ and $r_{i,j}, i = 0,\dots,m-1, j = 0,\dots,n-1$ tend to infinity, it causes the corresponding values from amongst $\widetilde{r}_i(t_j)$ and $r_j(\widetilde{t}_i), i = 0,\dots,m-1, j = 0,\dots,n-1$ respectively to approach infinity. Hence the shape parameters in equation(25) are chosen in such a way that one shape parameter is associated with each interval.
The surface by local support property is defined as:

$$p(\widetilde{t},t) = \sum_{k=i-1}^{i+2}\sum_{l=j-1}^{j+2} P_{k,l}\widetilde{B}_k(\widetilde{t},t)B_l(t,\widetilde{t}) \qquad (27)$$

where $\widetilde{t}_i \le \widetilde{t} < \widetilde{t}_{i+1}; \quad t_j \le t < t_{j+1}, i = 0,\dots,m-1, j = 0,\dots,n-1$
Substitution of the Bernstein-Bezier form of the rational B-splines gives the piecewise defined rational Bernstein-Bezier representation:

$$p(\widetilde{t},t) = \sum_{k=0}^{3}\sum_{l=0}^{3} X_{k,l}^{i,j}(\widetilde{t},t)R_k(\widetilde{\theta};\widetilde{r}_i(t))R_l(\theta;r_j(\widetilde{t})), \quad (28)$$

where the Bernstein-Bezier points $X_{k,l}^{i,j}(t,t)$ can be computed from the rational B-spline vertices $P_{i,j}$ as

$$X_{i,j} = \widetilde{Y}_i.Z_{i,j}.Y_j^T \qquad (29)$$

where

$$\mathrm{X}_{i,j} = \begin{bmatrix} X_{0,0}^{i,j} & X_{0,1}^{i,j} & \dots & X_{0,3}^{i,j} \\ X_{1,0}^{i,j} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ X_{3,0}^{i,j} & \dots & \dots & X_{3,3}^{i,j} \end{bmatrix}$$

$$\mathrm{Z}_{i,j} = \begin{bmatrix} P_{i-1,j-1} & P_{i-1,j} & \dots & P_{i-1,j+2} \\ P_{i,j-1} & .. & \dots & \dots \\ \dots & \dots & \dots & \dots \\ P_{i+2,j-1} & \dots & & P_{i+2,j+2} \end{bmatrix}$$

and the matrix Y_j is given as in equation(16) as well as $\widetilde{Y}_i$ provided tildes are put where appropriate. $\widetilde{Y}_i$ and Y_j^T now depend on $\widetilde{t}$ and t respectively.
Tension Properties The rational B-spline surface representation equation(28) satisfies the *global tension* property and the *local tension* property as proved in [5].

7 Interpolatory Rational B-Spline Surfaces

Expanding equation(29) we get the points through which the freeform rational B-spline surface passes, for given control points say P's. Here in our case we need to find P's (the new control points) given the data points F's, through which the interpolatory rational B-spline surface should pass. Let us denote it by F:

$$F_{i,j} = X = (\widetilde{\lambda}_i[P_{i-1,j-1}\lambda_j + P_{i-1,j}(1-\lambda_j-\mu_j) + P_{i-1,j+1}\mu_j]) + \quad (30)$$
$$((1-\widetilde{\lambda}_i-\widetilde{\mu}_i)[P_{i,j-1}\lambda_j + P_{i,j}(1-\lambda_j-\mu_j) + P_{i,j+1}\mu_j]) +$$
$$(\mu_i[P_{i+1,j-1}\lambda_j + P_{i+1,j}(1-\lambda_j-\mu_j) + P_{i+1,j+1}\mu_j])$$

For $i = 1,..,n$ and $j = 1, \dots, m.$
We can observe from the above that the sum of the coefficients of P's equal to unity. Which means that the resulting interpolatory surface satisfies the convex *hull property.* Let

$$A_{i,j} = P_{i,j-1}\lambda_j + P_{i,j}(1-\lambda_j-\mu_j) + P_{i,j+1}\mu_j \quad (31)$$

Equation(30) can be expressed as :

$$F = T.A \qquad (32)$$

Since T is invertible, we get from equation(32)

$$A = T^{-1}.F \qquad (33)$$

Moreover, from equation(31) A can be expressed as:

$$A = D.P \qquad (34)$$

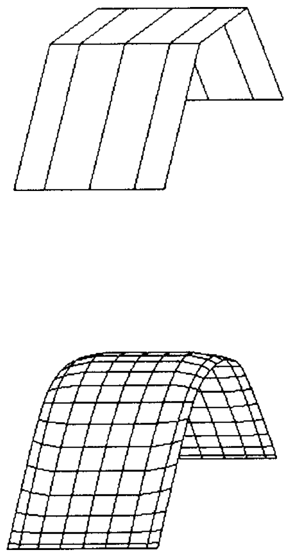
The process of calculating the new control points P's, is carried out in two stages, first the entire matrix A is calculated. Then the new control points P's can be calculated as

$$P = D^{-1}.A \qquad (35)$$

which when substituted in equation(28), gives the required interpolatory surface with interval tension which can be controlled both locally and globally.
Examples Consider a set of three dimensional data. The figures below show the effect of increase in tension, both locally and globally. Figure 7(a) is the control net. Surface in Figure 7(b) corresponds to the value $r = \widetilde{r} = 3$ (the bicubic case). Surface in Figure 8(a) converges to the control polyhedron as $r = \widetilde{r} = 50$. The local change of tension parameters is evident in Figure 8(b).

8 Concluding Remarks

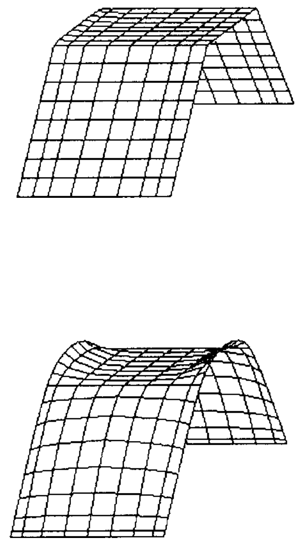
An analysis of a rational cubic tension spline has been developed with a view to its application in CAGD. We found it appropriate to construct a rational form which involves just one tension parameter per interval, although clearly the rational form defined by equation(1) could be generalized. One advantage of the use of C^2 parametric continuity, compared with that of the more general geometric GC^2 continuity, becomes apparent in the application of such a rational spline method to surfaces. In this case we followed the approach of [6], in the use of the spline blended methods of [7]. Nielson proposes a spline blended surfaces of GC^2



(a)

(b)

Figure 7. lnterpolatoty surfaces with interval tension



(a)

(b)

Figure 8. lnterpolatoty surfaces with interval tension

curves. However, the use of parametric C^2 curves in the blend will alleviate this loss of continuity.
The idea of C^2 freeform rational B-spline is extended to achieve a C^2 interpolatory parametric rational B-spline which can be controlled locally and globally. For CAGD applications, the developed interpolatory spline provides a parameter to control the shape of a curve on each interval.
The surface has been designed through the sum of the products of bivariate rational B-spline basis functions. The use of variable tensions allows shape control. This is not a tensor product surface but a tensor product surface can be recovered as a special case. This is not a NURBS representation either; the NURBS representation of the surface has some limitations regarding its shape control. Computation of the surface has been suggested through the Bernstein-Bezier representation which is quite convenient.

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