Chapter 5. Operations on Multiple Random Variables

0. Introduction
1. Expected Value of a Function of Random Variables
2. Jointly Gaussian Random Variables
3. Transformations of Multiple Random Variables
4. Linear Transformations of Gaussian Random Variables
5. Sampling and Some Limit Theorems

5.1 Expected Value of a Function of Random Variables

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dxdy$$

$$E[g(X)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x) f_{X,Y}(x,y)dxdy = \int_{-\infty}^{\infty} g(x)\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx$$

$$= \int_{-\infty}^{\infty} g(x) f_X(x)dx$$

$$E[g(X_1,\cdots,X_N)] = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\cdots,x_N) f_{X_1,\cdots,X_N}(x_1,\cdots,x_N)dx_1\cdots dx_N$$

5.1 Expected Value of a Function of Random Variables

Ex 5.1-1:

$$g(X_1,\cdots,X_N)=\sum_{i=1}^{N}\alpha_i X_i$$

$$E[g(X_1,\cdots,X_N)]=E[\sum_{i=1}^{N}\alpha_i X_i]=\sum_{i=1}^{N}E[\alpha_i X_i]=\sum_{i=1}^{N}\alpha_i E[X_i]$$

Joint moments

$$m_{nk}=E[X^n Y^k]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x^n y^k f_{X,Y}(x,y)dxdy$$

$$m_{10}=E[X] \qquad m_{01}=E[Y] \qquad m_{11}=E[XY]$$

Correlation of X & Y: $R_{XY}=E[XY]=m_{11}$

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5.1 Expected Value of a Function of Random Variables

X & Y -- uncorrelated $\Leftrightarrow$ $E[XY]=E[X]E[Y]$

X & Y -- orthogonal $\Leftrightarrow$ $E[XY]=0$

independent $\Rightarrow$ uncorrelated

$\not\Leftarrow$

$$E[XY]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf_{X,Y}(x,y)dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf_X(x)f_Y(y)dxdy$$

$$=\int_{-\infty}^{\infty}xf_X(x)dx\int_{-\infty}^{\infty}yf_Y(y)dy=E[X]E[Y]$$

Ex 5.1-2: $E[X]=3$ $\quad E[X^2]=11$ $\quad Y=-6X+22$

$E[Y]=-6E[X]+22=4$

$R_{XY}=E[XY]=E[X(-6X+22)]=-6E[X^2]+22E[X]=0$ $\Rightarrow$ orthogonal

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5.1 Expected Value of a Function of Random Variables

$R_{XY} = E[XY] \neq E[X]E[Y] = 12 \quad \Rightarrow \quad \text{NOT uncorrelated}$

$Y = aX + b$

$E[XY] = E[X(aX+b)] = aE[X^2] + bE[X]$

$E[X]E[Y] = E[X]E[aX+b] = a(E[X])^2 + bE[X]$

$a = 0 \quad \Rightarrow \quad E[XY] = E[X]E[Y] \quad \Rightarrow \quad \text{uncorrelated}$

$\frac{b}{a} = -\frac{E[X^2]}{E[X]} \quad \Rightarrow \quad \text{orthogonal}$

$m_{3,2,5} = E[X_1^3 X_2^2 X_3^5]$

5.1 Expected Value of a Function of Random Variables

Joint central moments

$\mu_{nk} = E[(X-\overline{X})^n (Y-\overline{Y})^k]$

$\mu_{10} = E[X-\overline{X}] = 0 \qquad \mu_{01} = E[Y-\overline{Y}] = 0$

$\mu_{20} = E[(X-\overline{X})^2] = \sigma_X^2 \qquad \mu_{02} = E[(Y-\overline{Y})^2] = \sigma_Y^2$

Covariance of X & Y: $C_{XY} = \mu_{11} = E[(X-\overline{X})(Y-\overline{Y})]$

$$C_{XY} = E[(X-\overline{X})(Y-\overline{Y})] = E[XY - \overline{X}Y - \overline{Y}X + \overline{X}\overline{Y}]$$
$$= E[XY] - \overline{X}E[Y] - \overline{Y}E[X] + \overline{X}\overline{Y} = R_{XY} - E[X]E[Y]$$

uncorrelated $\Leftrightarrow$ $C_{XY} = 0$

5.1 Expected Value of a Function of Random Variables

orthogonal $\Rightarrow$ $C_{XY} = -E[X]E[Y]$

Correlation coefficient of X & Y:

$$\rho = \frac{C_{XY}}{\sigma_X \sigma_Y} = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = E\left[\frac{(X-\bar{X})}{\sigma_X}\frac{(Y-\bar{Y})}{\sigma_Y}\right]$$

$$C_{XY} = \rho \sigma_X \sigma_Y$$

5.1 Expected Value of a Function of Random Variables

$$\rho = \frac{C_{XY}}{\sigma_X \sigma_Y} = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = E\left[\frac{(X-\bar{X})}{\sigma_X}\frac{(Y-\bar{Y})}{\sigma_Y}\right] \qquad -1 \le \rho \le 1$$

uncorrelated $\Leftrightarrow$ $\rho = 0$

$Y = X \Rightarrow \rho = 1$

$Y = -2X \Rightarrow \rho = -1$

5.1 Expected Value of a Function of Random Variables

Ex 5.1-3: $X = \sum_{i=1}^{N} \alpha_i X_i \qquad \sum_{i=1}^{N} \alpha_i = 1$

$$E[X] = E[\sum_{i=1}^{N} \alpha_i X_i] = \sum_{i=1}^{N} \alpha_i E[X_i] \qquad X - E[X] = \sum_{i=1}^{N} \alpha_i (X_i - \overline{X}_i)$$

$$\sigma_X^2 = E[(X-\overline{X})^2] = E\left[\sum_{i=1}^{N} \alpha_i (X_i - \overline{X}_i) \sum_{j=1}^{N} \alpha_j (X_j - \overline{X}_j)\right]$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j E[(X_i - \overline{X}_i)(X_j - \overline{X}_j)] = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j C_{X_i, X_j}$$

X_i's uncorrelated

$$C_{X_i,X_j} = \begin{cases} \sigma_{X_i}^2, & i = j \\ 0, & i \neq j \end{cases} \quad \Rightarrow \quad \sigma_X^2 = \sum_{i=1}^{N} \alpha_i^2 \sigma_{X_i}^2$$

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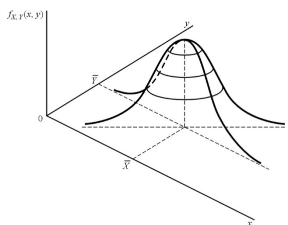
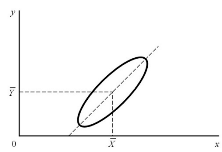
5.3 Jointly Gaussian Random Variables

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\overline{X})^2}{\sigma_X^2} - \frac{2\rho(x-\overline{X})(y-\overline{Y})}{\sigma_X\sigma_Y} + \frac{(y-\overline{Y})^2}{\sigma_Y^2}\right]}$$



(a)

$$f_{X,Y}(x,y) \le f_{X,Y}(\overline{X},\overline{Y})$$
$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$



(b)

$$E[X] = \overline{X} \qquad E[Y] = \overline{Y}$$
$$E[(X-\overline{X})^2] = \sigma_X^2$$
$$E[(X-\overline{X})(Y-\overline{Y})] = \rho\sigma_X\sigma_Y$$

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5.3 Jointly Gaussian Random Variables

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \cdots = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\bar{X})^2}{2\sigma_X^2}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \cdots = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(y-\bar{Y})^2}{2\sigma_Y^2}}$$

$$\rho = 0 \quad \Rightarrow \quad f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

jointly gaussian & uncorr. $\Rightarrow$ indep.

Ex 5.3-1:

$$Y_1 = X\cos\theta + Y\sin\theta \qquad Y_2 = -X\sin\theta + Y\cos\theta$$

5.3 Jointly Gaussian Random Variables

$$C_{Y_1,Y_2} = E[(Y_1 - \bar{Y}_1)(Y_2 - \bar{Y}_2)]$$

$$= E[\{(X-\bar{X})\cos\theta + (Y-\bar{Y})\sin\theta\}\{-(X-\bar{X})\sin\theta + (Y-\bar{Y})\cos\theta\}]$$

$$= (\sigma_Y^2 - \sigma_X^2)\sin\theta\cos\theta + C_{XY}[\cos^2\theta - \sin^2\theta]$$

$$= \frac{1}{2}(\sigma_Y^2 - \sigma_X^2)\sin 2\theta + C_{XY}\cos 2\theta = \frac{1}{2}(\sigma_Y^2 - \sigma_X^2)\sin 2\theta + \rho\sigma_X\sigma_Y\cos 2\theta$$

$$C_{Y_1,Y_2} = 0 \quad \Rightarrow \quad \theta = \frac{1}{2}\tan^{-1}\left[\frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2 - \sigma_Y^2}\right]$$

5.3 Jointly Gaussian Random Variables

$$x-\bar{X}=\begin{bmatrix} x_1-\bar{X}_1 \\ x_2-\bar{X}_2 \\ x_3-\bar{X}_3 \end{bmatrix} \qquad C_X=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3)=\frac{1}{(2\pi)^{N/2}\left|C_X\right|^{1/2}}e^{-\frac{1}{2}(x-\bar{X})^T C_X{}^{-1}(x-\bar{X})}$$

$$C_X=\begin{bmatrix} \sigma_{X_1}^2 & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix} \Rightarrow C_X^{-1}=\frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sigma_{X_1}^2} & \frac{-\rho}{\sigma_{X_1}\sigma_{X_2}} \\ \frac{-\rho}{\sigma_{X_1}\sigma_{X_2}} & \frac{1}{\sigma_{X_2}^2} \end{bmatrix}$$

$$\left|C_X\right|=\sigma_{X_1}^2\sigma_{X_2}^2(1-\rho^2)$$

5.3 Jointly Gaussian Random Variables

Properties of jointly gaussian r.v.'s $X_1, X_2, \& X_3$:

1. 1st & 2nd moments $\Rightarrow$ p.d.f
2. uncorr. $\Rightarrow$ indep.
3. linear transforms of gaussian r.v.'s is also jointly gaussian.
4. marginal density $f_{X_1,X_2}(x_1,x_2)$ is also jointly gaussian.
5. conditional density $f_{X_1,X_2|X_3}(x_1,x_2|x_3)$ is also jointly gaussian.

Recall: Leibniz's Rule

- Let $G(u) = \int_{\alpha(u)}^{\beta(u)} H(x,u)dx$
- Then

$$\frac{dG(u)}{du} = H[\beta(u),u]\left(\frac{d\beta(u)}{du}\right) - H[\alpha(u),u]\left(\frac{d\alpha(u)}{du}\right) + \int_{\alpha(u)}^{\beta(u)} \frac{\partial H(x,u)}{\partial u}dx$$

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5.4 Transformations of Multiple Random Variables

One function $\quad Y = g(X_1, X_2) \quad f_{X_1,X_2}(x_1,x_2)$

$$F_Y(y) = P[g(X_1,X_2) \le y] = \iint_{g(x_1,x_2)\le y} f_{X_1,X_2}(x_1,x_2)dx_1dx_2$$

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

Ex 5.4-1: positive r.v.'s X_1 & X_2 $\quad Y = \frac{X_1}{X_2}$

$$F_Y(y) = P[\frac{X_1}{X_2} \le y] = \int_0^\infty \int_0^{yx_2} f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \qquad y > 0$$

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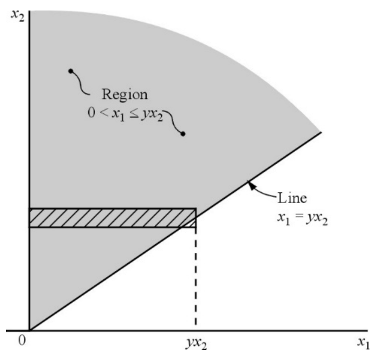
5.4 Transformations of Multiple Random Variables

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \int_0^{\infty} x_2 f_{X_1,X_2}(yx_2, x_2)dx_2 \qquad y > 0$$

Use Leibniz's Rule



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5.4 Transformations of Multiple Random Variables

Ex 5.4-2: $\quad Y = \sqrt{X_1^2 + X_2^2} \qquad f_{X_1,X_2}(x_1, x_2)$

$$F_Y(y) = P[\sqrt{X_1^2 + X_2^2} \le y] = \int_{-y}^{y} \int_{-\sqrt{y^2-x_2^2}}^{\sqrt{y^2-x_2^2}} f_{X_1,X_2}(x_1, x_2)dx_1dx_2 \; = \int_{-y}^{y} I(y, x_2)dx_2$$

$$I(y, x_2) = \int_{-\sqrt{y^2-x_2^2}}^{\sqrt{y^2-x_2^2}} f_{X_1,X_2}(x_1, x_2)dx_1$$

Using Leibniz's Rule

$$f_Y(y) = \frac{dF_Y(y)}{dy} = I(y, y) + I(y, -y) + \int_{-y}^{y} \frac{\partial I(y, x_2)}{\partial y} dx_2 = \int_{-y}^{y} \frac{\partial I(y, x_2)}{\partial y} dx_2$$

$$\frac{\partial I(y, x_2)}{\partial y} = f_{X_1,X_2}(\sqrt{y^2 - x_2^2}, x_2)\frac{y}{\sqrt{y^2 - x_2^2}} + f_{X_1,X_2}(-\sqrt{y^2 - x_2^2}, x_2)\frac{y}{\sqrt{y^2 - x_2^2}}$$

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5.4 Transformations of Multiple Random Variables

$$f_Y(y)=\int_{-y}^{y}\frac{\partial I(y,x_2)}{\partial y}dx_2$$

$$=\int_{-y}^{y}\{f_{X_1,X_2}(\sqrt{y^2-x_2^2},x_2)+f_{X_1,X_2}(-\sqrt{y^2-x_2^2},x_2)\}\frac{y}{\sqrt{y^2-x_2^2}}dx_2$$

HW: Solve Problem 5.4-3.

$$f_{X_1,X_2}(x_1,x_2)=\frac{1}{2\pi\sigma_X^2\sqrt{1-\rho^2}}e^{-\frac{x_1^2-2\rho x_1x_2+x_2^2}{2\sigma_X^2(1-\rho^2)}}$$

$\rho=0$ $\quad Y=\sqrt{X_1^2+X_2^2}$ $\quad \Rightarrow \quad f_Y(y)=?$

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5.4 Transformations of Multiple Random Variables

Multiple functions

$f_{X_1,X_2}(x_1,x_2)$

$$\begin{bmatrix}Y_1\\Y_2\end{bmatrix}=T(X_1,X_2)=\begin{bmatrix}T_1(X_1,X_2)\\T_2(X_1,X_2)\end{bmatrix}$$

$$f_{Y_1,Y_2}(y_1,y_2)=f_{X_1,X_2}(x_1,x_2)|J|$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=T^{-1}(y_1,y_2)=\begin{bmatrix}T_1^{-1}(y_1,y_2)\\T_2^{-1}(y_1,y_2)\end{bmatrix}$$

$$J=\begin{vmatrix}\frac{\partial T_1^{-1}}{\partial y_1} & \frac{\partial T_1^{-1}}{\partial y_2}\\ \frac{\partial T_2^{-1}}{\partial y_1} & \frac{\partial T_2^{-1}}{\partial y_2}\end{vmatrix}$$

jacobian

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5.4 Transformations of Multiple Random Variables

Ex 5.4-3:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = T(X_1, X_2) = \begin{bmatrix} T_1(X_1, X_2) \\ T_2(X_1, X_2) \end{bmatrix} = \begin{bmatrix} aX_1 + bX_2 \\ cX_1 + dX_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T^{-1}(y_1, y_2) = \begin{bmatrix} T_1^{-1}(y_1, y_2) \\ T_2^{-1}(y_1, y_2) \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$J = \begin{vmatrix} \frac{\partial T_1^{-1}}{\partial y_1} & \frac{\partial T_1^{-1}}{\partial y_2} \\ \frac{\partial T_2^{-1}}{\partial y_1} & \frac{\partial T_2^{-1}}{\partial y_2} \end{vmatrix} = \frac{1}{ad - bc} \qquad f_{Y_1,Y_2}(y_1, y_2) = \frac{f_{X_1,X_2}\left(\frac{dy_1 - by_2}{ad - bc}, \frac{-cy_1 + ay_2}{ad - bc}\right)}{|ad - bc|}$$

5.5 Linear Transformations of Gaussian Random Variables

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = AX = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} X \quad \Rightarrow X = A^{-1}Y$$

$$E[Y] = E[AX] = AE[X]$$

$$C_Y = E[(Y - \bar{Y})(Y - \bar{Y})^T] = E[A(X - \bar{X})(X - \bar{X})^T A^T]$$

$$= AE[(X - \bar{X})(X - \bar{X})^T]A^T = AC_X A^T \qquad \Rightarrow \quad C_Y^{-1} = A^{-T} C_X^{-1} A^{-1}$$

$$\det(C_Y) = \det(A)^2 \det(C_X)$$

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{(2\pi)^{N/2} |C_X|^{1/2}} e^{-\frac{1}{2}(x - \bar{X})^T C_X^{-1} (x - \bar{X})} \qquad f_{Y_1,Y_2}(y_1, y_2) = ?$$

5.5 Linear Transformations of Gaussian Random Variables

$$f_{Y_1,Y_2}(y_1,y_2)=\frac{1}{|\det(A)|}f_{X_1,X_2}(A^{-1}y)$$

$$=\frac{1}{(2\pi)^{N/2}|C_X|^{1/2}|\det(A)|}e^{-\frac{1}{2}(A^{-1}y-\bar{X})^T C_X^{-1}(A^{-1}y-\bar{X})}$$

$$(A^{-1}y-\bar{X})^T C_X^{-1}(A^{-1}y-\bar{X})=(y-\bar{Y})^T A^{-T}C_X^{-1}A^{-1}(y-\bar{Y})$$

$$=(y-\bar{Y})^T C_Y^{-1}(y-\bar{Y})$$

$$|C_X|^{1/2}|\det(A)|=|C_Y|^{1/2}$$

$$\Rightarrow \quad f_{Y_1,Y_2}(y_1,y_2)=\frac{1}{(2\pi)^{N/2}|C_Y|^{1/2}}e^{-\frac{1}{2}(y-\bar{Y})^T C_Y^{-1}(y-\bar{Y})} \qquad \text{(gaussian)}$$

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5.5 Linear Transformations of Gaussian Random Variables

Ex 5.5-1: $\begin{bmatrix}Y_1\\Y_2\end{bmatrix}=A\begin{bmatrix}X_1\\X_2\end{bmatrix}=\begin{bmatrix}1&-2\\3&4\end{bmatrix}\begin{bmatrix}X_1\\X_2\end{bmatrix}$ $\qquad \begin{bmatrix}E[X_1]\\E[X_2]\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$$C_X=\begin{bmatrix}4&3\\3&9\end{bmatrix} \qquad \begin{bmatrix}E[Y_1]\\E[Y_2]\end{bmatrix}=A\begin{bmatrix}E[X_1]\\E[X_2]\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_Y=AC_XA^T=\begin{bmatrix}1&-2\\3&4\end{bmatrix}\begin{bmatrix}4&3\\3&9\end{bmatrix}\begin{bmatrix}1&3\\-2&4\end{bmatrix}=\begin{bmatrix}28&-66\\-66&252\end{bmatrix}$$

$$\rho=\frac{C_{Y_1Y_2}}{\sigma_{Y_1}\sigma_{Y_2}}=\frac{-66}{\sqrt{28}\sqrt{252}}=-0.786$$

$$\Rightarrow \quad f_{Y_1,Y_2}(y_1,y_2)=\frac{1}{(2\pi)|C_Y|^{1/2}}e^{-\frac{1}{2}(y-\bar{Y})^T C_Y^{-1}(y-\bar{Y})}$$

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5.7 Sampling and Some Limit Theorems

sampling and estimation

$$\hat{\bar{x}}_N = \frac{1}{N}\sum_{n=1}^{N} x_n$$ -- estimate of average of N samples

estimator of mean of r.v. X

$$\hat{\bar{X}}_N = \frac{1}{N}\sum_{n=1}^{N} X_n$$

r.v.

(sample mean)

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5.7 Sampling and Some Limit Theorems

estimator of power of r.v. X $\qquad \widehat{\overline{X_N^2}} = \frac{1}{N}\sum_{n=1}^{N} X_n^2$

estimator of variance of r.v. X $\qquad \widehat{\sigma_X^2} = \frac{1}{N-1}\sum_{n=1}^{N} (X_n - \hat{\bar{X}}_N)^2$

(sample mean estimator is unbiased.)

$$E[\hat{\bar{X}}_N] = E[\frac{1}{N}\sum_{n=1}^{N} X_n] = \frac{1}{N}\sum_{n=1}^{N} E[X_n] = \bar{X}$$

$$E[\hat{\bar{X}}_N^2] = E[\frac{1}{N}\sum_{n=1}^{N} X_n \frac{1}{N}\sum_{n=1}^{N} X_n] = \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} E[X_n X_m]$$

$$= \frac{1}{N^2}\{NE[X^2] + (N^2 - N)\bar{X}^2\} = \frac{1}{N}\{\overline{X^2} + (N-1)\bar{X}^2\}$$

indep.

$n = m \quad \Rightarrow \quad E[X_n X_m] = E[X^2] \qquad n \neq m \quad \Rightarrow \quad E[X_n X_m] = \bar{X}^2$

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5.7 Sampling and Some Limit Theorems

$$\sigma^2_{\hat{\bar{X}}_N} = E[(\hat{\bar{X}}_N - \bar{X})^2] = E[\hat{\bar{X}}_N^2 - 2\bar{X}\hat{\bar{X}}_N + \bar{X}^2] = E[\hat{\bar{X}}_N^2] - \bar{X}^2$$
$$= \frac{1}{N}\{E[X^2] + (N-1)\bar{X}^2\} - \bar{X}^2 = \frac{1}{N}\{E[X^2] - \bar{X}^2\} = \frac{1}{N}\sigma_X^2$$

$$P\left(\left|\hat{\bar{X}}_N - \bar{X}\right| < \varepsilon\right) \geq 1 - \frac{\sigma^2_{\hat{\bar{X}}_N}}{\varepsilon^2} = 1 - \frac{\sigma_X^2}{N\varepsilon^2} \xrightarrow[N\to\infty]{} 1$$

(Chebychev's ineq.)

$$\hat{\bar{X}}_N \xrightarrow[N\to\infty]{} \bar{X} \quad \text{w.p. 1 (with probability 1)}$$

Ex 5.7-1: $\varepsilon = 0.05\bar{X}$ $\quad N = 50$

$$P\left(\left|\hat{\bar{X}}_N - \bar{X}\right| < \varepsilon\right) \geq 0.95 \;\Rightarrow\; 1 - \frac{\sigma_X^2}{50(0.05\bar{X})^2} \geq 0.95 \;\Rightarrow\; \bar{X} \geq \sqrt{160}\sigma_X$$

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5.7 Sampling and Some Limit Theorems

$$\widehat{\overline{X_N^2}} = \frac{1}{N}\sum_{n=1}^{N} X_n^2 \qquad \widehat{\sigma_X^2} = \frac{1}{N-1}\sum_{n=1}^{N}(X_n - \hat{\bar{X}}_N)^2$$

$$E[\widehat{\overline{X_N^2}}] = \frac{1}{N}\sum_{n=1}^{N} E[X_n^2] = \frac{1}{N}\sum_{n=1}^{N} E[X^2] = E[X^2] = \overline{X^2} \qquad \text{(unbiased)}$$

$$E[\widehat{\sigma_X^2}] = \frac{1}{N-1}\sum_{n=1}^{N} E[(X_n - \hat{\bar{X}}_N)^2] = \frac{1}{N-1}\sum_{n=1}^{N} E[X_n^2 - 2X_n\hat{\bar{X}}_N + \hat{\bar{X}}_N^2]$$

$$\sum_{n=1}^{N} E[X_n\hat{\bar{X}}_N] = \sum_{n=1}^{N} E[X_n \frac{1}{N}\sum_{m=1}^{N} X_m] = \frac{1}{N}\sum_{n=1}^{N}\sum_{m=1}^{N} E[X_n X_m] = \overline{X^2} + (N-1)\bar{X}^2$$

$$E[\widehat{\sigma_X^2}] = \frac{N\overline{X^2} - 2\{\overline{X^2} + (N-1)\bar{X}^2\} + \{\overline{X^2} + (N-1)\bar{X}^2\}}{N-1} = \overline{X^2} - \bar{X}^2 = \sigma_X^2$$

(unbiased)

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5.7 Sampling and Some Limit Theorems

Ex 5.7-2: $E[X] = 4\,V$ $\qquad \sigma_X^2 = 16$

11 samples $\quad (0.1, 0.4, 0.9, 1.4, 2.0, 2.8, 3.7, 4.8, 6.4, 9.2, 12.0\,V)$

sample mean

$$\hat{\bar{X}}_N = \frac{1}{N}\sum_{n=1}^{N} X_n = \frac{1}{11}(0.1 + 0.4 + 0.9 + \cdots + 12.0) = 3.973\,V$$

sample variance

$$\widehat{\sigma_X^2} = \frac{1}{N-1}\sum_{n=1}^{N}(X_n - \hat{\bar{X}}_N)^2 = \frac{1}{10}[(0.1 - 3.973)^2 + \cdots + (12.0 - 3.973)^2]$$

$$= 14.75\,V^2$$

5.7 Sampling and Some Limit Theorems

Weak Law of Large Numbers

$$\lim_{N\to\infty} P[\left|\hat{\bar{X}}_N - E[X]\right| < \varepsilon] = 1, \quad \forall \varepsilon > 0$$

Strong Law of Large Numbers

$$P\{\lim_{N\to\infty} \hat{\bar{X}}_N = E[X]\} = 1$$