Chapter 9

Graphs

Sec. 9.2: Terminology

- Let G=(V, E) be an undirected graph & e={u,v} be an edge, then
  - u and v are adjacent
  - e is incident to u and v
  - u and v are connected
  - u and v are endpoints of e

Definition

- Degree of a vertex v in an undirected graph G is denoted by deg(v) and is defined to be the number of edges incident to it where a loop contributes twice to the degree.

Definition

- If deg(v) =0 then v is called isolated
- If deg(v)=1 then v is called pendant

The Hand Shaking Theorem

- Let G be an undirected graph (simple, multi, or pseudo) with e edges. Then

The sum of all degrees = 2 e

$$\sum_{v} \deg(v) = 2\,e$$

- Notice that 2e is even number

Theorem

- Any undirected graph has an even number of vertices of odd degree
    - Because
        - the sum of all vertices of even degrees is even
        - The sum of all degrees is even

Definition

- Let G be a directed graph & e=(u,v) be an edge, then
  - u is adjacent to v
  - v is adjacent from u
  - u is initial vertex of e
  - v is terminal (end) vertex of e

Definition

- In-deg(v) = $deg^{-}(v)$ = # of edges with v as terminal vertex
- Out-deg(v) = $deg^{+}(v)$ = # of edges with v as initial vertex

Theorem

- Let G be a directed graph, then

The sum of all in-degrees

= the sum of all out-degrees

= total number of edges

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|$$

Special graphs

- Complete graph on n vertices K_n: is a simple graph with all of the C(n,2) edges
- Cycle C_n , $n \geq 3$
- Wheel W_n , $n \geq 3$
- n-Cubes Q_n , $n \geq 1$

Special Graphs

Graph	Vertices	Edges	Degree
K_n	n	C(n,2)	n-1
C_n	n	n	2
W_n	n+1	2n	3 and n
Q_n	2^n	$n2^{n-1}$	n

Bipartite Graph

- A simple graph G=(V,E) is a bipartite if $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ such that no edge in E connects two vertices V_1 in or two vertices in V_2 ,
- Note that C_n is bipartite iff n is even

Complete Bipartite Graph

- $K_{m,n}$ is a complete bipartite iff $|V_1| = m$ and $|V_2| = n$ and every vertex in V_1 is connected to every vertex in V_2 and vice versa.

Sec 9.3: Graphs Representations

Sec 8.3: Graphs Representations

- There are two ways to represent graphs

1. Adjacency Lists: each vertex has a linked list that contains vertices adjacent to it.
2. Adjacency Matrix: which is defined as follows

Adjacency Matrix

- For any simple graph G, we have an adjacency matrix A_G =[a_{ij}], where

  a_{ij} = 1, if $\{v_i, v_j\} \in E$, and 0 otherwise.

- Notice that A_G is symmetric if G is simple and for directed graphs it may not be.
- A_G is called sparse if it has few 1s.

Adjacency Matrix

- For multigraphs or pseudographs A_G can be represented by

a_{ij} = k, if there are k edges between v_i and v_j.

Isomorphism

Isomorphism

- Two graphs G_1 & G_2 are isomorphic iff there is a bijective function $f : V_1 \rightarrow V_2$ such that whenever

  $\{a,b\} \in E$ then $\{f(a), f(b)\} \in E$.

- This means that G1 can be redrawn to get G2 with the same number of vertices, edges and degrees.

- See examples in textbook.

Isomorphism

- There are some graphical properties that are invariant (preserved) under isomorphism, e.g.
  - # of vertices
  - # of edges
  - Degrees
  - Cycles
  - Euler cycles
  - Hamilton cycles

Definitions

- A subgraph of G =(V, E) is a graph H=(U,F) such that $U \subseteq V$ and $F \subseteq E$.
- A graph is called regular iff every vertex has the same degree.
- A graph is called n-regular iff every vertex has deg=n.

Sec. 9.4: Connectivity

Definitons

- A path is sequence of edges {a,b}, {b,c}, {c,d}, {d,e}, ...
- The length of a path $e_1, e_2, e_3, e_4, .. e_n$ is the number of edges in the path =n.
- A path is a circuit (cycle) when it begins and ends with the same vertex and n>0.
- The path is said to passes through the vertices a, b, c, .. Or traverses the edges $e_1, e_2, e_3, e_4, .. e_n$

Definition

- A path is simple if it doesn’t contain the same edge more than once.
- An undirected graph is connected iff there is a simple path between any two vertices.
- A directed graph is strongly connected iff there is a 2-way path between any two vertices.
- A directed graph is weakly connected iff there is a path between any two vertices inits underlying undirected graph.

Definition

- A cut vertex in a connected graph is a vertex whose removal produces unconnected graph
- A cut edge (bridge) in a connected graph is an edge whose removal produces unconnected graph

Definitions

- The connected components in any graph are the maximum connected subgraphs
- The strongly connected components in any directed graph are the maximum strongly connected subgraphs

Sec. 9.5: Euler and Hamilton Paths and Circuits

Definitions

- Euler path in a graph G is a simple path containing every edge in G.
  - Start from a vertex & walk on every edge exactly once but you don't have to return back to the same vertex
- Euler circuit in a graph G is a simple circuit containing every edge in G.
  - Start from a vertex & walk on every edge exactly once and return back to the same vertex.

Theorem

- A connected multigraph has an Euler circuit iff every vertex has an even degree.
- A connected multigraph has an Euler path but not an Euler circuit iff there exists exactly two vertices of odd degrees and the others are even degrees.

Definition

- A Hamilton path in G is a path that visits every vertex exactly once
- A Hamilton circuit in G is a circuit that visits every vertex exactly once
- Note that every K_n has Hamilton Circuit because it has C_n

Remark

- Some graphs could have Euler path but no circuits and same for Hamiltonian.

Sec 9.7: Planar graphs

Definition

- A graph is called planar if it can be drawn in the plane without any edge crossing. Any planar graph split the plane into regions.

- E.g., K_4 and Q_3 are planar

Euler Formula

- Let G be any connected simple planar graph. Then the number of regions

$$r = |E| - |V| + 2$$

Sec. 9.8: Graph Coloring

Problem

- Color a map of regions with common borders such that no two neighbor regions have the same color.

- Maps can be modeled by a graph called the dual graph where
  - Vertices represent regions
  - Edges represent common borders

Definition

- A coloring of a simple graph is an assignment of colors of vertices of G such that adjacent vertices are colored with different colors.

- The chromatic number of a graph G is the least number of colors needed to color the graph correctly.

The 4 colors Theorem

- The chromatic number of

  a planar graph $\leq$ 4.

- Recall that a graph is called planar if it can be drawn in the plane without any edge crossing.

Examples

- $\chi(K_n) = n$
- $\chi(C_n) = 2$ if n is even and 3 otherwise
- $\chi(W_n) = 3$ if n is even and 4 otherwise

Chapter 10: Trees

Definition

- A tree is a connected undirected acyclic (with no cycle) simple graph
- A collection of trees is called forest.

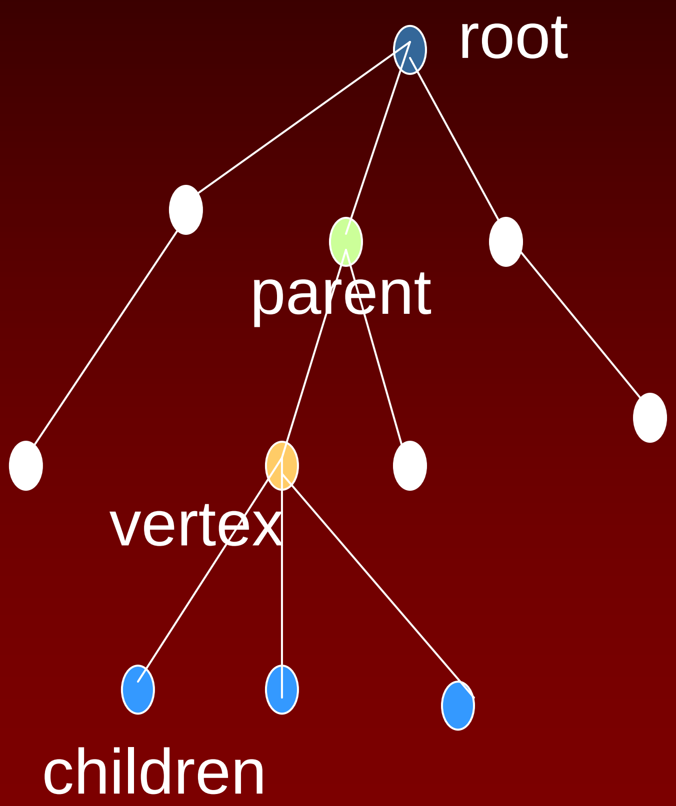
Theorem

- An undirected graph is a tree iff there is a unique simple path between any two vertices.

Definition

- A rooted tree is a tree in which one vertex is specified as the root & every edge is drawn away from the root

Definition



Definition

- Root: a vertex with no parent
- Leaf: a vertex with no children
- Internal node: vertex with children
- Descendants: all children and children of children
- Ancestors: parent and parents of parents
- Siblings: vertices with the same parent

Definition

- An m-ary tree is a rooted tree where the number of children of any internal vertex $\leq$ m
- A full m-ary tree is an m-ary tree where the number of children of any internal vertex = m
- if m=2, we call it binary tree

Definition

- In an ordered rooted tree the children are ordered.
- For example, in an ordered binary tree, a vertex may have left child and right child

Properties

- Number of edges in a tree of size n = n-1
- Any full m-ary tree with i internal vertices has m i + 1 vertices

Properties

A full m-ary tree with

- *n vertices has $i = (n-1)/m$ internal and*

$l = ((m-1)n+1)/m$ leaves

- *i internal has $n = mi+1$ vertices and*

$l = (m-1)i+1$ leaves

- *l leaves has $n = (ml - 1)/(m-1)$ vertices and*

$i = (l-1)/(m-1)$ internal