ICS 252–Discrete Structures

Major Exam 2

Date: Nov. 26th, 2005 Duration: 90 minutes

Please write your name, ID and section numbers.

Name:

ID#:

Section #:

This is a closed-book and closed-notes exam. No calculators are allowed. You may use any of the following notations in your final answers without any farther simplifications:

$$\lfloor n/m \rfloor,\ \lceil n/m \rceil,\ \binom{n}{r},\ C(n,r),\ P(n,r),\ n!\,.$$

Q.#	Marks	Scores	Remarks
1	20		
2	20		
3	12		
4	12		
5	12		
6	16		
7	8		
8	**8**		**Bonus**
Total	100 + 8		

Question 1: [2×10 **marks**]

a) What is the probability that a randomly chosen integer from the first 100 positive integers

1. is 36?

2. is even?

3. is greater than 50?

4. is not a multiple of 10?

5. is not divisible by 7?

b) What is the probability of having an even number when a die is rolled given that

1. any outcome is equally likely?

2. the number 4 is twice as likely to appear as each of the other five numbers on the die?

c) What is the probability that a permutation chosen randomly from the set of all possible permutations of $\{1, \ldots, n\}$

1. has the numbers sorted decreasingly or increasingly?

2. starts with 1 and ends with n?

3. has 1 immediately before 2?

Question 2: [2×10 **marks**]

a) Suppose a card is selected randomly from a dick. Let A be the event that the card is a diamond, and B be the event that the card is a queen. Find the following probabilities, and state whether A and B are independent or not.

1. $P(A) =$

2. $P(B) =$

3. $P(A \cap B) =$

4. $P(A \cup B) =$

5. $P(A \,|\, B) =$

b) What is the probability that a five-card poker hand

1. contains the king of hearts?

2. contains all different kings?

3. does not contain any king?

4. contains at least two different suits?

5. contains exactly three different kinds?

Question 3: [2×6 **marks**]

a) For each of the following recurrence relations write the first 5 numbers.

1. $a_0 = 0$, and $a_n = 2a_{n-1} + n$, for $n \geq 1$.

2. $a_0 = 0$, $a_1 = 1$, and $a_n = na_{n-1} - a_{n-2}$, for $n \geq 2$.

3. $a_0 = 1$, $a_1 = 2$, and $a_n = a_{n-1} + (-1)^n a_{n-2}$, for $n \geq 2$.

b) For each of the following sequences, write the next number in the sequence, and find a recurrence relation for the n-th term.

1) 1, 3, 5, 7, 9, 11, 13, 15,...

2) 1, 2, 3, 1, 2, 3, 1, 2, 3, ...

3) 0, 1, 2, 3, 6, 11, 20, 37, ...

Question 4: [4×3 **marks**]

A young pair of rabbits (one of each sex) is placed on an island. Suppose that during the first month the pair of rabbits produces only one rabbit; during the second month two new rabbits are produced, and so on–during the n-th month n new rabbits are produced. Let R_n be the total number of rabbits exactly after n months.

a) Write a recurrence relation for R_n.

b) Find an explicit formula for R_n?

c) What is the total number of rabbits on the island exactly after one year?

Question 5: [4×3 **marks**]

Find the solution for each of the following recurrence relations.

a) $a_0 = 1$, and $a_n = -2a_{n-1}$, for $n \geq 1$.

b) $a_0 = 1$, $a_1 = 1$, and $a_n = 9a_{n-2}$, for $n \geq 2$.

c) $a_0 = 3$, $a_1 = 2$, and $2a_n = a_{n+1} + a_{n-1}$, for $n \geq 2$.

Question 6: [8×2 **marks**]

a) Draw the following graphs: C_5, W_5, K_5 and Q_3.

b) Write the number of vertices and the number of edges for each of the following graphs.

Graph	# of vertices	# of edges
K_n		
C_n		
W_n		
Q_n		

Question 7: [2×4 **marks**]

a) Is there an undirected graph with 19 vertices each of degree 3? Explain your answer.

b) Is there a directed graph with 19 vertices each of in-degree 3 and out-degree 2? Explain your answer.

c) How many edges does an undirected graph have if the degrees of its vertices are 4, 3, 2, 5, 1, and 7?

d) How many edges does a directed graph have if the in-degrees of its vertices are 4, 3, 2, 5, 1, and 7?

Question 8: [2×4 **bonus marks**]

Consider the following process to generate a random simple graph G with n vertices. We start with the complete simple graph K_n. *For each edge* $\{u, v\}$ in K_n, we flip an independent coin and if it comes up a tail we remove the edge from the graph, otherwise the edge stays. The remaining graph at the end of this process is G. Suppose that each coin comes up a head with probability $p \in (0, 1)$. Let X be the number of edges in the random graph G. Notice that $0 \leq X \leq N$, where N is the number of edges in K_n.

a) What is the probability that X is not 0?

b) What is the probability that $G = K_n$?

c) Let u be a certain fixed vertex in G.

1. What is the probability that u is not isolated?

2. What is the probability that u is pendant?