Predicates and Quantifiers

Predicate Calculus

Section 1.3

Definition

- Propositional Function P(x) is a statement that has a variable x.
- Examples: P(x)

  P(x) = “The Course x is difficult”

  P(x) = “x+2 < 5”

- Note: a propositional function is not a proposition because it depends on the value of x.
- A propositional function is called a predicate

Examples

- If P(x) = "x > 3", then P(4) is true but not P(1).

- It also possible to have more than one variable in one statement, e.g.,

  Q(x,y) ="x = y -2"

Remark

- P(x) is a function that takes a value for x and produce either true or false.
- Example: P(x) = "$x^2 > 2$"

  $P : \text{Domain} \rightarrow \{T, F\}$

- Domain of the values of x sometime is called the domain of discourse

Quantifiers

- A predicate (propositional functions) could be made a proposition by either assigning values to the variables or by quantification.

- Predicate Calculus: Is the science of studying predicate logic with quantifiers.

Quantifications

1. Universal quantifier: P(x) is true for all (every) x in the domain.

   We write $\forall$ **x P(x)**

2. Existential quantifier: there exists at least one x in the domain s.t. P(x) is true. We write $\exists$ **x P(x)**

3. Others: there is a unique x s.t P(x) is true. We write $\exists$**! x P(x)**

Predicates and Negations

Predicate	When true	When false	Negation
$\forall x \; P(x)$	For all x, P(x) is true	There is at least one x s.t. P(x) is false	$\exists x \neg P(x)$
$\exists x \; P(x)$	There is at least one x s.t. P(x) is true	For all x, P(x) is false	$\forall x \neg P(x)$

Logically Equivalence

Def. Two statements involving predicates & quantifiers are logically equivalence

if and only if

they have the same truth values independent of the domains and the predicates.

Examples:

- $\neg(\forall x \; P(x)) \equiv \exists x \; \neg P(x)$
- $\neg(\exists x \; P(x)) \equiv \forall x \; \neg P(x)$

Domain (or Universe) of Discourse

- It is the domain of values of x for which the proposition p(x) could be true or false.

- It is important to specify the domain.

Examples

- P(x) = “x+1 = 2” Domain =R

Proposition	Truth Value
$\forall x \; P(x)$	F
$\forall x \; \neg P(x)$	F
$\exists x \; P(x)$	T
$\exists x \; \neg P(x)$	T
$\exists ! x \; P(x)$	T
$\exists ! x \; \neg P(x)$	F

Examples

- P(x) = “$x^2 > 0$”

Domain	Proposition	Truth Value
R	$\forall x \; P(x)$	F
Z	$\forall x \; P(x)$	F
Z - {0}	$\forall x \; P(x)$	T
Z	$\exists ! x \; \neg P(x)$	T
N={1,2, ..}	$\exists x \; \neg P(x)$	F

Theorem

If the domain of discourse is finite, say Domain = $\{x_1, x_2, .., x_n\}$, then

$$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

$$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

More Examples

True or False

Proposition	Truth Value
$\forall x \in R \quad (x^2 \geq x)$	F
$\exists !\, x \in R \quad (x^2 < x)$	F
$\forall x \in (0,1) \quad (x^2 < x)$	T
$\forall x \in \{0,1\} \quad (x^2 = x)$	T
$\forall x \in \varnothing \quad P(x)$	T

Precedence

$\forall$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

$\exists$ $\oplus$

Example:

$$\Big((\forall x\ P(x)) \vee \big((\exists! x\ Q(x)) \wedge (\forall x \neg P(x)) \big) \Big)$$

Wrong Equivalences

- $\forall x (P(x) \vee Q(x)) \equiv \forall x P(x) \vee \forall x Q(x)$
- One can construct an example that makes the above equivalence false. Consider the following

x	1	2	3	4	5	6	7
P(x)	T	F	F	T	F	T	T
Q(x)	F	T	T	F	T	T	F

- $\forall x (P(x) \vee Q(x)) \equiv T$ but
- $\forall x P(x) \vee \forall x Q(x) \equiv F$

Examples

- $(\forall x \; P(x)) \vee Q(x) \neq \forall x \; (P(x) \vee Q(x))$
- Notice that $F(x) = (\forall x \; P(x)) \vee Q(x)$ is not even a proposition. It is a predicate.

Wrong Equivalences

- $\exists x\ (P(x) \land Q(x)) \equiv \exists x\ P(x) \land \exists x\ Q(x)$
- One can construct an example that makes the above equivalence false. Consider the following

x	1	2	3	4	5	6	7
P(x)	T	F	F	T	F	T	T
Q(x)	F	T	T	F	T	F	F

- $\exists x\ (P(x) \land Q(x)) \equiv F$ but
- $\exists x\ P(x) \land \exists x\ Q(x) \equiv T$

Correct Equivalences

- $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

- $\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$