Chapter 7. Random Process – Spectral Characteristics

7.1 Power density spectrum and its properties

Goal: Find the frequency component of RP X(t).

For deterministic signal $x(t)$ it is simple

Fourier transform $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

You can also get the time domain representation of the signal using

Inverse FT $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

This is not the case with RP X(t). Why?
How to find the frequency spectrum of RP X(t)?

7.1 Power density spectrum and its properties

$$x_T(t) = \begin{cases} x(t), & -T < t < T \\ 0, & o/w \end{cases}$$

Assume $\int_{-T}^{T} |x_T(t)| dt < \infty$, for all finite T.

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt = \int_{-T}^{T} x(t) e^{-j\omega t} dt$$

Energy contained in $x(t)$ in the interval $(-T, T)$ ← Energy saving property

$$E(T) = \int_{-\infty}^{\infty} x_T(t)^2 dt = \int_{-T}^{T} x(t)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega$$

average power in $x(t)$ over the interval $(-T, T)$

$$P(T) = \frac{1}{2T}\int_{-T}^{T} x(t)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T} d\omega \quad ☆$$

← Power=Energy/time

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7.1 Power density spectrum and its properties

For a RP X(t), we find the average power using the same equation as ☆ but with two modifications:

1. We take the expectation of X(t) to include all possible sample values.
2. We let T→ infinity to cover the whole time domain

Thus we can write the average power of RP $X(t)$ as

power density spectrum

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[X(t)^2] dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T} d\omega$$

$$P_{XX} = A\{E[X(t)^2]\}$$

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega$$

$$S_{XX} = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

← Power Density Spectrum
Power Spectral Density
PSD

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7.1 Power density spectrum and its properties

$$P_{XX} = A\{E[X(t)^2]\}$$

$$\text{w.s.s.} \quad \Rightarrow \quad P_{XX} = R_{XX}(0)$$

Example 7.1-1: $X(t) = A_0\cos(\omega_0 t + \Theta)$ Θ -- uniformly distributed on $(0, \frac{\pi}{2})$

$$E[X(t)^2] = E[A_0^2\cos^2(\omega_0 t + \Theta)] = E[\frac{A_0^2}{2} + \frac{A_0^2}{2}\cos(2\omega_0 t + 2\Theta)]$$

$$= \frac{A_0^2}{2} + \frac{A_0^2}{2}\int_0^{\frac{\pi}{2}} \frac{2}{\pi}\cos(2\omega_0 t + 2\theta)d\theta = \frac{A_0^2}{2} + \frac{A_0^2}{2\pi}\sin(2\omega_0 t + 2\theta)\Big|_{\theta=0}^{\frac{\pi}{2}}$$

$$= \frac{A_0^2}{2} - \frac{A_0^2}{\pi}\sin(2\omega_0 t)$$

$$P_{XX} = A\{E[X(t)^2]\} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\frac{A_0^2}{2} - \frac{A_0^2}{\pi}\sin(2\omega_0 t)]dt = \frac{A_0^2}{2}$$

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7.1 Power density spectrum and its properties

Example 7.1-2: $X(t) = A_0\cos(\omega_0 t + \Theta)$

$$X_T(\omega) = \int_{-T}^{T} A_0\cos(\omega_0 t + \Theta)e^{-j\omega t}dt = \int_{-T}^{T} A_0\frac{1}{2}[e^{j\Theta}e^{j\omega_0 t} + e^{-j\Theta}e^{-j\omega_0 t}]e^{-j\omega t}dt$$

$$= \frac{A_0}{2}e^{j\Theta}\int_{-T}^{T} e^{j(\omega_0-\omega)t}dt + \frac{A_0}{2}e^{-j\Theta}\int_{-T}^{T} e^{-j(\omega_0+\omega)t}dt$$

$$= A_0 T e^{j\Theta}\frac{\sin[(\omega-\omega_0)T]}{(\omega-\omega_0)T} + A_0 T e^{-j\Theta}\frac{\sin[(\omega+\omega_0)T]}{(\omega+\omega_0)T}$$

$$\int_{-T}^{T} e^{j\beta t}dt = \frac{1}{j\beta}e^{j\beta t}\Big|_{t=-T}^{T} = \frac{e^{j\beta T} - e^{-j\beta T}}{j\beta} = 2T\frac{\sin(\beta T)}{\beta T}$$

$$S_{XX} = \lim_{T\to\infty}\frac{E[|X_T(\omega)|^2]}{2T}$$ power density spectrum

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7.1 Power density spectrum and its properties

$$X_T(\omega) = A_0 T e^{j\Theta} \frac{\sin[(\omega-\omega_0)T]}{(\omega-\omega_0)T} + A_0 T e^{-j\Theta} \frac{\sin[(\omega+\omega_0)T]}{(\omega+\omega_0)T}$$

$$X_T(\omega)^* = A_0 T e^{-j\Theta} \frac{\sin[(\omega-\omega_0)T]}{(\omega-\omega_0)T} + A_0 T e^{j\Theta} \frac{\sin[(\omega+\omega_0)T]}{(\omega+\omega_0)T}$$

$$|X_T(\omega)|^2 = X_T(\omega) X_T(\omega)^* = A_0^2 [T^2 \frac{\sin^2[(\omega-\omega_0)T]}{(\omega-\omega_0)^2 T^2} + T^2 \frac{\sin^2[(\omega+\omega_0)T]}{(\omega+\omega_0)^2 T^2}]$$

$$+ A_0^2 T^2 (e^{j2\Theta} + e^{-j2\Theta}) \frac{\sin[(\omega-\omega_0)T]}{(\omega-\omega_0)T} \frac{\sin[(\omega+\omega_0)T]}{(\omega+\omega_0)T}$$

$$E[e^{j2\Theta} + e^{-j2\Theta}] = E[2\cos 2\Theta] = \int_0^{\frac{\pi}{2}} \frac{2}{\pi} 2\cos 2\theta d\theta = \frac{2}{\pi} \sin 2\theta \Big|_0^{\pi/2} = 0$$

$$\frac{E[|X_T(\omega)|^2]}{2T} = \frac{A_0^2 \pi}{2} [\frac{T}{\pi} \frac{\sin^2[(\omega-\omega_0)T]}{(\omega-\omega_0)^2 T^2} + \frac{T}{\pi} \frac{\sin^2[(\omega+\omega_0)T]}{(\omega+\omega_0)^2 T^2}]$$

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7.1 Power density spectrum and its properties

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi \qquad \text{(C-54)}$$

$$\int_{-\infty}^{\infty} \frac{T}{\pi} \frac{\sin^2(\alpha T)}{(\alpha T)^2} d\alpha = \int_{-\infty}^{\infty} \frac{T}{\pi} \frac{\sin^2 x}{x^2} \frac{1}{T} dx = 1 \qquad \text{(a)}$$

$$\lim_{T\to\infty} \frac{T}{\pi} \frac{\sin^2(\alpha T)}{(\alpha T)^2} = \begin{cases} \infty, & \text{if } \alpha = 0 \\ 0, & \text{if } \alpha \neq 0 \end{cases} \qquad \text{(b)}$$

$$\text{(a) \& (b)} \Rightarrow \quad \lim_{T\to\infty} \frac{T}{\pi} \frac{\sin^2(\alpha T)}{(\alpha T)^2} = \delta(\alpha)$$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T} = \frac{A_0^2 \pi}{2} [\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A_0^2 \pi}{2} [\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] d\omega = \frac{A_0^2}{2}$$

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7.1 Power density spectrum and its properties

Properties of the power density spectrum:

(1) $S_{XX}(\omega) \geq 0$

(2) $X(t)$ real $\Rightarrow$ $S_{XX}(-\omega) = S_{XX}(\omega)$

(3) $S_{XX}(\omega)$ is real

(4) $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = A\{E[X(t)^2]\}$

$$S_{XX}(\omega) = \lim_{T\to\infty}\frac{E[|X_T(\omega)|^2]}{2T}$$

PF of (2): $X_T(\omega) = \int_{-T}^{T} X(t)e^{-j\omega t}dt$

$$X_T(\omega)^* = \int_{-T}^{T} X(t)^* e^{j\omega t}dt = \int_{-T}^{T} X(t)e^{j\omega t}dt = X_T(-\omega)$$

$$S_{XX}(-\omega) = \lim_{T\to\infty}\frac{E[X_T(-\omega)X_T(-\omega)^*]}{2T} = \lim_{T\to\infty}\frac{E[X_T(\omega)^* X_T(\omega)]}{2T} = S_{XX}(\omega)$$

7.1 Power density spectrum and its properties

Properties of the power density spectrum

(5) $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$ $\qquad \frac{d}{dt}X(t) = \lim_{\varepsilon\to 0}\frac{X(t+\varepsilon)-X(t)}{\varepsilon}$

PF of (5):

$$\dot{X}_T(t) = \begin{cases} \lim_{\varepsilon\to 0}\frac{X(t+\varepsilon)-X(t)}{\varepsilon}, & -T < t < T \\ 0, & \text{o/w} \end{cases}$$

$$f(t-a) \xleftrightarrow{FT} F(\omega)e^{-j\omega a}$$

$$\dot{X}_T(t) \xleftrightarrow{\text{FT}} \lim_{\varepsilon\to 0}\frac{X_T(\omega)e^{j\omega\varepsilon} - X_T(\omega)}{\varepsilon} = j\omega X_T(\omega)$$

$$S_{\dot{X}\dot{X}}(\omega) = \lim_{T\to\infty}\frac{E[|\dot{X}_T(\omega)|^2]}{2T} = \lim_{T\to\infty}\frac{E[|j\omega X_T(\omega)|^2]}{2T} = \omega^2\lim_{T\to\infty}\frac{E[|X_T(\omega)|^2]}{2T} = \omega^2 S_{XX}(\omega)$$

7.1 Power density spectrum and its properties

Bandwidth of the power density spectrum

$X(t)$ real $\Rightarrow$ $S_{XX}(\omega)$ even

$S_{XX}(\omega)$ lowpass form $\Rightarrow$ $$W_{\text{rms}}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega}$$

root mean square Bandwidth

$S_{XX}(\omega)$ bandpass form $\Rightarrow$ $$\bar{\omega}_0 = \frac{\int_0^{\infty} \omega S_{XX}(\omega) d\omega}{\int_0^{\infty} S_{XX}(\omega) d\omega}$$ mean frequency

$$W_{\text{rms}}^2 = \frac{4\int_0^{\infty} (\omega - \bar{\omega}_0)^2 S_{XX}(\omega) d\omega}{\int_0^{\infty} S_{XX}(\omega) d\omega}$$ rms BW

7.1 Power density spectrum and its properties

Example 7.1-3: $S_{XX}(\omega) = \dfrac{10}{[1+(\omega/10)^2]^2}$ $S_{XX}(\omega)$ lowpass form

$$\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = \int_{-\infty}^{\infty} \frac{10}{[1+(\omega/10)^2]^2} d\omega = \int_{-\pi/2}^{\pi/2} \frac{10}{[1+\tan^2\theta]^2} 10\sec^2\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{100}{\sec^2\theta} d\theta = \int_{-\pi/2}^{\pi/2} 100\cos^2\theta \, d\theta = \int_{-\pi/2}^{\pi/2} 100 \frac{1+\cos 2\theta}{2} d\theta = 50\pi$$

$$\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega = \int_{-\infty}^{\infty} \frac{10\omega^2}{[1+(\omega/10)^2]^2} d\omega = 5000\pi$$

$$W_{\text{rms}}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega} = 100$$

rms BW $W_{\text{rms}} = 10$ rad/sec

7.2 Relationship between power spectrum and autocorrelation function

(6) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = A[R_{XX}(t,t+\tau)]$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)]e^{-j\omega\tau}d\tau$$

$$S_{XX}(\omega) = \lim_{T\to\infty}\frac{E[X_T(\omega)^* X_T(\omega)]}{2T} = \lim_{T\to\infty}\frac{1}{2T}E[\int_{-T}^{T} X(t_1)e^{j\omega t_1}dt_1 \int_{-T}^{T} X(t_2)e^{-j\omega t_2}dt_2]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[X(t_1)X(t_2)]e^{j\omega(t_1-t_2)}dt_2 dt_1$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{j\omega(t_1-t_2)}dt_2 dt_1$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{j\omega(t_1-t_2)}dt_2 dt_1\, e^{j\omega\tau}d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau+t_1-t_2)}d\omega dt_2 dt_1$$

7.2 Relationship between power spectrum and autocorrelation function

$$\delta(t) \xleftrightarrow{FT} 1$$

$$\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt = 1$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}d\omega$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\delta(\tau+t_1-t_2)dt_2 dt_1$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XX}(t_1,t_1+\tau)dt_1 = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XX}(t,t+\tau)dt$$

$$= A[R_{XX}(t,t+\tau)]$$

$$A[R_{XX}(t,t+\tau)] \xleftrightarrow{FT} S_{XX}(\omega)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)]e^{-j\omega\tau}d\tau$$

7.2 Relationship between power spectrum and autocorrelation function

$X(t)$ w.s.s. $\Rightarrow A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$

$$R_{XX}(\tau) \overset{FT}{\longleftrightarrow} S_{XX}(\omega)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

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7.2 Relationship between power spectrum and autocorrelation function

Example 7.2-1: $X(t) = A\cos(\omega_0 t + \Theta)$

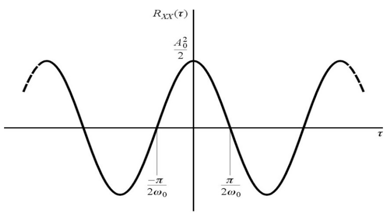
by Ex 6.2-1, $R_{XX}(\tau) = \dfrac{A_0^2}{2}\cos(\omega_0 \tau)$

$$R_{XX}(\tau) = \frac{A_0^2}{4}(e^{j\omega_0\tau} + e^{-j\omega_0\tau})$$

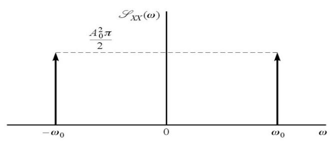
$$x(t)e^{j\alpha t} \overset{FT}{\longleftrightarrow} X(\omega - \alpha)$$

$$1 \overset{FT}{\longleftrightarrow} 2\pi\delta(\omega)$$

$$S_{XX}(\omega) = \frac{A_0^2 \pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$



(a)



(b)

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7.2 Relationship between power spectrum and autocorrelation function

Example 7.2-2: $X(t)$ -- w.s.s.

$$R_{XX}(\tau)=\begin{cases} A_0[1-\dfrac{|\tau|}{T}], & -T<\tau<T \\ 0, & \text{elsewhere} \end{cases}$$

$$S_{XX}(\omega)=\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau = A_0\int_{-T}^{0}(1+\frac{\tau}{T})e^{-j\omega\tau}d\tau + A_0\int_{0}^{T}(1-\frac{\tau}{T})e^{-j\omega\tau}d\tau$$

$$\int_{-T}^{0}(1+\frac{\tau}{T})e^{-j\omega\tau}d\tau = \int_{T}^{0}(1-\frac{\alpha}{T})e^{j\omega\alpha}(-d\alpha) = \int_{0}^{T}(1-\frac{\tau}{T})e^{j\omega\tau}d\tau$$

$$S_{XX}(\omega)=A_0\int_0^T(1-\frac{\tau}{T})(e^{j\omega\tau}+e^{-j\omega\tau})d\tau = 2A_0\int_0^T(1-\frac{\tau}{T})\cos(\omega\tau)d\tau$$

$$v=1-\frac{\tau}{T} \;\Rightarrow\; v'=\frac{-1}{T} \qquad u'=\cos(\omega\tau) \;\Rightarrow\; u=\frac{1}{\omega}\sin(\omega\tau)$$

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7.2 Relationship between power spectrum and autocorrelation function

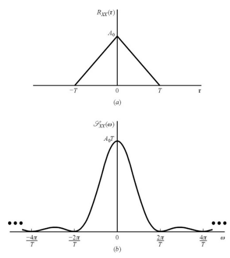
$$S_{XX}(\omega)=2A_0(1-\frac{\tau}{T})\frac{\sin(\omega\tau)}{\omega}\Bigg|_0^T$$

$$+\frac{2A_0}{T}\int_0^T\frac{\sin(\omega\tau)}{\omega}d\tau$$

$$=\frac{2A_0}{T}\frac{-\cos(\omega\tau)}{\omega^2}\Bigg|_0^T = \frac{2A_0}{\omega^2 T}[1-\cos(\omega T)]$$

$$=4A_0\frac{\sin^2(\omega T/2)}{\omega^2 T} = A_0 T\frac{\sin^2(\omega T/2)}{(\omega T/2)^2}$$

$$=A_0 T\,\mathrm{Sa}^2(\omega T/2)$$



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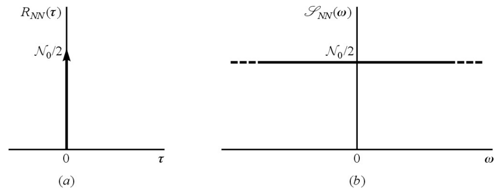
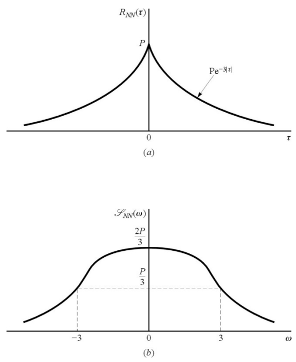
7.6 Some noise definitions and other topics

white noise & colored noise $N(t)$



white noise



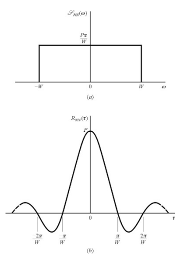
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)d\omega = \infty \quad \Rightarrow \quad \text{unrealizable}$$

colored noise

7.6 Some noise definitions and other topics

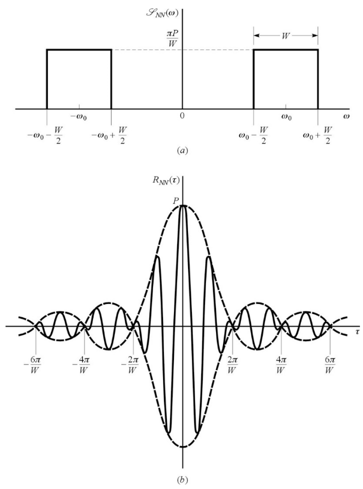
band-limited white noise

lowpass type



$$R_{NN}(\tau) = P\frac{\sin(W\tau)}{W\tau}$$

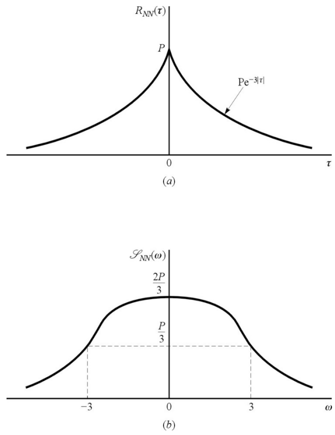
bandpass type



$$R_{NN}(\tau) = P\frac{\sin(W\tau/2)}{(W\tau/2)}\cos(\omega_0\tau)$$

7.6 Some noise definitions and other topics

Example 7.6-1: w.s.s. $N(t)$



$$R_{NN}(\tau) = Pe^{-3|\tau|}$$

$$S_{NN}(\omega) = \int_{-\infty}^{\infty} Pe^{-3|\tau|} e^{-j\omega\tau} d\tau$$

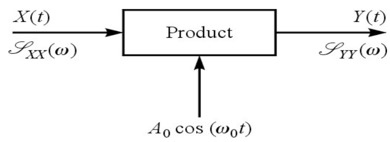
$$= \int_{-\infty}^{0} Pe^{(3-j\omega)\tau} d\tau + \int_{0}^{\infty} Pe^{-(3+j\omega)\tau} d\tau$$

$$= P\frac{1}{3-j\omega} e^{(3-j\omega)\tau}\Big|_{-\infty}^{0} + P\frac{-1}{3+j\omega} e^{-(3+j\omega)\tau}\Big|_{0}^{\infty}$$

$$= P[\frac{1}{3-j\omega} + \frac{1}{3+j\omega}] = \frac{6P}{9+\omega^2}$$

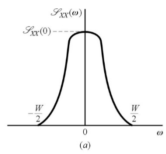
7.6 Some noise definitions and other topics

$$Y(t) = X(t)A_0 \cos(\omega_0 t)$$



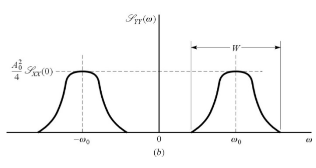
$$R_{YY}(t,t+\tau) = E[Y(t)Y(t+\tau)] = E[A_0^2 X(t)X(t+\tau)\cos(\omega_0 t)\cos(\omega_0 t + \omega_0\tau)]$$

$$= \frac{A_0^2}{2} R_{XX}(t,t+\tau)[\cos(\omega_0\tau) + \cos(2\omega_0 t + \omega_0\tau)]$$



$X(t)$ -- w.s.s. $\Rightarrow$ $Y(t)$ -- NOT w.s.s.

$$A[R_{YY}(t,t+\tau)] = \frac{A_0^2}{2} R_{XX}(\tau)\cos(\omega_0\tau)$$



$$S_{YY}(\omega) = \frac{A_0^2}{4}[S_{XX}(\omega-\omega_0) + S_{XX}(\omega+\omega_0)]$$

7.6 Some noise definitions and other topics

Example 7.6-2:

$$Y(t) = X(t)A_0\cos(\omega_0 t)$$

$$S_{XX}(\omega) = \begin{cases} N_0/2, & |\omega+\omega_0| < W_{RF}/2 \\ N_0/2, & |\omega-\omega_0| < W_{RF}/2 \\ 0, & \text{o/w} \end{cases}$$

$$S_{YY}(\omega) = \begin{cases} N_0A_0^2/8, & |\omega+2\omega_0| < W_{RF}/2 \\ N_0A_0^2/4, & |\omega| < W_{RF}/2 \\ N_0A_0^2/8, & |\omega-2\omega_0| < W_{RF}/2 \\ 0, & \text{o/w} \end{cases}$$

After lowpass filtering, $S_{YY}(\omega) = \begin{cases} N_0A_0^2/4, & |\omega| < W_{RF}/2 \\ 0, & \text{o/w} \end{cases}$

output noise power = $\frac{1}{2\pi}\int_{-W_{RF}/2}^{W_{RF}/2} \frac{N_0A_0^2}{4} d\omega = \frac{N_0A_0^2W_{RF}}{8\pi}$

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