In the previous chapters, we discussed the different communication methods that are used for continuous–time signals. Starting with chapter, we will talk about digital communication techniques. Before we can study the digital communication techniques, we need to study how to convert analog signals to digital signals. The first step in this process is called SAMPLING. In sampling, we convert continuous–time analog signals (signals that are defined at all time instants and have amplitudes that may take any real value) to discrete–time analog signals (signals that are defined at specific instants of time but still have amplitudes that may take any real value).

Sampling

A continuous–time analog or digital signal is defined at all time instants. On the other hand, a discrete–time analog or digital signal is defined only at some time instants. A simple method to sample a continuous–time signal at a specific time instant is to multiply this signal by a delta function that occurs at the time instant of interest. For example, let the signal $g(t)$ be a continuous–time signal with bandwidth $2\pi B$ rad/s (B Hz). The signal $g_s(t)$ given by

$$g_s(t) = g(t)\cdot\delta(t-t_0) = g(t_0)\cdot\delta(t-t_0)$$

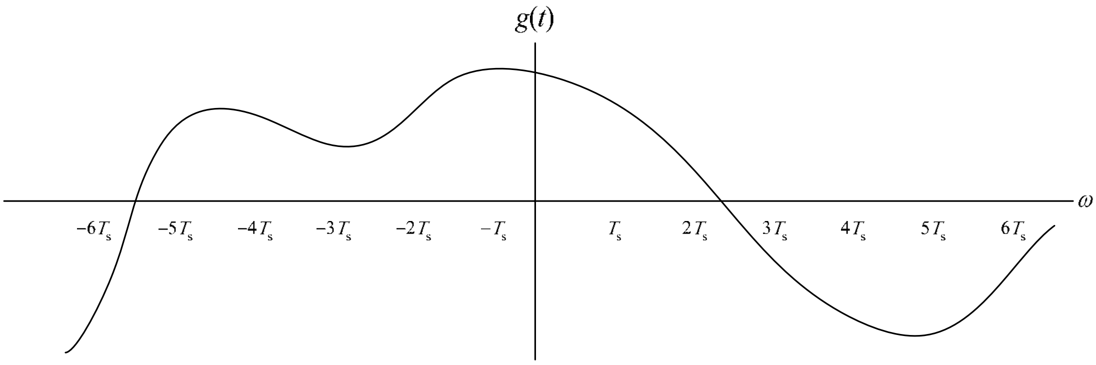
is zero everywhere and a delta function at time t_0 with an area (we will also call it magnitude) that is equal to the value of $g(t)$ evaluated at t_0, that is a magnitude of value $g(t_0)$. This represents getting only one sample of $g(t)$. If we want to sample $g(t)$ periodically every T_s then we can repeat this process periodically. That is, multiply the signal $g(t)$ by a train of delta functions that occur every T_s seconds. A train of delta function $\delta_{T_s}(t)$ that occur every T_s is given by

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT_s).$$

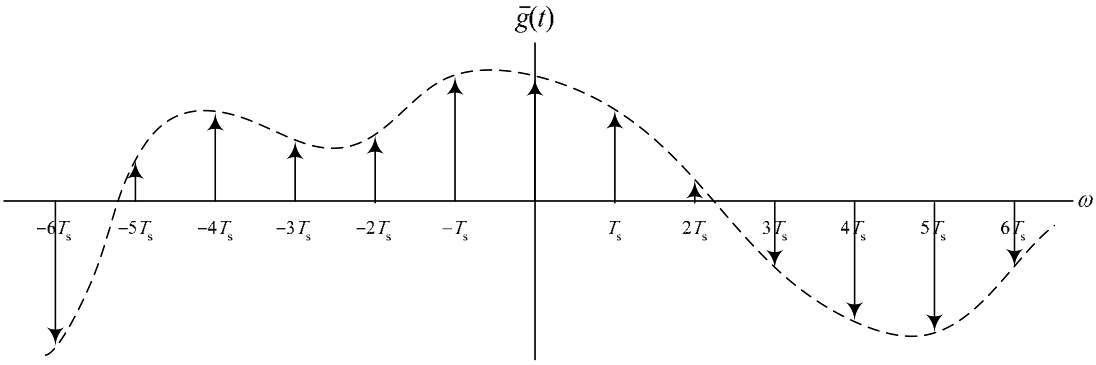
Therefore, the sampled signal $\bar{g}(t)$ is given by

$$\begin{aligned}\bar{g}(t) &= g(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\\ &= \sum_{n=-\infty}^{\infty} g(t)\cdot\delta(t-nT_s)\\ &= \sum_{n=-\infty}^{\infty} g(nT_s)\cdot\delta(t-nT_s).\end{aligned}$$

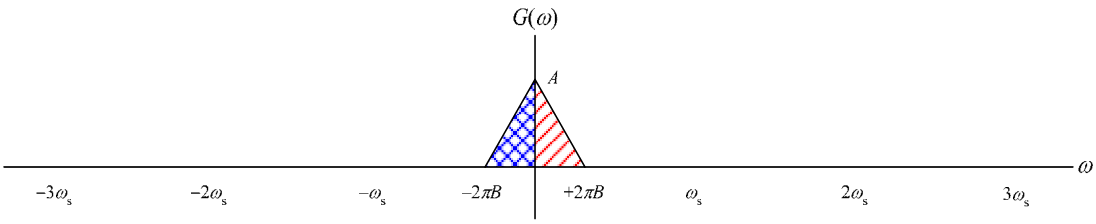
So, the sampled signal is a sum of delta functions that have magnitudes equal to the value of $g(t)$ at the time instants that the delta functions occur. The following figure shows a signal $g(t)$.



and the following figure show the sampled signal $\bar{g}(t)$ where the amplitude of the deltas follows the original signal $g(t)$.



Assume that the spectrum of $g(t)$ is given by $G(\omega)$ shown below.



We can get the spectrum of $\bar{g}(t)$ by find the spectrum of the train of delta functions and convolving it with $G(\omega)$, or by decomposing the train function into sine and cosine functions and then taking the Fourier transform of each element independently. Since the train of delta functions $\delta_{T_s}(t)$ is periodic, we can decompose it using the Fourier series as

$$
\begin{aligned}
\delta_{T_s}(t) &= \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \\
&= a_0 + a_1 \cos(\omega_s t) + a_2 \cos(2\omega_s t) + a_3 \cos(3\omega_s t) + \ldots \\
&\quad + b_1 \sin(\omega_s t) + b_2 \sin(2\omega_s t) + b_3 \sin(3\omega_s t) + \ldots
\end{aligned}
$$

where T_s and ω_s are related by

$$\omega_s = \frac{2\pi}{T_s}.$$

The value of a_0 is

$$a_0 = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta_{T_s}(t)dt = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)dt = \frac{1}{T_s},$$

and the value of the a_n for any $n \geq 1$ is

$$a_0 = \frac{2}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta_{T_s}(t)\cdot\cos(\omega_s t)dt = \frac{2}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)\cos(\omega_s t)dt = \frac{2}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)\cos(0)dt = \frac{2}{T_s}.$$

Also, the value of b_n for any $n \geq 1$ is

$$b_0 = \frac{2}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta_{T_s}(t)\cdot\sin(\omega_s t)dt = \frac{2}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)\sin(\omega_s t)dt = \frac{2}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)\sin(0)dt = 0.$$

Therefore,

$$\begin{aligned}\delta_{T_s}(t) &= \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \\ &= \frac{1}{T_s} + \frac{2}{T_s}\cos(\omega_s t) + \frac{2}{T_s}\cos(2\omega_s t) + \frac{2}{T_s}\cos(3\omega_s t) + \ldots\end{aligned}$$

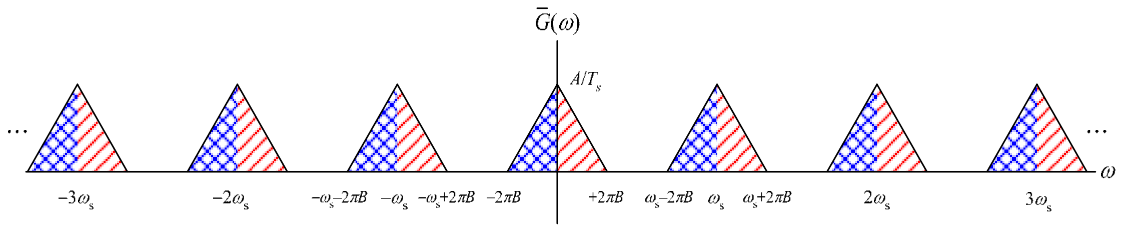
and

$$\begin{aligned}\bar{g}(t) &= g(t)\sum_{n=-\infty}^{\infty} \delta(t - nT_s) \\ &= \frac{1}{T_s}g(t) + \frac{2}{T_s}g(t)\cos(\omega_s t) + \frac{2}{T_s}g(t)\cos(2\omega_s t) + \frac{2}{T_s}g(t)\cos(3\omega_s t) + \ldots.\end{aligned}$$

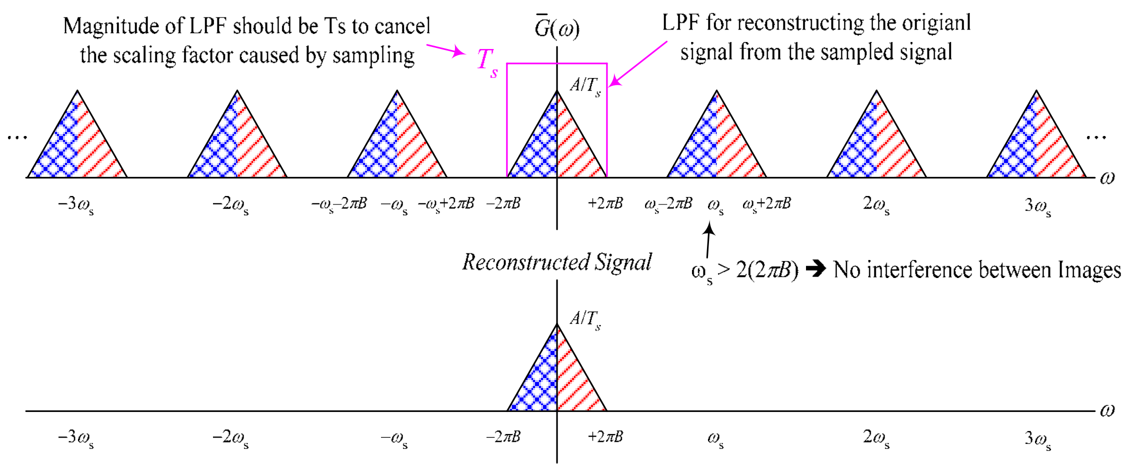
So, by taking the Fourier transform of each term of the above independently, we see that the spectrum $\bar{G}(\omega)$ is given by

$$\bar{G}(\omega)=\frac{1}{T_s}G(\omega)+\frac{1}{T_s}\left[G(\omega-\omega_s)+G(\omega+\omega_s)\right]+\frac{1}{T_s}\left[G(\omega-2\omega_s)+G(\omega+2\omega_s)\right]$$
$$+\frac{1}{T_s}\left[G(\omega-3\omega_s)+G(\omega+3\omega_s)\right]+\ldots.$$
$$=\frac{1}{T_s}\sum_{n=-\infty}^{\infty}G(\omega-n\omega_s)$$

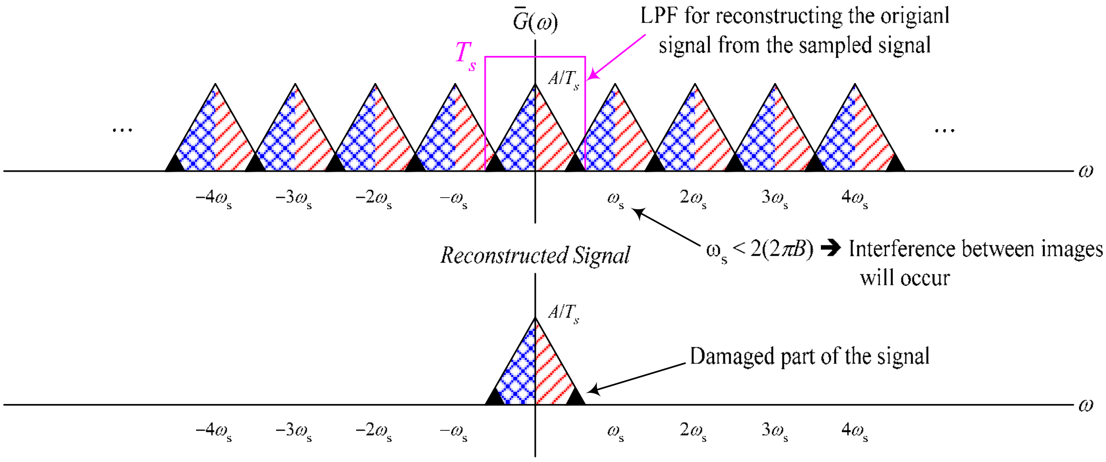
or simply scaled copies of the spectrum of the original continuous–time signal at multiples of the sampling frequency ω_s. Therefore, the spectrum of the sampled signal would be



To extract the original signal from the sampled signal, it is clear that using a LPF with bandwidth equal to the bandwidth of the original signal $g(t)$ (which is $2\pi B$ rad/s in this case) will do the job. However, this is true only if the signal was sampled at a sampling rate that is greater than twice the bandwidth of the signal.



If the signal was sampled at a sampling rate lower that 2 times the bandwidth of the signal (called the **<u>NYQUIST SAMPLING RATE</u>**), the different spectral components of the sampled signal (called IMAGES) will interfere with each other and reconstructing the original signal will be impossible. This is illustrated in the following figure. The dark parts in the figure represent parts of the sampled signal and reconstructed signal that have been damaged.



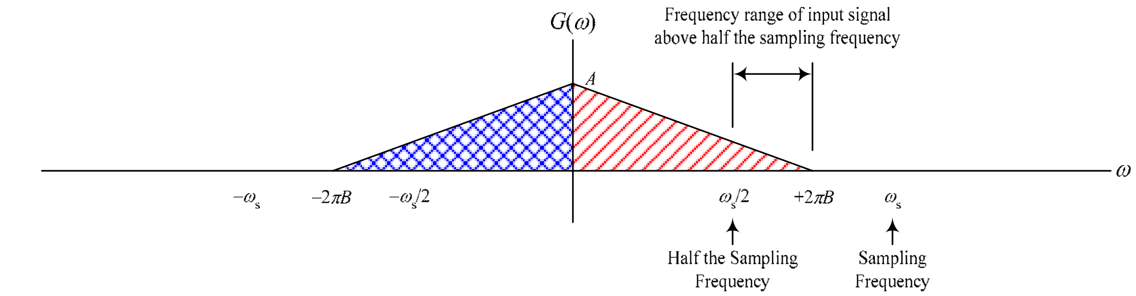
Examples of Aliasing in Real Life

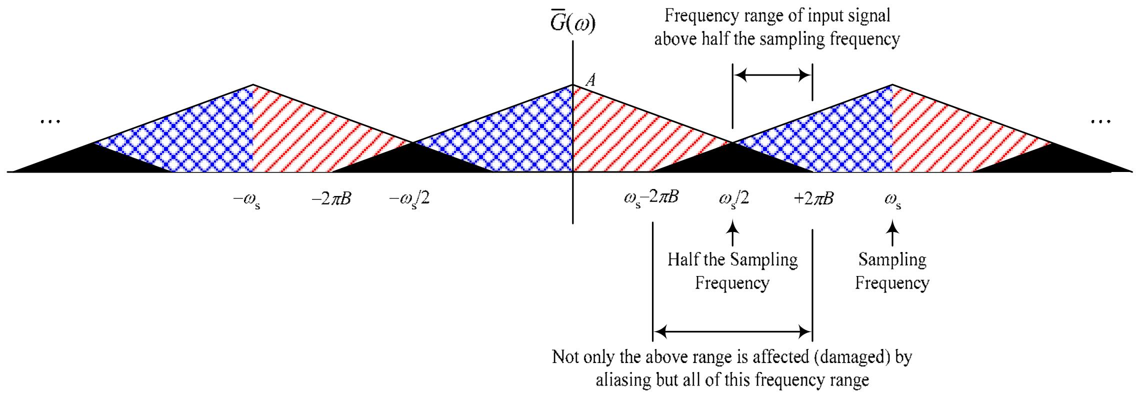
There are many real–life phenomena that result from aliasing, but many do not know that these are actually caused by aliasing. Here we will list two examples.

1. When video taping a TV or a PC monitor, sometimes wide black lines appear moving at some speed from top to bottom or vice versa across the screen. This results because of the difference in sampling (number of pictures per second) of the video camera and the number of frames the TV or PC monitor display per second.
2. When looking at something that rotates at high speed (such as a fan or a car's wheel), you sometimes see that it is rotating in the opposite direction. This also happens because the human eye works like a video camera where it also takes pictures at a rate close to 24 pictures per second. If the rotating object rotates at a high speed that by the time the eye takes the next picture that object has revolved slightly less than one rotations, this object will appear as if it is rotating in the opposite direction.

Anti–Aliasing Filters

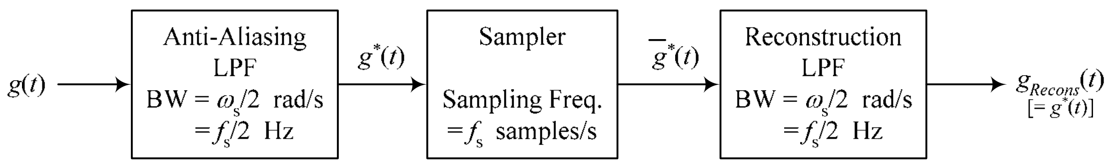
So now we know that when the whenever the bandwidth of the input signal to a sampler is greater than half the sampling frequency (in other words, the sampling frequency is less than twice the bandwidth of the input signal), aliasing will occur. Unfortunately, aliasing does not only destroy the part of the input signal that has frequency greater than half the sampling frequency, but also an equal part of input signal that is below half the sampling frequency. This is illustrated in the figure below.



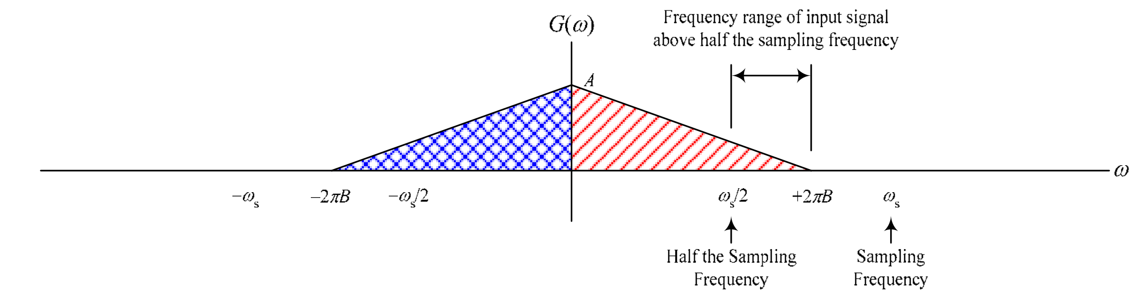


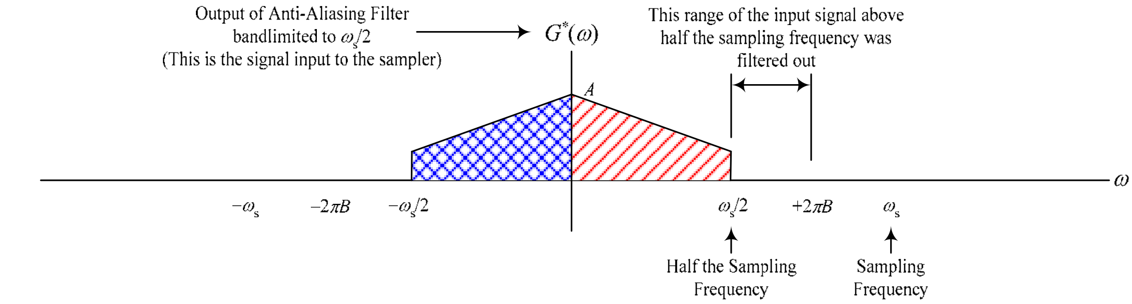
So, it is clear that not only the range of the input signal [$\omega_s/2$, $2\pi B$] gets affected by aliasing, but all the range from [ω_s–$2\pi B$, $2\pi B$] is affected by aliasing.

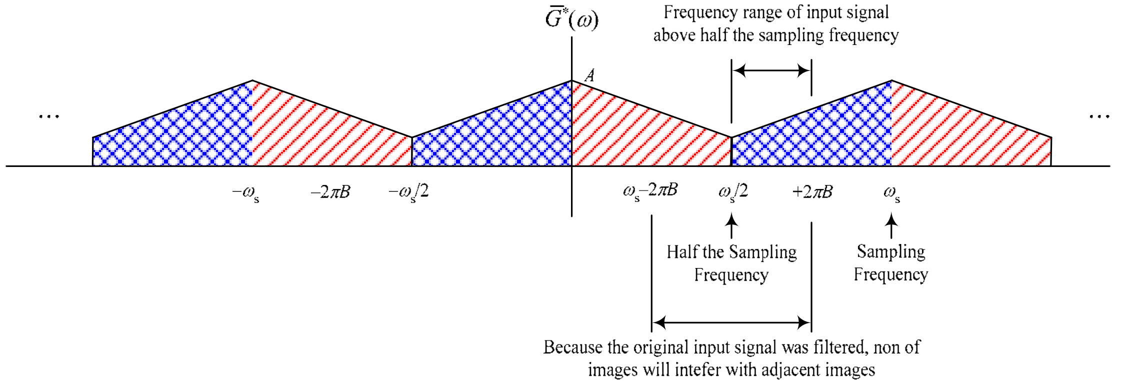
To SAVE HALF of the signal in the frequency range [ω_s–$2\pi B$, $2\pi B$], we can pass the input signal before sampling into a LPF that will cut all the part that is above $\omega_s/2$ so that the input signal to the sampling device has a bandwidth of exactly $\omega_s/2$. This means that a LPF with bandwidth $\omega_s/2$ called ANTI–ALIASING filter must be used. If the input signal to the sampler (which was produced by the anti–aliasing filter) has exactly half the sampling frequency, there will be no aliasing at all (but we will require an ideal LPF with bandwidth $\omega_s/2$ to reconstruct the continuous–time signal from the samples). Notice that the original input signal cannot be reconstructed back exactly because we removed part of it to avoid aliasing. Therefore, the block diagram of a practical sampling system is shown below.

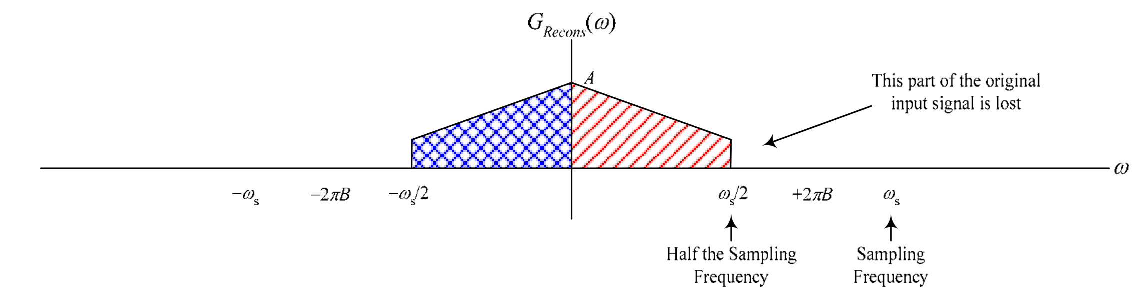


The signals in the above block diagram will be as follows.


G(ω)
Frequency range of input signal above half the sampling frequency
A
ω
−ωs
−2πB
−ωs/2
ωs/2
+2πB
ωs
Half the Sampling Frequency
Sampling Frequency


Output of Anti-Aliasing Filter bandlimited to ωs/2 (This is the signal input to the sampler)
G*(ω)
This range of the input signal above half the sampling frequency was filtered out
A
ω
−ωs
−2πB
−ωs/2
ωs/2
+2πB
ωs
Half the Sampling Frequency
Sampling Frequency


Ḡ*(ω)
Frequency range of input signal above half the sampling frequency
A
…
…
−ωs
−2πB
−ωs/2
ωs−2πB
ωs/2
+2πB
ωs
Half the Sampling Frequency
Sampling Frequency
Because the original input signal was filtered, non of images will intefer with adjacent images


G_Recons(ω)
A
This part of the original input signal is lost
ω
−ωs
−2πB
−ωs/2
ωs/2
+2πB
ωs
Half the Sampling Frequency
Sampling Frequency