Ch. 2 Basic Structures

Section 1

Sets

Principles of Inclusion and Exclusion

- $| A \cup B | = | A | + | B | - | A \cap B|$

- $| A \cup B \cup C |$
  $= | A | + | B | + | C | - | A \cap B| - | A \cap C |$
  $- | B \cap C | + | A \cap B \cap C |$

- $| A^c | = | U | - | A |$

Disjoint sets

- A & B are disjoint sets iff $A \cap B = \varnothing$

- Example:
- $A = \{1, 2, 3\}$
- $B = \{ 4, 5, 6\}$

Power Set

- P(A) = The power set of a set A = the set of all subsets of A.

- A $\in$ P(S) iff A $\subseteq$ S
- $|\ P(\ \{\{a, \{b\}\}, a, \{b\}\ \}\ )\ | = 2^3$
- There is no set A s.t. |P(A)| = 0
- There is a set A s.t. |P(A)| = 1
- There is no set A s.t. |P(A)| = 3

Power Set

A = B iff P(A) = P(B)

Proof

"$\Rightarrow$" Suppose that A=B. Then P(A) = P(B)

"$\Leftarrow$" If P(A) = P(B), then A $\in$ P(A) = P(B) and so A $\subseteq$ B

Similarly, B $\in$ P(B) = P(A) and so B $\subseteq$ A

Therefore A =B

Power Set

- $P(\ ?\ ) = \{\emptyset, \{a\}, \{a, \emptyset\}\}$

Note that:

- $\emptyset \in \{\emptyset\}, \emptyset \subseteq \{\emptyset\}, \emptyset \subset \{\emptyset\}, \emptyset \neq \{\emptyset\}$

- $\{2\} \in \{2, \{2\}\}, \{2\} \subset \{2, \{2\}\}, 2 \in \{2, \{2\}\}$
- $\{\{2\}\} \in \{2, \{2\}\}, \{2\} \notin \{2\}$
- $a \notin \{\{a\}, \{a, \{a\}\}\}$

Some facts

- $A \cup B = A \Leftrightarrow B \subseteq A$
- $A - B = A \Leftrightarrow A \cap B = \emptyset$
- $A - B = B - A \Leftrightarrow A = B$
- $A = A - B \cup (A \cap B)$

Ch. 2 Basic Structures

Section 2

Set Operations

Set Operations

1. Union $A \cup B$
2. Intersection $A \cap B$
3. Complement $A^c = \bar{A} = U - A$

   complement of A w.r.t. to U
4. Difference $A - B = A \cap B^c$
5. Generalized operations $\bigcup_{i=1}^{n}$ & $\bigcap_{i=1}^{n}$

Example

- A = set of students who live 1 mile of school
- B = set of students who walk to school

1. $A \cup B$ = set of students who live 1 mile of school OR walk to it.
2. $A \cap B$ = set of students who live 1 mile of school AND walk to it.
3. $A - B$ = set of students who live 1 mile of school but not walk to it.

Set Identities

- See Table 1 in page 124
- Identity Laws:

$A \cup \varnothing = A$

$A \cap U = A$

- Dominations Laws:

$A \cup U = U$

$A \cap \varnothing = \varnothing$

Set Identities

- Idempotent Laws:

$A \cup A = A$

$A \cap A = A$

- Complementation Law:

$(A^c)^c = A$

- Commutative Laws:

$A \cup B = B \cup A$

$A \cap B = B \cap A$

Set Identities

- Associative Laws:

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

- Distributive Laws:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- De Morgan’s Laws:

$(A \cup B)^c = B^c \cap A^c$

$(A \cap B)^c = B^c \cup A^c$

Set Identities

- Absorption Laws:

$A \cup (A \cap B) = A$

$A \cap (B \cup C) = A$

- Complement Laws:

$A \cap A^c = \varnothing$

$A \cup A^c = U$

Proof of A=B

- Show that $A \subseteq B$ and $B \subseteq A$
- Show that $\forall x \ (x \in A \leftrightarrow x \in B)$
- Direct proof

Examples

$(A \cap B)^c = B^c \cup A^c$

Proof

$$
\begin{aligned}
(A \cap B)^c &= \{ x \in U \mid x \notin A \cap B \} \\
&= \{ x \in U \mid \neg (x \in A \cap B) \} \\
&= \{ x \in U \mid \neg x \in A \vee \neg x \in B \} \\
&= \{ x \in U \mid x \notin A \vee x \notin B \} \\
&= \{ x \in U \mid x \notin A \} \cup \{ x \in U \mid x \notin B \} \\
&= A^c \cup B^c
\end{aligned}
$$

Another proof of $(A \cap B)^c = B^c \cup A^c$

$x \in (A \cap B)^c \Leftrightarrow x \notin A \cap B$

$\Leftrightarrow x \notin A$ or $x \notin B$

$\Leftrightarrow x \in A^c$ or $x \in B^c$

$\Leftrightarrow x \in A^c \cup B^c$