Chapter 6. Random Process – Temporal Characteristics

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1. The Random Process Concept
2. Stationary and Independence
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5. Gaussian Random Process
6. Poisson Random Process

6.1 The Random Process Concept

R.P. = collection of r.v.'s

$$X[n], \quad n = 0,1,2,\cdots \qquad X[n], \quad n \in Z^{+} = \{0,1,2,\cdots\} = \{0\} \cup N$$

$$X[0], \quad X[1], \quad X[2], \quad \cdots \quad \text{are r.v.'s}$$

Ex 1: $X[n], \quad n \in Z,$ where $\cdots, \quad X[-1], \quad X[0], \quad X[1], \quad \cdots$

are iid r.v.'s with $f_X(x;n) = f_{X[n]}(x) = u(x) - u(x-1)$

Ex 2: $X[n], \quad n \in Z,$ where $X[n] = A$ and A is a r.v. with $f_A(a)$

6.1 The Random Process Concept

continuous time

$$X(t), \quad -\infty < t < \infty$$

Ex 3: $X(t) = A, \quad -\infty < t < \infty$

where A is a r.v. with $f_A(a)$

Ex 4: $X(t), \quad -\infty < t < \infty$ (Poisson R.P.)

$$P[X(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \cdots$$

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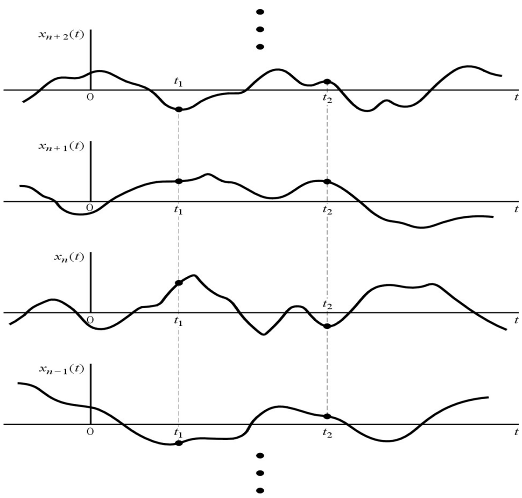
6.1 The Random Process Concept

R.P. $X(t)$

$x(t, s)$

$X(t_1) = X_1$ -- r.v.

$x(t)$ -- a sample function



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6.1 The Random Process Concept

Classification of R.P.

1. continuous random process
2. discrete random process
3. continuous random sequence
4. discrete random sequence

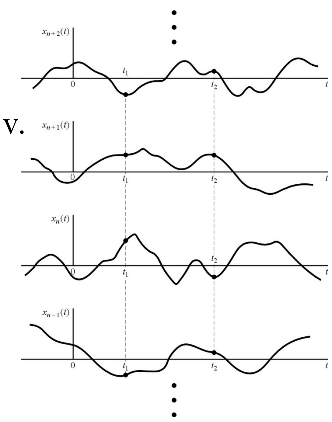
$$\text{continuous time } t \Rightarrow X(t) = \text{R. Process}$$
$$\text{discrete time } n \Rightarrow X[n] = \text{R. Sequence}$$

6.1 The Random Process Concept

continuous random process

t -- continuous

$X(t)$ -- continuous r.v.

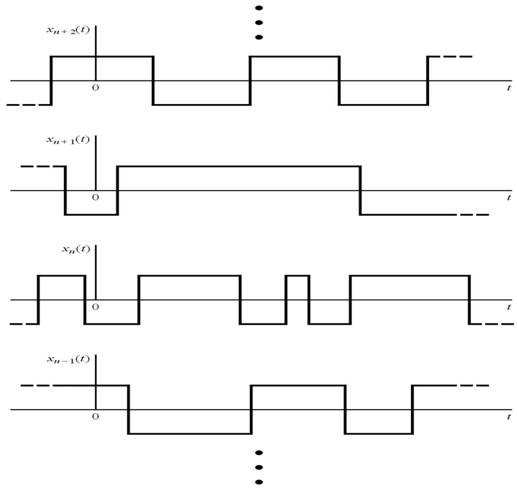


6.1 The Random Process Concept

discrete random process

t -- continuous

$X(t)$ -- discrete r.v.

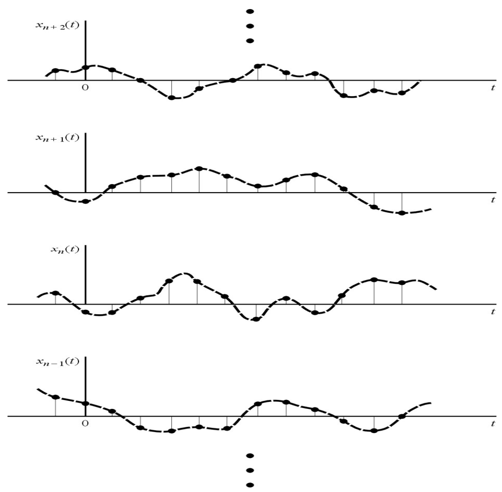


6.1 The Random Process Concept

continuous random sequence

n -- discrete

$X[n]$ -- continuous r.v.

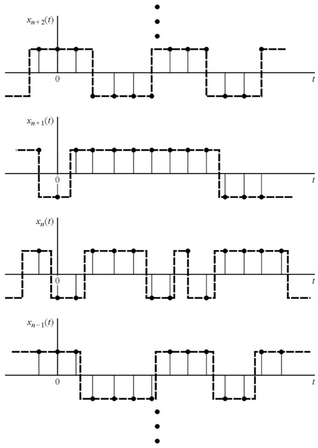


6.1 The Random Process Concept

discrete random sequence

n -- discrete

$X[n]$ -- discrete r.v.



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6.1 The Random Process Concept

Nondeterministic R.P. -- future values of any sample function can not be predicted exactly from the past values.

Most R.P.s are non-deterministic

Deterministic R.P. -- future values of any sample function can be predicted exactly from the past values.

Ex: $X(t) = A\cos(\omega_0 t + \theta), \quad A, \omega_0, \theta :$ r.v.'s

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6.2 Stationary and Independence

Stationary R.P. -- all its statistical properties do not change with time.

Nonstationary R.P. -- not stationary

$$F_X(x_1;t_1) = P\{X(t_1) \le x_1\}$$ 1st-order distribution

$$f_X(x_1;t_1) = \frac{d}{dx_1} F_X(x_1;t_1)$$ 1st-order density

$$F_X(x_1,x_2;t_1,t_2) = P\{X(t_1) \le x_1,\ X(t_2) \le x_2\}$$ 2nd-order joint distribution

$$f_X(x_1,x_2;t_1,t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1,x_2;t_1,t_2)$$ 2nd-order joint density

6.2 Stationary and Independence

Nth-order joint distribution & density

$$F_X(x_1,\cdots,x_N;t_1,\cdots,t_N) \qquad f_X(x_1,\cdots,x_N;t_1,\cdots,t_N)$$

Ex 3: $X(t) = A, \quad -\infty < t < \infty \qquad f_A(a)$

$$f_X(x_1;t_1) = f_A(x_1) \qquad f_X(x_1,x_2;t_1,t_2) = f_A(x_1)\delta(x_1 - x_2)$$

Mean (ensemble average)
$$E[X(t)] = \int_{-\infty}^{\infty} x f_X(x;t)dx$$
$$E[X(t)] = \sum_i x_i P\{X(t) = x_i\}$$

Autocorrelation function of $X(t)$

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1,x_2;t_1,t_2)dx_1 dx_2$$

6.2 Stationary and Independence

Autocovariance function of $X(t)$

$$C_{XX}(t_1,t_2) = E[\{X(t_1) - E[X(t_1)]\}\{X(t_2) - E[X(t_2)]\}]$$
$$= R_{XX}(t_1,t_2) - E[X(t_1)]E[X(t_2)]$$

Cross-correlation function of $X(t)$ & $Y(t)$

$$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_1 f_{XY}(x_1,y_1;t_1,t_2)dx_1 dy_1$$

Cross-covariance function of $X(t)$ & $Y(t)$

$$C_{XY}(t_1,t_2) = E[\{X(t_1) - E[X(t_1)]\}\{Y(t_2) - E[Y(t_2)]\}$$
$$= R_{XY}(t_1,t_2) - E[X(t_1)]E[Y(t_2)]$$

6.2 Stationary and Independence

$X(t)$ & $Y(t)$ -- independent $\Leftrightarrow$

$$f_{XY}(x_1,\cdots,x_N,y_1,\cdots,y_M;t_1,\cdots,t_N,\tau_1,\cdots,\tau_M)$$
$$= f_X(x_1,\cdots,x_N;t_1,\cdots,t_N)f_Y(y_1,\cdots,y_M;\tau_1,\cdots,\tau_M)$$

$X(t)$ & $Y(t)$ -- uncorrelated $\Leftrightarrow$ $C_{XY}(t_1,t_2) = 0$

$X(t)$ & $Y(t)$ -- orthogonal $\Leftrightarrow$ $R_{XY}(t_1,t_2) = 0$

indep. $\Rightarrow$ uncorr.

6.2 Stationary and Independence

1st-order stationary (stationary to order one)

$$f_X(x_1;t_1) = f_X(x_1;t_1+\Delta), \quad \forall t_1, \Delta$$

$$E[X(t_1+\Delta)] = \int_{-\infty}^{\infty} x f_X(x;t_1+\Delta)dx = \int_{-\infty}^{\infty} x f_X(x;t_1)dx = E[X(t_1)]$$

$$E[X(t)] = \overline{X} = \text{constant}$$

6.2 Stationary and Independence

2nd-order stationary (stationary to order two)

$$f_X(x_1,x_2;t_1,t_2) = f_X(x_1,x_2;t_1+\Delta,t_2+\Delta), \quad \forall t_1,t_2,\Delta$$

$$f_X(x_1,x_2;t_1,t_2) = f_X(x_1,x_2;0,t_2-t_1)$$

2nd-order stationary $\Rightarrow$ 1st-order stationary

$$f_X(x_1;t_1) = \int_{-\infty}^{\infty} f_X(x_1,x_2;t_1,t_2)dx_2 = \int_{-\infty}^{\infty} f_X(x_1,x_2;t_1+\Delta,t_2+\Delta)dx_2 = f_X(x_1;t_1+\Delta)$$

2nd-order stationary $\Rightarrow$ $R_{XX}(t_1,t_1+\tau) = E[X(t_1)X(t_1+\tau)] = R_{XX}(\tau)$

6.2 Stationary and Independence

widesense stationary $\Leftrightarrow$
$$E[X(t)] = \overline{X} = const$$
$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

2nd-order stationary $\underset{\not\Leftarrow}{\Rightarrow}$ widesense stationary

Nth-order stationary

$$f_X(x_1,\cdots,x_N;t_1,\cdots,t_N) = f_X(x_1,\cdots,x_N;t_1+\Delta,\cdots,t_N+\Delta)$$

Nth-order stationary $\Rightarrow$ kth-order stationary ($k < N$)

strict-sense stationary = stationary to all orders

6.2 Stationary and Independence

Ex 6.2-1: $X(t) = A\cos(\omega_0 t + \Theta)$ A, ω_0 : constants

$$f_\Theta(\theta) = \frac{1}{2\pi}\{u(\theta) - u(\theta - 2\pi)\}$$

$$E[X(t)] = \int_0^{2\pi} A\cos(\omega_0 t + \theta)\frac{1}{2\pi}d\theta = 0$$

$$R_{XX}(t,t+\tau) = E[X(t)X(t+\tau)] = A^2 E[\cos(\omega_0 t + \Theta)\cos(\omega_0(t+\tau)+\Theta)]$$

$$= \frac{A^2}{2}E[\cos(2\omega_0 t + \omega_0\tau + 2\Theta) + \cos(\omega_0\tau)]$$

$$= \frac{A^2}{2}E[\cos(\omega_0\tau)] + \frac{A^2}{2}E[\cos(2\omega_0 t + \omega_0\tau + 2\Theta)] = \frac{A^2}{2}\cos(\omega_0\tau)$$

$\Rightarrow$ $X(t)$ -- w.s.s. (widesense stationary)

6.2 Stationary and Independence

jointly widesense stationary

$$X(t) \quad \& \quad Y(t) \quad \text{w.s.s.}$$

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$$

Time average

$$A[\bullet] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\bullet] dt$$

$$\overline{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt$$

time autocorrelation function

$$\Re_{xx}(\tau) = A[x(t)x(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau) dt$$

6.2 Stationary and Independence

time cross-correlation function

$$\Re_{xy}(\tau) = A[x(t)y(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t+\tau) dt$$

ergodic $\Leftrightarrow$ $\overline{x} = \overline{X}$, $\Re_{xx}(\tau) = R_{XX}(\tau)$

jointly ergodic $\Leftrightarrow$ ergodic $X(t)$ and $Y(t)$, $\Re_{xy}(\tau) = R_{XY}(\tau)$

Mean-ergodic => $\overline{x} = \overline{X}$

Correlation-ergodic => $\Re_{xx}(\tau) = R_{XX}(\tau)$

6.3 Correlation Functions

Properties of autocorrelation function of w.s.s. r.p.:

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

(1) $|R_{XX}(\tau)| \le R_{XX}(0)$

(2) $R_{XX}(-\tau) = R_{XX}(\tau)$

(3) $R_{XX}(0) = E[X(t)^2]$

(4) stationary & ergodic $X(t)$ with no periodic components

$$\Rightarrow \lim_{|\tau|\to\infty} R_{XX}(\tau) = \overline{X}^2$$

(5) stationary $X(t)$ has a periodic component

$\Rightarrow$ $R_{XX}(\tau)$ has a periodic component with the same period.

6.3 Correlation Functions

Ex 6.3-1: stationary, ergodic with no periodic component

$$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \overline{X}^2 \quad \Rightarrow \quad E[X(t)] = \overline{X} = \pm\sqrt{25} = \pm 5$$

$$\sigma_X^2 = E[X^2(t)] - (E[X(t)])^2 = R_{XX}(0) - \overline{X}^2 = 29 - 25 = 4$$

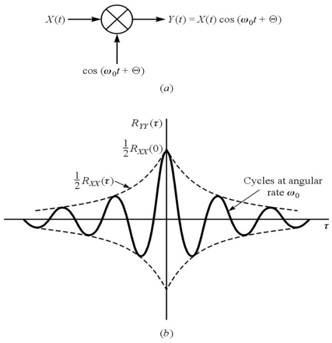
Ex 6.3-2: w.s.s. $X(t)$ $R_{XX}(\tau) = e^{-a|\tau|}, \quad a > 0$

$Y(t) = X(t)\cos(\omega_0 t + \Theta)$ $X(t)$ & Θ indep.

$\omega_0 =$ constant $f_\Theta(\theta) = \dfrac{1}{2\pi}\{u(\theta+\pi) - u(\theta-\pi)\}$

6.3 Correlation Functions

$$R_{YY}(t,t+\tau)=E[Y(t)Y(t+\tau)]$$
$$=E[X(t)X(t+\tau)\cos(\omega_0 t+\Theta)\cos(\omega_0 t+\omega_0\tau+\Theta)]$$
$$=E[X(t)X(t+\tau)]E[\cos(\omega_0 t+\Theta)\cos(\omega_0 t+\omega_0\tau+\Theta)]$$
$$=R_{XX}(\tau)\frac{1}{2}E[\cos(2\omega_0 t+\omega_0\tau+2\Theta)+\cos(\omega_0\tau)]$$
$$=\frac{1}{2}R_{XX}(\tau)\cos(\omega_0\tau)=R_{YY}(\tau)$$



6.3 Correlation Functions

Properties of cross-correlation function of jointly w.s.s. r.p.'s:

$$R_{XY}(\tau)=E[X(t)Y(t+\tau)]$$

(1) $R_{XY}(-\tau)=R_{YX}(\tau)$

(2) $|R_{XY}(\tau)|\le\sqrt{R_{XX}(0)R_{YY}(0)}$

(3) $|R_{XY}(\tau)|\le\frac{1}{2}[R_{XX}(0)+R_{YY}(0)]$

$$E[\{Y(t+\tau)+\alpha X(t)\}^2]\ge 0,\quad\forall\alpha$$

$$\sqrt{R_{XX}(0)R_{YY}(0)}\le\frac{1}{2}[R_{XX}(0)+R_{YY}(0)]$$

6.3 Correlation Functions

Ex 6.3-3: A, B : r.v.'s $\quad \omega_0 = \text{const}$

$$E[A] = E[B] = 0, \quad E[AB] = 0, \quad E[A^2] = E[B^2] = \sigma^2$$

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t), \quad Y(t) = B\cos(\omega_0 t) - A\sin(\omega_0 t)$$

$$E[X(t)] = E[A\cos(\omega_0 t) + B\sin(\omega_0 t)] = E[A]\cos(\omega_0 t) + E[B]\sin(\omega_0 t) = 0$$

$$\begin{aligned} R_{XX}(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= E[A^2\cos(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) + AB\cos(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau) \\ &\quad + AB\sin(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) + B^2\sin(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau)] \\ &= \sigma^2\{\cos(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) + \sin(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau)\} = \sigma^2\cos(\omega_0 \tau) \end{aligned}$$

$\Rightarrow \quad X(t)$: w.s.s.

6.3 Correlation Functions

$Y(t)$: w.s.s.

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] \\ &= E\{[A\cos(\omega_0 t) + B\sin(\omega_0 t)][B\cos(\omega_0 (t+\tau)) - A\sin(\omega_0 (t+\tau))]\} \\ &= E[AB\cos(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) + B^2\sin(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) \\ &\qquad - A^2\cos(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau) - AB\sin(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau)] \\ &= \sigma^2[\sin(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) - \cos(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau)] \\ &= -\sigma^2\sin(\omega_0 \tau) \end{aligned}$$

$\Rightarrow \quad X(t)$ & $Y(t)$: jointly w.s.s.

6.3 Correlation Functions

Random Sequence (=Discrete-time R.P)

$$X(nT_s) = X[n]$$

Mean $= E(X[n])$

$$R_{XX}(n, n+k) = E(X[n]X[n+k])$$

$$C_{XX}(n, n+k) = E\{(X[n]-\overline{X}[n])(X[n+k]-\overline{X}[n+k])\}$$
$$= R_{XX}(n, n+k) - \overline{X}[n]\overline{X}[n+k]$$

$$R_{XY}(n, n+k) = E(X[n]Y[n+k])$$

$$C_{XY}(n, n+k) = E\{(X[n]-\overline{X}[n])(Y[n+k]-\overline{Y}[n+k])\}$$
$$= R_{XY}(n, n+k) - \overline{X}[n]\overline{Y}[n+k]$$

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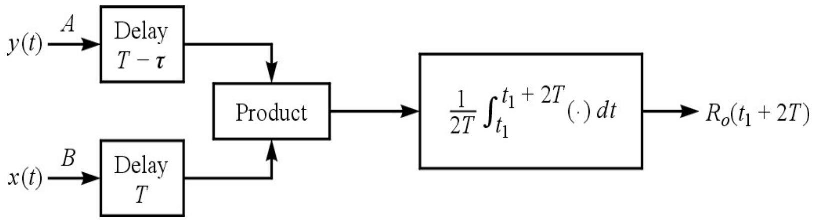
6.4 Measurement of Correlation Functions



$$R_0(t_1+2T) = \frac{1}{2T}\int_{t_1}^{t_1+2T} x(t-T)y(t-T+\tau)dt = \frac{1}{2T}\int_{t_1-T}^{t_1+T} x(t)y(t+\tau)dt$$

$t_1 = 0$ & T large

$$R_0(2T) = \frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)dt \approx \Re_{xy}(\tau) = R_{XY}(\tau)$$

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6.4 Measurement of Correlation Functions

Ex 6.4-1: $X(t) = A\cos(\omega_0 t + \Theta)$ Θ: uniform on $(-\pi, \pi)$

$$R_0(2T) = \frac{1}{2T}\int_{-T}^{T} A^2\cos(\omega_0 t + \theta)\cos(\omega_0 t + \theta + \omega_0 \tau)dt$$

$$= \frac{A^2}{4T}\int_{-T}^{T}[\cos(\omega_0 \tau) + \cos(2\omega_0 t + 2\theta + \omega_0 \tau)]dt$$

$$R_0(2T) = R_{XX}(\tau) + \varepsilon(T) \qquad R_{XX}(\tau) = \frac{A^2}{2}\cos(\omega_0 \tau)$$

$$\varepsilon(T) = \frac{A^2}{2}\cos(\omega_0 \tau + 2\theta)\frac{\sin(2\omega_0 T)}{2\omega_0 T}$$

$$|\varepsilon(T)| < 0.05 R_{XX}(0) \quad \Rightarrow \quad \frac{1}{2\omega_0 T} < 0.05 \quad \Rightarrow \quad T > \frac{10}{\omega_0}$$

6.5 Gaussian Random Process

- continuous r.p. $X(t), \quad -\infty < t < \infty$

$$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N) = \frac{1}{\sqrt{(2\pi)^N |C_X|}}\exp\{-\frac{1}{2}[x - \bar{X}]^t C_X^{-1}[x - \bar{X}]\}$$

$$\bar{X}_i = E[X(t_i)] \qquad C_{ik} = C_{XX}(t_i, t_k)$$

stationary $\Rightarrow$ $E[X(t)] = \bar{X}$ (const) & $R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

w.s.s. Gaussian $\Rightarrow$ strictly stationary

6.5 Gaussian Random Process

Ex 6.5-1: w.s.s. gaussian r.p. $X(t)$

$$\bar{X} = 4 \qquad R_{XX}(\tau) = 25e^{-3|\tau|} \qquad t_i = t_0 + \frac{i-1}{2}, \quad i = 1,2,3.$$

$$C_{ik} = C_{XX}(t_i, t_k) = R_{XX}(t_i, t_k) - \bar{X}^2 = 25e^{-3\frac{|k-i|}{2}} - 16$$

$$C_X = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 25-16 & 25e^{-\frac{3}{2}}-16 & 25e^{-3}-16 \\ 25e^{-\frac{3}{2}}-16 & 25-16 & 25e^{-\frac{3}{2}}-16 \\ 25e^{-3}-16 & 25e^{-\frac{3}{2}}-16 & 25-16 \end{bmatrix}$$

6.6 Poisson Random Process

-- integer-valued discrete r.p. $X(t), \quad -\infty < t < \infty$

$$X(0) = 0 \qquad t_b < t_a \quad \Rightarrow \quad X(t_b) \le X(t_a)$$

$$P[X(t_a) - X(t_b) = k] = \frac{[\lambda(t_a - t_b)]^k}{k!} e^{-\lambda(t_a - t_b)}, \quad k = 0,1,2,\cdots$$

$t_d < t_c \le t_b < t_a \quad \Rightarrow \quad X(t_a) - X(t_b)$ & $X(t_c) - X(t_d)$ are indep.

$$\bar{X}(t) = E[X(t)] = \lambda t \qquad R_{XX}(t,t) = E[X(t)^2] = \lambda t + (\lambda t)^2$$

$$C_{XX}(t,t) = \lambda t$$

6.6 Poisson Random Process

$0 < t_1 < t_2 \quad \Rightarrow$

$$P[X(t_1)=k_1, X(t_2)=k_2] = P[X(t_1)=k_1, X(t_2)-X(t_1)=k_2-k_1]$$

$$= \begin{cases} \dfrac{(\lambda t_1)^{k_1}}{k_1!} e^{-\lambda t_1} \dfrac{[\lambda(t_2-t_1)]^{(k_2-k_1)}}{(k_2-k_1)!} e^{-\lambda(t_2-t_1)}, & k_2 \geq k_1 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{(\lambda t_1)^{k_1}[\lambda(t_2-t_1)]^{(k_2-k_1)}}{k_1!(k_2-k_1)!} e^{-\lambda t_2}, & k_2 \geq k_1 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

6.6 Poisson Random Process

$0 < t_1 < t_2 \quad \Rightarrow$

$$P[X(t_2)=k_2 \mid X(t_1)=k_1] = P[X(t_2)-X(t_1)=k_2-k_1 \mid X(t_1)=k_1]$$

$$= P[X(t_2)-X(t_1)=k_2-k_1]$$

$$= \begin{cases} \dfrac{[\lambda(t_2-t_1)]^{(k_2-k_1)}}{(k_2-k_1)!} e^{-\lambda(t_2-t_1)}, & k_2 \geq k_1 \\ 0, & \text{otherwise} \end{cases}$$

6.6 Poisson Random Process

Ex 6.6-1: $X(t) = \text{Poisson r.p.}$

$$0 < t_1 < t_2 < t_3$$

$0 \le k_1 \le k_2 \le k_3 \quad \Rightarrow$

$$\begin{aligned}
&P[X(t_1) = k_1, X(t_2) = k_2, X(t_3) = k_3] \\
&\quad = P[X(t_1) = k_1, X(t_2) - X(t_1) = k_2 - k_1, X(t_3) - X(t_2) = k_3 - k_2] \\
&\quad = P[X(t_1) = k_1] P[X(t_2) - X(t_1) = k_2 - k_1] P[X(t_3) - X(t_2) = k_3 - k_2] \\
&\quad = \frac{(\lambda t_1)^{k_1}}{k_1!} e^{-\lambda t_1} \frac{[\lambda(t_2 - t_1)]^{(k_2 - k_1)}}{(k_2 - k_1)!} e^{-\lambda(t_2 - t_1)} \frac{[\lambda(t_3 - t_2)]^{(k_3 - k_2)}}{(k_3 - k_2)!} e^{-\lambda(t_3 - t_2)} \\
&\quad = \frac{(\lambda t_1)^{k_1} [\lambda(t_2 - t_1)]^{(k_2 - k_1)} [\lambda(t_3 - t_2)]^{(k_3 - k_2)}}{k_1!(k_2 - k_1)!(k_3 - k_2)!} e^{-\lambda t_3}
\end{aligned}$$

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