GEOPHYSICS, VOL. 50, NO. 4 (APRIL 1985); P. 609-614, 8 FIGS., 2 TABLES.

A simplification of the Zoeppritz equations

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ABSTRACT

The compressional wave reflection coefficient $R(\theta)$ given by the Zoeppritz equations is simplified to the following:

$$R(\theta) = R_0 + \left[A_0 R_0 + \frac{\Delta\sigma}{(1-\sigma)^2} \right] \sin^2 \theta + \frac{1}{2} \frac{\Delta V_p}{V_p} (\tan^2 \theta - \sin^2 \theta).$$

The first term gives the amplitude at normal incidence ($\theta = 0$), the second term characterizes $R(\theta)$ at intermediate angles, and the third term describes the approach to critical angle. The coefficient of the second term is that combination of elastic properties which can be determined by analyzing the offset dependence of event amplitude in conventional multichannel reflection data. If the event amplitude is normalized to its value for normal incidence, then the quantity determined is

$$A = A_0 + \frac{1}{(1-\sigma)^2} \frac{\Delta\sigma}{R_0}.$$

A_0 specifies the normal, gradual decrease of amplitude with offset; its value is constrained well enough that the main information conveyed is $\Delta\sigma/R_0$, where $\Delta\sigma$ is the contrast in Poisson's ratio at the reflecting interface and R_0 is the amplitude at normal incidence. This simplified formula for $R(\theta)$ accounts for all of the relations between $R(\theta)$ and elastic properties first described by Koefoed in 1955.

INTRODUCTION

Recently the dependence of seismic reflection amplitude upon the offset between source and receiver has been intensely investigated (Ostrander, 1984; Sherwood et al., 1983; Gassaway and Richgels, 1983). At the core of the matter are the Zoeppritz equations, which give the reflection and transmission coefficients for plane waves as a function of angle of incidence and *six* independent elastic parameters, three on each side of the reflecting interface. The inverse problem is to make inferences about the elastic parameters from observation of reflection amplitude as a function of angle.

This problem was fairly definitively investigated in the pioneering work of Koefoed (1955). His method was laborious computation of reflection coefficient versus angle out to 30 degrees for 17 different sets of elastic properties. Koefoed took the three elastic parameters for each medium to be longitudinal velocity V_p, density ρ, and Poisson's ratio σ. He gave his conclusions as follows.

(a) 'When the underlying medium has the greater longitudinal velocity and other relevant properties of the two strata are equal to each other, an increase of Poisson's ratio for the underlying medium causes an increase of the reflection coefficient at the larger angles of incidence.
(b) When, in the above case, Poisson's ratio for the incident medium is increased, the reflection coefficient at the larger angles of incidence is thereby decreased.
(c) When, in the above case, Poisson's ratios for both media are increased and kept equal to each other, the reflection coefficient at the larger angles of incidence is thereby increased.
(d) The effect mentioned in (a) becomes more pronounced as the velocity contrast becomes smaller.
(e) Interchange of the incident and the underlying medium affects the shape of the curves only slightly, at least up to values of the angle of incidence of about 30°.'

The precise meaning of some of these five rules may be unclear without Koefoed's figures or subsequent similar parametric studies. I give my understanding of each rule in a later section of this paper.

While various authors have presented approximations to the Zoeppritz equations (e.g., Bortfeld, 1961), to my knowledge they have not been simplified to the point where both (1) Koefoed's rules are displayed analytically and (2) the inverse problem—elastic properties from curve shape—is done analytically. This paper presents such a simplification. The work was done before I became aware of Koefoed's paper.

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DERIVATION

Provided the percentage changes in elastic properties are small, the P-wave reflection amplitude $R(\theta)$ is given approximately by Aki and Richards (1980, p. 153) as

$$R(\theta) \approx \frac{1}{2}\left(1 - 4\frac{V_s^2}{V_p^2}\sin^2\theta\right)\frac{\Delta\rho}{\rho} + \frac{\sec^2\theta}{2}\frac{\Delta V_p}{V_p} - \frac{4V_s^2}{V_p^2}\sin^2\theta\,\frac{\Delta V_s}{V_s}. \tag{1}$$

The elastic properties in equation (1) are related as follows to those on each side of the interface:

$$\Delta V_p = (V_{p2} - V_{p1}), \quad V_p = (V_{p2} + V_{p1})/2, \tag{2}$$

$$\Delta V_s = (V_{s2} - V_{s1}), \quad V_s = (V_{s2} + V_{s1})/2, \tag{3}$$

$$\Delta\rho = (\rho_2 - \rho_1),$$

and

$$\rho = (\rho_2 + \rho_1)/2, \tag{4}$$

where the incident and reflected waves are on side 1 and the transmitted wave is on side 2. The angle θ is the average of incidence and transmission angles,

$$\theta = (\theta_2 + \theta_1)/2. \tag{5}$$

These two angles are related by Snell's law,

$$p = \frac{\sin\theta_1}{V_1} = \frac{\sin\theta_2}{V_2}. \tag{6}$$

I modified equation (1) by eliminating the properties V_s, ΔV_s in favor of σ, $\Delta\sigma$. The latter are defined as above by

$$\Delta\sigma = (\sigma_2 - \sigma_1),$$

and

$$\sigma = (\sigma_2 + \sigma_1)/2. \tag{7}$$

The substitution is effected by the equation

$$V_s^2 = V_p^2\,\frac{1 - 2\sigma}{2(1 - \sigma)} \tag{8}$$

and also by the differential of this equation. This change of variable was motivated by the general perception [embodied in Koefoed's rules (a), (b), and (c)] that Poisson's ratio is the elastic property most directly related to angular dependence of reflection coefficient.

A further modification was to factor out R_0, the amplitude at normal incidence. Because the practical problems in recovering absolute reflection amplitude seem more severe than the problems of recovering the relative variation of reflection amplitude with offset, it is appropriate to consider the information content of the relative curve $R(\theta)/R_0$. The result of these manipulations is

$$R(\theta)/R_0 \approx 1 + A\sin^2\theta + B(\tan^2\theta - \sin^2\theta), \tag{9}$$

where

$$R_0 \approx \frac{1}{2}\left(\frac{\Delta V_p}{V_p} + \frac{\Delta\rho}{\rho}\right), \tag{10}$$

$$A = A_0 + \frac{1}{(1-\sigma)^2}\frac{\Delta\sigma}{R_0}, \tag{11}$$

$$A_0 = B - 2(1 + B)\frac{1 - 2\sigma}{1 - \sigma}, \tag{12}$$

and

$$B = \frac{\Delta V_p/V_p}{\Delta V_p/V_p + \Delta\rho/\rho}. \tag{13}$$

Equation (9) displays which combinations of elastic properties are effective in successive ranges of angle θ. The third term vanishes as θ^4, so it does not normally contribute for $\theta < 30$ degrees. However, at large angles it dominates. For consideration of absolute instead of relative amplitude, equation (9) should be multiplied through by R_0, i.e.,

$$R(\theta) \approx R_0 + \left[A_0 R_0 + \frac{\Delta\sigma}{(1-\sigma)^2}\right]\sin^2\theta + \frac{1}{2}\frac{\Delta V_p}{V_p}(\tan^2\theta - \sin^2\theta). \tag{14}$$

DISCUSSION

Equation (14) has some similarity to the approximation of $R(\theta)$ given in Bortfeld (1961) and discussed recently in numerous unpublished lectures by Hilterman (1983). The Bortfeld approximation has about the same accuracy as equation (1) or equation (14), but the three expressions differ in the philosophy of grouping terms. Equation (14) displays which combinations of elastic properties are effective in successive ranges of angle θ. Equation (1) is arranged to separate the effects of the three variables $\Delta\rho$, ΔV_p, and ΔV_s. The Bortfeld arrangement is designed to contrast elastic and acoustic reflection coefficients. It separates two terms, the first of which (fluid factor) involves only velocity and density and is the same as $R(\theta)$ for a fluid-fluid contact. The linearization of this fluid factor is

$$R_f(\theta) \approx R_0 + \frac{1}{2}\frac{\Delta V_p}{V_p}\tan^2\theta \tag{15}$$

which corresponds to the first term and part of the third term in equation (14).

Equations (9) and (14) diagonalize the relationship between elastic properties and $R(\theta)$ in the sense that certain features are related to certain combinations of elastic properties without significant coupling between the variables. As is apparent from Figure 1, the dimensionless parameter A controls whether the amplitude initially increases ($A > 0$) or decreases ($A < 0$), while the dimensionless parameter B controls the sign at large angles. From equations (9) through (14) I perceive three quasi-independent connections between $R(\theta)$ and elastic properties:

(1) Normal incidence.—The magnitude R_0 is the average of fractional changes in V_p and ρ [cf., equation (10)]. Alternatively, R_0 is half the change in natural logarithm of impedance ρV_p since the approximation

$$R_0 \approx \frac{1}{2}\frac{\Delta\rho V_p}{\rho V_p} = \frac{1}{2}\Delta\ln(\rho V_p) \tag{16}$$

is also valid to first order in change of elastic properties.

(2) Intermediate angles ($0 < \theta < 30$ degrees).—The reflection amplitude at intermediate angles relative to that at normal incidence connects to the parameter A, which is the sum of two terms [equation (11)]. I argue in Appendix A that the first term A_0 can be accurately predicted just from an approximate value of average Poisson's ratio σ, provided only that the parameter B is in its normal range. The real information in A is in the ratio $\Delta\sigma/R_0$. In Appendix B I consider the degenerate case $R_0 = 0$.

(3) Wide angles.—The reflection amplitude at wide angles relates only to the change in velocity. At sufficiently large angles where the third term in equation (14) dominates the first two, it becomes

$$R(\theta) \approx \frac{1}{2}\frac{\Delta V_p}{V_p}(\tan^2\theta - \sin^2\theta). \tag{17}$$

Analysis of wide-angle seismic records for velocity is a highly developed subject to which equation (17) probably contributes nothing. However, its derivation does indicate the relation to analysis of amplitude at intermediate angles.

ACCURACY AND FURTHER APPROXIMATION

The basis of the derivation of equations (1) and (9) from the exact Zoeppritz equations is that the percentage change in elastic properties is small, i.e., $\Delta V_p/V_p$, $\Delta V_s/V_s$, and $\Delta\rho/\rho$ are all small compared to unity. For the vast majority of exploration

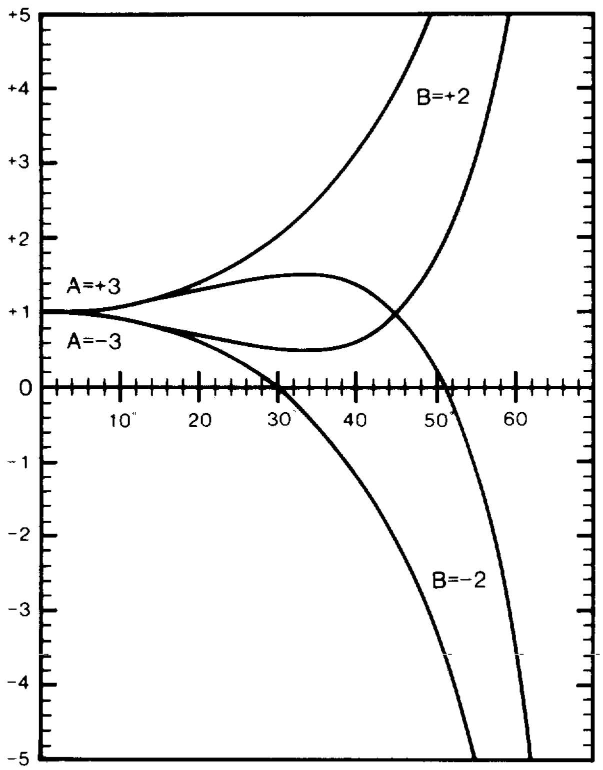

FIG. 1. The four possible variations of amplitude with angle. Plotted are curves $R(\theta)/R_0$ according to equation (9), for the four possible combinations of $A = \pm 3$, $B = \pm 2$. As discussed in Appendix A, the case $B < 0$ is rare.

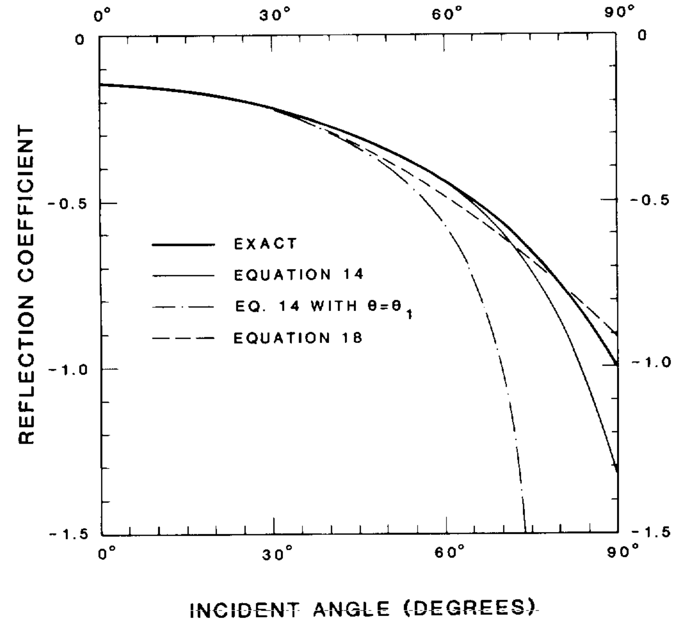

FIG. 2. Approximations in an instance with $\Delta V < 0$. The elastic properties correspond to an actual gas sand in the Gulf of Mexico: $V_{p1} = 7\,570$ ft/s, $\rho_1 = 2.15$, and $\sigma_1 = 0.40$ for the overlying shale and $V_{p2} = 6\,400$ ft/s, $\rho_2 = 1.95$, and $\sigma_2 = 0.10$ for the gas sand.

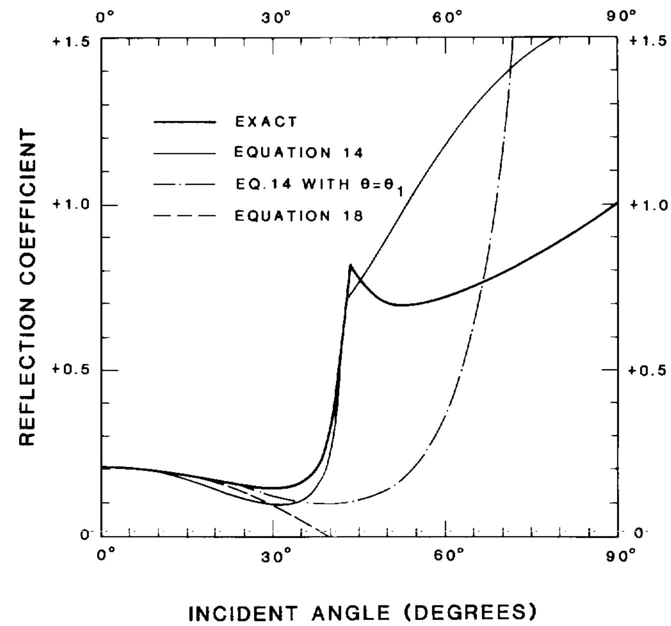

FIG. 3. Approximations in an instance with $\Delta V > 0$. The elastic properties on the upper side are the same as for Figure 2, while the underlying medium has $V_{p2} = 11\,350$ ft/s, $\rho_2 = 2.20$, $\sigma_2 = 0.30$. These values are appropriate to a chalk with 30 percent porosity. Beyond critical angle (42 degrees) the reflection coefficient becomes complex and the absolute value is shown for the exact solution and for equation (14).

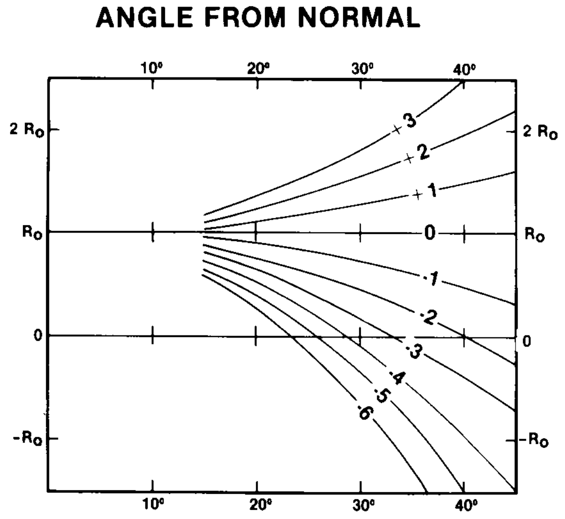

FIG. 4. Parabolic approximation to $R(\theta)$ for different values of the dimensionless parameter A [equation (18)].

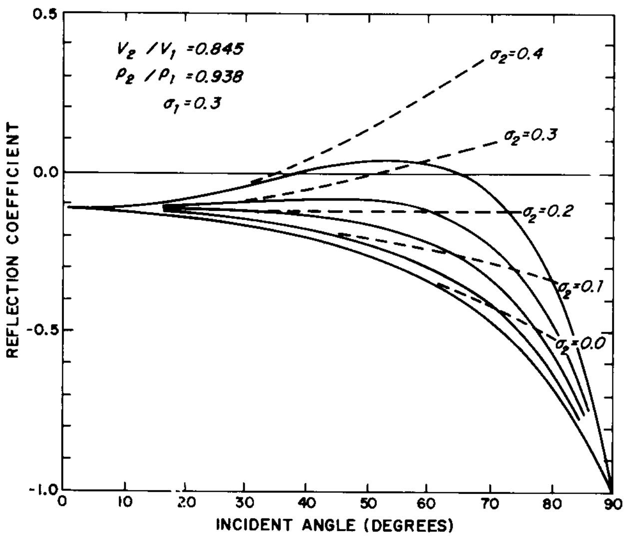

FIG. 5. Comparison of exact reflection coefficients (solid line) with the parabolic approximation [equation (18)] in a case of velocity-decrease.

situations, in which the reflection coefficient R_0 does not exceed 0.2, this is no problem. However, the derivation also assumes that both θ_1 and θ_2 are real and less than 90 degrees. This means that the approximation is good only to about $\theta_1 = 80$ degrees in the case of a velocity decrease (Figure 2) and to θ_1 about 10 degrees less than critical in the case of a velocity increase (Figure 3). It is not necessary that $\Delta\sigma/\sigma$ be small; indeed, it has the value 1.2 for Figure 2.

At intermediate angles further simplifications are possible (Figures 2 and 3). The incident angle θ_1 can be used instead of average angle θ in equation (14). This saves the work of calculating transmitted angle θ_2 by equation (6) and then averaging by equation (5). The resulting curve has the right shape but rises too soon in the case of a velocity decrease ($\Delta V < 0$, Figure 2) and too late in the case of a velocity increase ($\Delta V > 0$, Figure 3). The error is not significant in the intermediate angles $0 < \theta_1 < 30$ degrees.

A further approximation is to omit entirely the third term in equation (9) or equation (14). I argued above that the third term was unnecessary for incident angles less than 30 degrees. One caution is that in the case $A < 0$, $B > 0$ omission of the third term may lead to false prediction of a zero crossing at wide angles. Figures 1, 3, 5, and 6 all illustrate this situation.

A final approximation is to replace $\sin\theta_1$ by θ_1, which is accurate for $\theta_1 < 30$ degrees. The concatenation of all these approximations is

$$R = R_0(1 + A\theta_1^2). \tag{18}$$

Figure 4 shows this parabolic form of variation in more detail for a range of values of parameter A. Figures 5 and 6 compare the parabolic approximation to the exact solution of the Zoeppritz equations for a range of $\Delta\sigma$ and both signs of ΔV. These figures do not include an instance with $B < 0$, but that is rare as discussed in Appendix A.

DERIVATION OF KOEFOED'S RULES

The five observations made numerically by Koefoed can be established analytically using equations (9) through (14). Equation (14) states that an increase (decrease) of Poisson's ratio for the underlying medium produces an increase (decrease) in the reflection coefficient at larger angles of incidence. This agrees with Koefoed's rules (a) and (b) but without the qualification that "the underlying medium has the greater longitudinal velocity and other relevant properties of the two strata are equal to each other."

When the Poisson's ratios for the two media are equal ($\Delta\sigma = 0$), then this Poisson's ratio enters into $R(\theta)$ only through A_0. Koefoed's rule (c) is equivalent to saying A_0 increases as σ increases. The derivative of equation (12) is

$$\frac{\partial A_0}{\partial \sigma} = \frac{2(1 + B)}{(1 - \sigma)^2}, \tag{19}$$

so we see rule (c) is true except for $B < -1$. Figure 8 illustrates the increase of A_0 with σ for the normal range of B. Koefoed only considered the case $B = 1$.

Koefoed's rule (d) is derived from equation (11). The smaller is R_0; the larger is the effect of a given $\Delta\sigma$ upon A. This is apparently what Koefoed meant by rule (d).

Rule (e) follows from the observation that equation (14) [or equation (9)] is linear in the three differences ΔV_p, $\Delta\sigma$, and $\Delta\rho$. Therefore $R(\theta)$ simply changes sign when the two sets of elastic properties "1" and "2" are interchanged. Rule (e) breaks down at large angles when the difference between θ_1 and θ [equation (5)] cannot be neglected.

INVERSION FOR ELASTIC PROPERTIES

A current problem is inversion of the reflection coefficient $R(\theta)$ to obtain information about elastic properties (Gassaway and Richgels, 1983; Rosa, 1976). The diagonalization accomplished in equation (9) or equation (14) can simplify this inversion and clarify which combinations of elastic properties are well determined and which are poorly determined.

Because the practical problems in recovering absolute reflection amplitude differ from the problems of recovering the relative variation of reflection amplitude with offset, it is appropriate to consider the information content of the relative curve $R(\theta)/R_0$. If reflections are not recorded much past 30 degrees then equation (18) or Figure 4 could be used to extract a value of A from the input $R(\theta)/R_0$. If wider angles are involved, then $R(\theta)/R_0$ could be fit to the trigonometric series in equation (9). In either case, the dimensionless parameter A is the only information to be extracted other than critical-angle behavior. Figure 7 shows that A can be transformed to $\Delta\sigma/R_0$ provided very approximate information is available or hypothesized about average Poisson's ratio σ and velocity/impedance ratio B. In brief, the single piece of information about elastic properties available from analysis of relative change of reflection amplitude with offset is $\Delta\sigma/R_0$. The multidimensional, nonlinear inversion problem discussed previously (Rosa, 1976; Gassaway and Richgels, 1982) is reduced to a one-dimensional linear problem.

As I pointed out, R_0 is half the change in natural logarithm of impedance [equation (16)]. Thus the information $\Delta\sigma/R_0$ is equivalent to slope on a crossplot of Poisson's ratio versus impedance ρV_p on a logarithmic scale. This could be the basis of a graphic procedure to relate information about $R(\theta)$ to information about lithology.

SUMMARY

I took the known linearization of the P-wave reflection coefficient $R(\theta)$ [equation (1)] and transformed variables from V_s to σ to display analytically the effect of $\Delta\sigma$, the contrast in Poisson's ratio. The result [equations (9) or (14)] was arranged into three terms which contribute to three distinct features of the $R(\theta)$ curve: (1) the normal-incidence magnitude, (2) the behavior at intermediate angles of about 30 degrees, and (3) the approach to critical angle. Thus I have approximately diagonalized the multivariate relationship between elastic properties and curve features. The coefficient for intermediate angles also has two terms: one term is proportional to $\Delta\sigma$, the contrast in Poisson's ratio; and the other term is A_0, which describes the bland decrease of $R(\theta)$ in the absence of contrast in Poisson's ratio. When angles approaching critical are not included, $R(\theta)$ may be adequately approximated by a parabola [equation (18)]. The approximation provides an analytic basis for the systematics first described by Koefoed, and also for a simple inversion of $R(\theta)$ to $\Delta\sigma$.

ACKNOWLEDGMENTS

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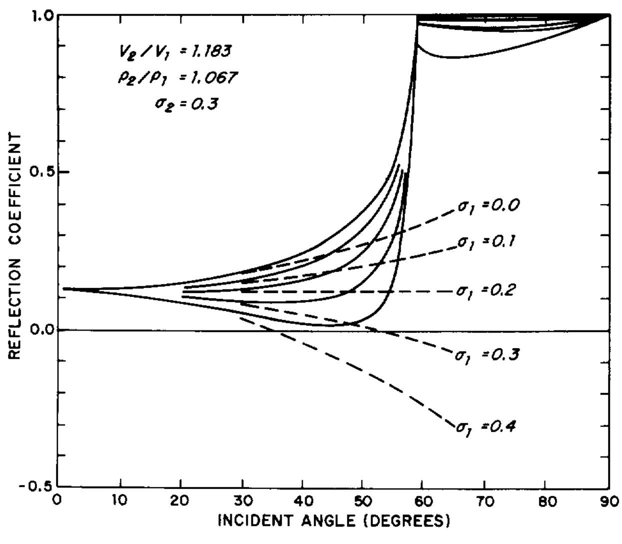

FIG. 6. As in Figure 5 but incident on the opposite side of the interface.

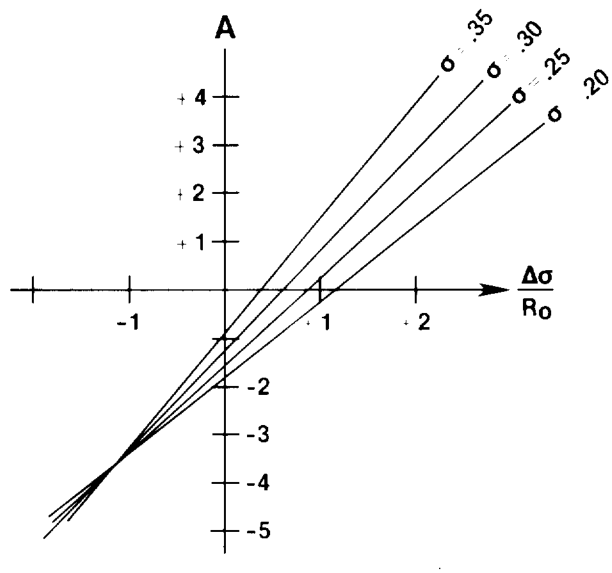

FIG. 7. Linear dependence of amplitude curvature A on the ratio $\Delta\sigma/R_0$, according to equation (11). The intercept A_0 is computed using $B = 2/3$ in equation (12).

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APPENDIX A
NO CONTRAST IN POISSON'S RATIO

The quantity A_0, given by equation (11), specifies the variation of $R(\theta)$ in the approximate range $0 < \theta < 30$ degrees for the case of no contrast in Poisson's ratio. It depends upon average Poisson's ratio σ and the ratio B [equation (13)]. Previous investigations, by Muskat and Meres (1940) found $|R(\theta)|$ to be slowly decreasing in this case, i.e., A_0 is negative and small in magnitude. Study of equation (12) confirms this is usually but not always true.

Parameter B is the ratio of the fractional change in velocity to the fractional change in impedance [equation (13)]. When the velocity change and density change have the same sign, the ratio B is in the range $0 < B < 1$, the limit $B = 0$ corresponding to no velocity change and the limit $B = 1$ corresponding to no density change. The density-velocity correlation introduced in Gardner et al. (1974) corresponds to $B = 0.8$. Frequency distributions (histograms) for B can be derived from well logs. Typically they peak near $B = 0.7$ and have only slight tails outside the range $0 < B < 1$. Figure A-1 shows that for values of B in this range, A_0 is more dependent on σ than on B, ranging from about 0.0 for very high σ to -2.0 for very low σ. At the intermediate value $\sigma = 1/3$, A_0 is -1.0 for all values of B.

Average Poisson's ratio σ can be estimated from a hypothesis about the lithologies involved and from published laboratory work such as Gregory (1976). For consolidated rocks the value $\sigma = 0.25$ has long been a standard. Unconsolidated, water-saturated clastics have a higher Poisson's ratio, approaching 0.5 for ocean-floor sediments (Hamilton, 1979). Qualitative knowledge of average lithology allows estimation of σ with an uncertainty of ± 0.05, for instance. Then, provided B is in its normal range $0 < B < 1$, Figure A-1 shows A_0 can be estimated with an uncertainty of ± 0.4.

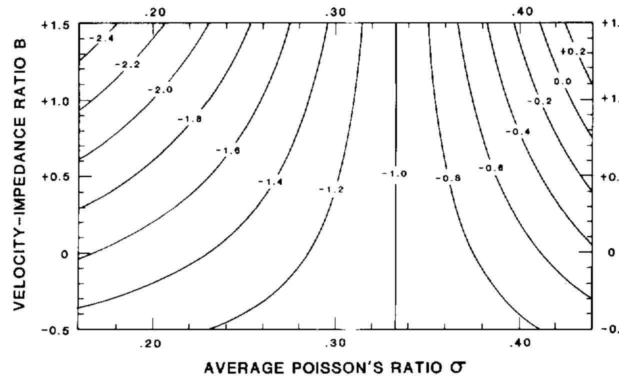

FIG. A-1. Contours of A_0 [equation (12)] as a function of average Poisson's ratio σ and velocity/impedance ratio B.

APPENDIX B
REFLECTIONS WHICH VANISH AT NORMAL INCIDENCE

This paper focuses on $R(\theta)/R_0$, the *PP* reflection amplitude at intermediate angles relative to that at normal incidence. A separate analysis applies in the degenerate case $R_0 = 0$, i.e., *PP* reflections which vanish at normal incidence. Equation (14) for absolute amplitude should be used instead of equation (9) for relative amplitude. The first term of equation (14) vanishes in the degenerate case being considered, while the third is negligible for $0 < 30$ degrees. After multiplying the product $A_0 R_0$ using equations (10), (12), and (13), I get

$$R(\theta) \approx \left[\frac{\Delta V}{V}\frac{3\sigma - 1}{2(1-\sigma)} + \frac{\Delta\sigma}{(1-\sigma)^2}\right]\sin^2\theta. \tag{B-1}$$

The brackets in equation (B-1) give the combination of elastic properties connected to the absolute amplitude of a degenerate *PP* reflection. Converted *PS* reflections have the same offset dependence, but they might be distinguished by finding the *PP* reflection from the same interface and possibly by use of shear-wave detectors.

I suggest that degenerate *PP* reflections are rare but not nonexistent. Two cases come to mind of lithologic interfaces for which R_0 [equation (10)] could be much smaller than the bracketed quantity in equation (B-1): (1) dense limestone against chert, e.g., in the Paleozoic of the western United States, and (2) salt against consolidated clastics in the deep Gulf of Mexico.