Data-aided SNR estimation in time-variant Rayleigh channels

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Outline

Signal model

$$y_n = \sqrt{\rho} a_n h_n + n_n, \ n = 1, \ldots, N$$

- a_k: a known transmitted signals with $|a_n|^2 = 1$
- $\rho \overset{\text{def}}{=} \frac{E_b}{N_0}$: Average SNR
- n_k: I.I.D noise samples with $n_k \sim \mathcal{N}(0,1)$
- h_k: Rayleigh fading gain

Complex gain: Rayleigh model with Jakes' power spectrum

- Jakes' model: $h_k \sim \mathcal{N}(0,1)$ with correlation matrix $\mathbf{R}_h^J(k,l) \overset{\text{def}}{=} \mathrm{E}(h_k h_l^*) = J_0(2\pi f_d T|k-l|)$
- Variations of gain can be well approached by AR 1 model.
  $h_k = \gamma h_{k-1} + e_k$, with $e_k \sim \mathcal{N}(0, 1-\gamma^2)$ and $\gamma \overset{\text{def}}{=} J_0(2\pi f_d T)$.
  Correlation matrix: $\mathbf{R}_h^{\mathrm{AR}}(k,l) = \mathrm{E}(h_k h_l^*) = \gamma^{|k-l|}$

- f_d : Doppler frequency.

Samples vector representation

$$\mathbf{y} = \sqrt{\rho}\mathbf{A}\mathbf{h} + \mathbf{n}$$

- $\mathbf{y} \overset{\text{def}}{=} (y_1, \ldots, y_N)^T$; $\mathbf{n} \overset{\text{def}}{=} (n_1, \ldots, n_N)^T \sim \mathcal{N}(0, \mathbf{I})$
- $\mathbf{A} \overset{\text{def}}{=} \text{Diag}(a_1, \ldots, a_N)$; $\mathbf{h} \overset{\text{def}}{=} (h_1, \ldots, h_N)^T \sim \mathcal{N}(0, \mathbf{R}_h)$

Covaraince matrix

$$\mathbf{R}_y \overset{\text{def}}{=} \mathrm{E}(\mathbf{y}\mathbf{y}^H) = \rho\mathbf{A}\mathbf{R}_h\mathbf{A}^H + \mathbf{I}$$

- $\mathbf{R}_h \overset{\text{def}}{=} \mathrm{E}(\mathbf{h}\mathbf{h}^H)$: fading-channel correlation matrix $\mathbf{R}_h = \mathbf{R}_h^{\text{J}}$ or $\mathbf{R}_h = \mathbf{R}_h^{\text{AR}}$

Probability density function

$$p(\mathbf{y};\rho) = p(\mathbf{z};\rho) = \frac{1}{\pi^N \det(\mathbf{R}_z)} e^{-\mathbf{z}^H\mathbf{R}_z^{-1}\mathbf{z}}$$

- $\mathbf{R}_z \overset{\text{def}}{=} \mathbf{A}^H\mathbf{R}_y\mathbf{A} = \rho\mathbf{R}_h + \mathbf{I}$ is the covariance matrix of $\mathbf{z} \overset{\text{def}}{=} \mathbf{A}^H\mathbf{y}$

Objective: Estimation of SNR, ρ from $\mathbf{y}$.

RESULT 1

The CRB on SNR estimation under time-variant flat Rayleigh fading channel and known modulating symbols is given by:

$$\mathrm{CRB}(\rho) = \frac{1}{\mathrm{Tr}(\mathbf{R}_z^{-2}\mathbf{R}_h^2)}$$

RESULT 2

The CRB on SNR estimation over slow and uncorrelated fading channel are given by

$$\begin{aligned}\mathrm{CRB}^{\mathrm{Slow}}(\rho) &= \frac{(N\rho+1)^2}{N^2}\\ \mathrm{CRB}^{\mathrm{Uncor}}(\rho) &= \frac{(\rho+1)^2}{N}.\end{aligned}$$

Properties

1. If $\mathbf{R}_h$ is positive semidefinite, the CRB on the SNR estimation is lower bounded by C_h

$$\mathrm{CRB}(\rho) \geq C_h$$

where the constant C_h depends on the channel parameters only, and is defined by

$$C_h \stackrel{\text{def}}{=} \frac{1}{\mathrm{Tr}(\mathbf{R}_h^2)}.$$

2. For uncorrelated and slow fading channel, we have

$$\begin{aligned}\mathrm{CRB}^{\mathrm{Slow}}(\rho) &\leq \mathrm{CRB}^{\mathrm{Uncor}}(\rho),\ \rho \leq 1/\sqrt{N}\\ \mathrm{CRB}^{\mathrm{Slow}}(\rho) &\geq \mathrm{CRB}^{\mathrm{Uncor}}(\rho),\ \rho \geq 1/\sqrt{N}\end{aligned}$$

High SNR expression

Proposition 1

For high SNR, the CRB on SNR estimation under time-variant flat Rayleigh fading channel is given by

$$\mathrm{CRB}^{\mathrm{High}}(\rho) = \frac{\rho^2}{N}\left(1 + \frac{2}{\rho}\frac{\alpha_1}{N}\right), \text{with} \quad \alpha_1 = \mathrm{Tr}(\mathbf{R}_h^{-1})$$

For AR1 channel model: $\alpha_1 = \frac{N+(N-2)\gamma^2}{1-\gamma^2}$

$$\Downarrow$$

$$\mathrm{CRB}^{\mathrm{High}}(\rho) = \frac{\rho^2}{N}\left(1 + \frac{2}{\rho(1-\gamma^2)}\left((1 + \left(\frac{N-2}{N}\right)\gamma^2\right)\right)$$

Property 1

For sufficiently large N and SNR, the CRB is a monotonically decreasing function of the channel correlation parameter γ which varies from the slow fading bound ($\gamma = 1$) to the uncorrelated fading bound ($\gamma = 0$).

Low SNR expression

Proposition 2

For low SNR, the CRB on SNR estimation over time-variant flat Rayleigh fading channel is given by

$$\mathrm{CRB}^{\mathrm{Low}}(\rho) = \frac{1}{\beta_2}\left(1 + 2\rho\frac{\beta_3}{\beta_2}\right), \quad \beta_k = \mathrm{Tr}(\mathbf{R}_h^k), k = 1, 2, 3.$$

Property 2

For very low SNR, the CRB for SNR estimation under fast fading channel is upper bounded [resp. lower bounded] by the associated CRB for uncorrelated fading channel [resp. slow fading channel].

$$\mathrm{CRB}^{Slow}(\rho) \leq \mathrm{CRB}(\rho) \leq \mathrm{CRB}^{Uncor}(\rho).$$

ML ESTIMATION OF SNR

- **Definition of the ML estimation of the parameter ρ**

$$\hat{\rho} = Arg\ max_{\tau}\ L(\tau)$$

  - $L(\rho)$ is a function depends nonlinearly on ρ

$$L(\rho) = -\left(\log(\det(\mathbf{R}_z)) + \mathrm{Tr}(\mathbf{R}_z^{-1}\mathbf{Z})\right) \text{ with } \mathbf{Z} \stackrel{\text{def}}{=} \mathbf{z}\mathbf{z}^H$$

- **Difficulty to find analytical SNR ML estimate**
  - For high and low SNR, $\frac{\partial L(\rho)}{\partial \rho}$ becomes linear in $\rho \Rightarrow$ Closed-form expressions for the estimate of ρ can be found.

Result 3

In the high and low SNR environments, the ML estimate of the SNR, ρ, for the fast fading Rayleigh channel are given by:

$$\hat{\rho} = \frac{\mathrm{Tr}(\mathbf{R}_h\mathbf{Z}) - \mathrm{Tr}(\mathbf{R}_h)}{\mathrm{Tr}(\mathbf{R}_h^2)} \quad \text{for low SNR}$$

$$\hat{\rho} = \frac{\mathrm{Tr}(\mathbf{R}_h^{-1}\mathbf{Z}) - \mathrm{Tr}(\mathbf{R}_h^{-1})}{N} \quad \text{for high SNR}$$

and in the particular cases of slow (i.e., $\mathbf{R}_h = \mathbf{1}\mathbf{1}^T$) and uncorrelated (i.e., $\mathbf{R}_h = \mathbf{I}$) fading channels, we have

$$\hat{\rho} = \frac{\mathrm{Tr}(\mathbf{R}_h\mathbf{Z}) - N}{N\mathrm{Tr}(\mathbf{R}_h\mathbf{Z})} \quad \text{for slow fading}$$

$$\hat{\rho} = \frac{1}{N}(\mathrm{Tr}(\mathbf{Z}) - N) \quad \text{for uncorrelated fading}$$

SIMULATION PARAMETERS

- The transmitted sequence of known symbols is the length N.
- The channel is simulated according to the Jakes and AR1 correlation model with doppler-time product of $f_d T$.
- MSE obtained by averaging over 2500 Monte Carlo independent runs

COMPARISON OF THE CRB FOR SNR ESTIMATION CALCULATED WITH THE JAKES' AND THE AR1 CORRELATION MODELS

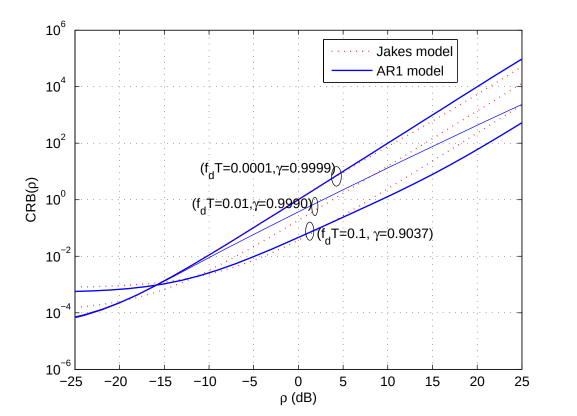

FIGURE: Exact CRB for the Jakes and AR1 correlation model versus SNR for three values of $f_d T$ with $N = 200$.

ANALYSIS: APPROXIMATED CRB

HIGH AND LOW APPROXIMATIONS OF THE CRB

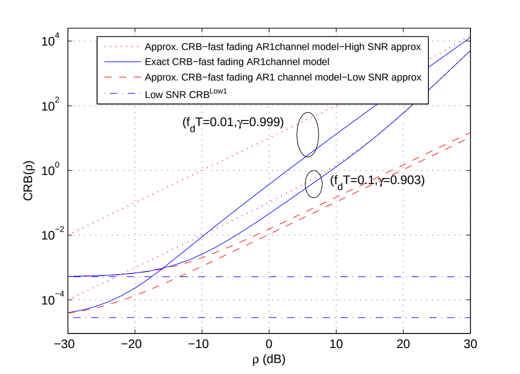

FIGURE: Exact CRB and its approximations for the AR1 correlation model versus SNR for two values of $f_d T$ with $N = 200$

ANALYSIS: SNR ESTIMATION

MSE OF THE SNR ESTIMATION VERSUS $f_d T$

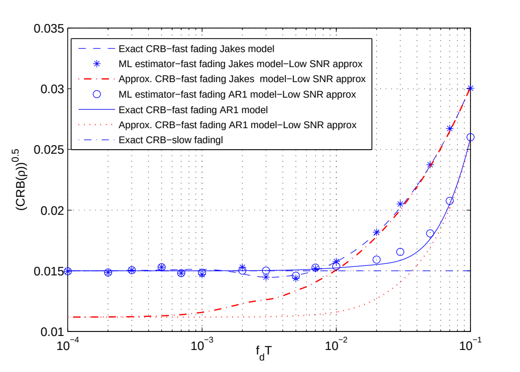

FIGURE: Exact $\mathrm{CRB}(\rho)$ with a fast fading and its approximations for the Jakes and AR1 correlation model, $\mathrm{CRB}^{\mathrm{Slow}}(\rho)$ and estimated MSE $\mathrm{E}(\hat{\rho}-\rho)^2$ given by the ML estimator, versus $f_d T$ for SNR$=-20dB$ and $N=200$.

Analysis: SNR estimation

MSE of the SNR estimation versus SNR

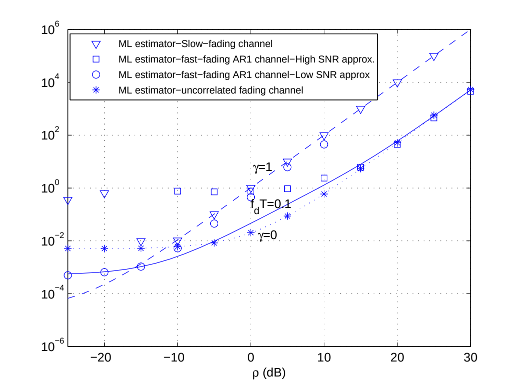

FIGURE: Exact $\mathrm{CRB}(\rho)$ with a fast fading, $\mathrm{CRB}^{\mathrm{Slow}}(\rho)$, $\mathrm{CRB}^{\mathrm{Uncor}}(\rho)$ and estimated MSE $\mathrm{E}(\hat{\rho}-\rho)^2$ given by the ML estimator, versus SNR for $N = 200$.

CONCLUSION

- DA CRB for SNR parameter over time variant flat Rayleigh fading channel:
    - Modelisation of the flat fading channel using the well-known Jakes' model and using an AR1 model.
    - Analytical closed-form expressions for the CRB over correlated, slow and uncorrelated flat fading channels.
    - Analytical approximate expressions for the CRB on SNR estimation for low and high SNR.
- Data-aided ML estimation for SNR parameter:
    - Derivation of the DA ML SNR estimators for slow fading and uncorrelated fading channel.
    - Two approximate solutions of DA ML SNR estimators for high and low SNR in the case of fast-fading channel.
    - The dependence of performance on the channels time variation.
- The simulation and theoretical results show good agreement.
- Perspectives: extension to NDA estimation, and in the case of unknown channel parameters, ...