Proof Methods & strategies

Section 1.7

Proof by Cases

- We use the rule of inference

$$P_1 \vee P_2 \vee \ldots \vee P_n \rightarrow q$$
$$\equiv (P_1 \rightarrow q) \wedge (P_2 \rightarrow q) \wedge .. \wedge (P_n \rightarrow q)$$

Exhaustive Proof

- A special type of proof by cases where each case can be checked by examining an example.

- Good for computers.

Examples

$(n+1)^3 \geq 3^n$, for n=1, 2, 3, 4

Proof: (Exhaustive)

- For n=1, $2^3 = 8 \geq 3$
- For n=2, $3^3 = 27 \geq 3^2 = 9$
- For n=3, $4^3 = 64 \geq 3^3 = 27$
- For n=4, $5^3 = 125 \geq 3^4 = 81$

Examples

The only consecutive positive integers < 100 that are perfect powers are 8 and 9

Proof:

- Perfect squares < 100 are 1, 4, 9, 16, 25, 36, 49, 64, 81
- Perfect cubes < 100 are 1, 8, 27, 64
- Perfect 4^{th} powers < 100 are 1, 16, 81
- Perfect 5^{th} powers < 100 are 1, 32
- Perfect 6^{th} powers < 100 are 1, 64
- So all perfect powers < 100 are

  1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81
- So the consecutive perfect powers are 8 & 9

Examples

$n^2 \geq n$ for all integers.

Proof:

- Case I: when $n=0$, then $n^2=0 \geq n=0$
- Case II: when $n \geq 1$, then

  $n^2=n \times n \geq n \times 1 = n$
- Case III: when $n \leq -1$, then

  $n^2 \geq 0 \geq -1 \geq n$

Examples

$\forall$ x, y $\in$ R, $|x\ y| = |x|\ |y|$

Proof:

- Case I: x $\geq$0 & y $\geq$0, then

  $|x\ y| = x\ y = |x|\ |y|$

- Case II: x $\geq$0 & y <0, then

  $|x\ y| = -\ x\ y = x\ (-y) = |x|\ |y|$

- Case III: x < 0 & y $\geq$ 0, then same as II
- Case IV: x < 0 & y < 0, then

  $|x\ y| = x\ y = (-x)\ (-y) = |x|\ |y|$

Common Error

- Not all cases are considered.
- Example: if $x \in R$, then $x^2 > 0$
- Case I: x> o, then ..
- Case II: x< 0, then ..
- But Case III is forgotten!

Uniqueness Proof

Existence first, then uniqueness

$\exists ! x P(x)$

$\equiv \exists x (P(x) \wedge \forall y \ (y \neq x \rightarrow \neg P(y)))$

Example

If $a, b \in R$ & $a \neq 0$, then

$$\exists !\, r \in R \text{ s.t. } ar + b = 0$$

Proof:

- Clearly $r=-b/a$ (existence)
- Suppose $as+b=0$, then $ar+b=as+b$, i.e. $as = ab$, and hence $s=r$ because $a \neq 0$.