Chapter 4. Multiple Random Variables

0. Introduction
1. Vector Random Variables
2. Joint Distribution and Its Properties
3. Joint Density and Its Properties
4. Conditional Distribution and Density
5. Statistical Independence
6. Distribution and Density of a Sum of Random Variables
7. Central Limit Theorem

4.1 Vector Random Variables

vector r.v.'s = r. vector

r.v. $X: S \to R$ r.v. $Y: S \to R$

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix} : S \to R^2 \quad \text{r. vector}$$

Ex: $S = \{HH, HT, TH, TT\}$ $P(\{HH\}) = \cdots = P(\{TT\}) = \frac{1}{4}$

$X(HH) = 2, \quad X(HT) = 1, \quad X(TH) = 1, \quad X(TT) = 0$

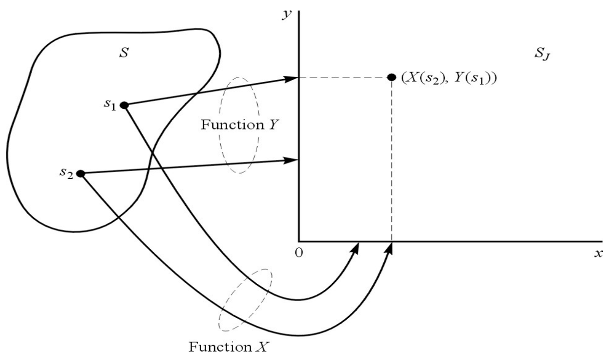
$Y(HH) = 1, \quad Y(HT) = -1, \quad Y(TH) = -1, \quad Y(TT) = 1$

$P(X = 2, Y = 1) = P(\{HH\}) = 0.25$ joint probability

$P(X = 1, Y = 1) = P(\{HH, TT\} \cap \{HT, TH\}) = P(\phi) = 0$

$P(X < 1, Y > 0) = P(\{TT\} \cap \{HH, TT\}) = P(\{TT\}) = 0.25$

4.1 Vector Random Variables



4.1 Vector Random Variables

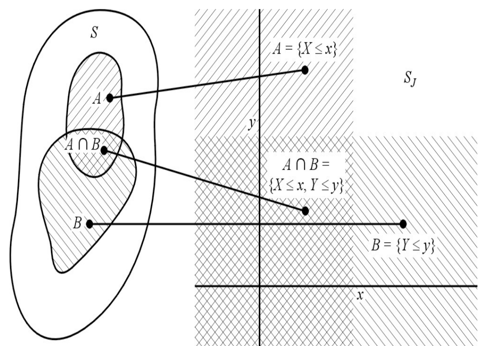
$A = \{X \le x\}$

$B = \{Y \le y\}$

$P(X \le x, Y \le y) = P(A \cap B)$

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_N \end{bmatrix} : S \to R^N$$

$N-$dim r. vector



4.2 Joint Distribution and Its Properties

Joint probability distribution function of X & Y

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

Ex 4.2-1:

$P(X=1,Y=1)=0.2 \qquad P(X=2,Y=1)=0.3 \qquad P(X=3,Y=3)=0.5$

$$F_{X,Y}(2,1.3) = P(X \le 2, Y \le 1.3) = P[\{(1,1),(2,1)\}] = 0.2+0.3=0.5$$

$P(Y=1)=P[\{(1.1),(2,1)\}]=0.5 \qquad P(Y=3)=P[\{(3.3)\}]=0.5$

$P(X=1)=P[\{(1.1)\}]=0.2 \qquad P(X=2)=0.3 \qquad P(X=3)=0.5$

$$F_{X_1,X_2,X_3}(x_1,x_2,x_3) = P(X_1 \le x_1,\ X_2 \le x_2,\ X_3 \le x_3)$$

4.2 Joint Distribution and Its Properties

Properties:

(1) $F_{X,Y}(-\infty,-\infty)=0, \quad F_{X,Y}(-\infty,y)=0, \quad F_{X,Y}(x,-\infty)=0$

(2) $F_{X,Y}(\infty,\infty)=1$

(3) $0 \le F_{X,Y}(x,y) \le 1$

(4) $F_{X,Y}(x,y)$ -- nondecreasing function of both x & y

(5) $P(x_1 < X \le x_2,\ y_1 < Y \le y_2)$
$= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1)$

(6) $F_{X,Y}(x,\infty) = F_X(x) \qquad F_{X,Y}(\infty,y) = F_Y(y)$ marginal distribution
$F_{X,Y,Z,W}(x,\infty,z,\infty) = F_{X,Z}(x,z)$

4.2 Joint Distribution and Its Properties

Ex 4.2-2:

$P(X=1,Y=1)=0.2 \qquad P(X=2,Y=1)=0.3 \qquad P(X=3,Y=3)=0.5$

$$F_{X,Y}(x,y) = 0.2u(x-1)u(y-1) + 0.3u(x-2)u(y-1) + 0.5u(x-3)u(y-3)$$

marginal distribution

$$F_X(x) = F_{X,Y}(x,\infty) = 0.2u(x-1) + 0.3u(x-2) + 0.5u(x-3)$$

$$F_Y(y) = F_{X,Y}(\infty,y) = 0.2u(y-1) + 0.3u(y-1) + 0.5u(y-3)$$
$$= 0.5u(y-1) + 0.5u(y-3)$$

4.3 Joint Density and Its Properties

Joint probability density function of X & Y

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Properties:

(1) $f_{X,Y}(x,y) \ge 0 \qquad$ (2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

(3) $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1 d\xi_2$

(4) $F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)\,d\xi_2 d\xi_1 \qquad F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1 d\xi_2$

4.3 Joint Density and Its Properties

(5) $P(x_1 < X \le x_2, y_1 < Y \le y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)dxdy$

(6) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$ marginal density function of X

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$

$f_X(x) = \frac{dF_X(x)}{dx}$ $f_Y(y) = \frac{dF_Y(y)}{dy}$

Ex 4.3-1:

$$f_{X,Y}(x,y) = \begin{cases} be^{-x}\cos y, & 0 \le x \le 2 \;\&\; 0 \le y \le \pi/2 \\ 0, & \text{elsewhere} \end{cases}$$

$b = ?$

4.3 Joint Density and Its Properties

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = \int_0^{\pi/2}\int_0^2 be^{-x}\cos y\, dxdy = b\int_0^2 e^{-x}dx\int_0^{\pi/2}\cos y\, dy$$
$$= b(1-e^{-2}) = 1$$

$$\Rightarrow \quad b = \frac{1}{1-e^{-2}}$$

marginal density

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^{\pi/2} be^{-x}\cos y\, dy = \frac{1}{1-e^{-2}}e^{-x}, \quad 0 \le x \le 2$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \cdots = \cos y, \quad 0 \le y \le \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \qquad \int_{-\infty}^{\infty} f_Y(y)dy = 1$$

4.3 Joint Density and Its Properties

Ex 4.3-2: $f_{X,Y}(x,y) = u(x)u(y)xe^{-x(y+1)}$

marginal density

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{-\infty}^{\infty} u(x)u(y)xe^{-x(y+1)}\, dy$$
$$= u(x)xe^{-x}\int_0^{\infty} e^{-xy}\, dy = u(x)xe^{-x}\frac{1}{-x}e^{-xy}\Big|_{y=0}^{\infty} = u(x)e^{-x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_{-\infty}^{\infty} u(x)u(y)xe^{-x(y+1)}\, dx \qquad u' = e^{-x(y+1)} \quad \& \quad v = x$$

$$= u(y)\int_0^{\infty} xe^{-x(y+1)}\, dx = \frac{-xu(y)}{y+1}e^{-x(y+1)}\Big|_{x=0}^{\infty} + u(y)\int_0^{\infty}\frac{1}{y+1}e^{-x(y+1)}\, dx = \frac{u(y)}{(y+1)^2}$$

4.4 Conditional Distribution and Density

Conditional distribution of X, given event B

$$F_X(x|B) = P(X \le x|B) = \frac{P(\{X \le x\} \cap B)}{P(B)}$$

Conditional density

$$f_X(x|B) = \frac{dF_X(x|B)}{dx}$$

$$f_Y(y) \ne 0 \quad \Rightarrow \quad f_X(x|Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = f_{X|Y}(x|y)$$

$$f_X(x) \ne 0 \quad \Rightarrow \quad f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

4.4 Conditional Distribution and Density

$$F_X(x|Y=y)=\int_{-\infty}^{x} f_X(\xi|y)d\xi=\int_{-\infty}^{x}\frac{f_{X,Y}(\xi,y)}{f_Y(y)}d\xi=\frac{\int_{-\infty}^{x} f_{X,Y}(\xi,y)d\xi}{f_Y(y)}$$

Ex 4.4-2: $f_{X,Y}(x,y)=u(x)u(y)xe^{-x(y+1)}$

$$f_X(x)=u(x)e^{-x}$$

$$f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)}=u(x)u(y)xe^{-xy}$$

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x)dy=\int_{-\infty}^{\infty}\frac{f_{X,Y}(x,y)}{f_X(x)}dy=\frac{1}{f_X(x)}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy=\frac{f_X(x)}{f_X(x)}=1$$

4.4 Conditional Distribution and Density

Ex 4.4-1: discrete

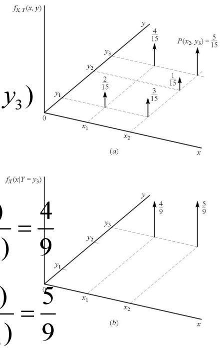
$$P(x_1,y_1)=\frac{2}{15},\quad \cdots$$

$$P(Y=y_3)=P(x_1,y_3)+P(x_2,y_3)$$

$$=\frac{9}{15}$$

$$P(X=x_1|Y=y_3)=\frac{P(x_1,y_3)}{P(Y=y_3)}=\frac{4}{9}$$

$$P(X=x_2|Y=y_3)=\frac{P(x_2,y_3)}{P(Y=y_3)}=\frac{5}{9}$$



4.4 Conditional Distribution and Density

$$B=\{y_a<Y\le y_b\}$$

$$F_X(x|y_a<Y\le y_b)=\frac{P(X\le x,y_a<Y\le y_b)}{P(y_a<Y\le y_b)}=\frac{\int_{y_a}^{y_b}\int_{-\infty}^{x} f_{X,Y}(\xi,y)d\xi dy}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{X,Y}(\xi,y)d\xi dy}$$

$$=\frac{F_{X,Y}(x,y_b)-F_{X,Y}(x,y_a)}{F_Y(y_b)-F_Y(y_a)}$$

$$f_X(x|y_a<Y\le y_b)=\frac{dF_X(x|y_a<Y\le y_b)}{dx}=\frac{\int_{y_a}^{y_b} f_{X,Y}(x,y)dy}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{X,Y}(\xi,y)d\xi dy}$$

4.4 Conditional Distribution and Density

Ex 4.4-3: $f_{X,Y}(x,y)=u(x)u(y)xe^{-x(y+1)}$ $\quad f_X(x|Y\le y)=?$

$$f_X(x|Y\le y)=\frac{\int_{-\infty}^{y} f_{X,Y}(x,\xi)d\xi}{\int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(\xi,y)d\xi dy}=\frac{\int_{-\infty}^{y} f_{X,Y}(x,\xi)d\xi}{F_Y(y)}$$

$$f_Y(y)=\frac{u(y)}{(y+1)^2}\quad\Rightarrow$$

$$F_Y(y)=\int_0^y\frac{1}{(\xi+1)^2}d\xi=\left.\frac{-1}{\xi+1}\right|_{\xi=0}^{y}=\frac{-1}{y+1}+1=\frac{y}{y+1},\quad y>0$$

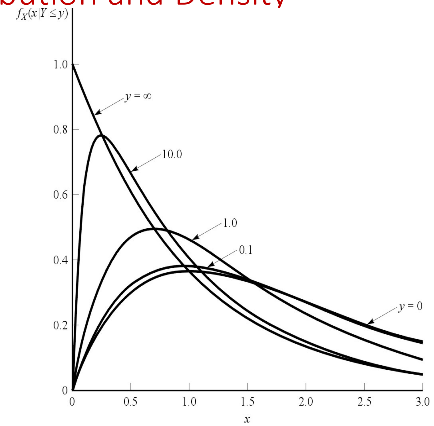
$$\int_{-\infty}^{y} f_{X,Y}(x,\xi)d\xi=u(x)\int_0^y xe^{-x(\xi+1)}d\xi=u(x)e^{-x}(1-e^{-xy}),\quad y>0$$

4.4 Conditional Distribution and Density

$$f_X(x|Y \le y)$$

$$= u(x)\frac{y+1}{y}e^{-x}(1-e^{-xy}), \quad y>0$$

$$= u(x)u(y)\frac{y+1}{y}e^{-x}(1-e^{-xy})$$



4.5 Statistical Independence

X & Y -- independent

$$\Leftrightarrow \quad P(X \le x, Y \le y) = P(X \le x)P(Y \le y), \quad \forall x, y$$

$$\Leftrightarrow \quad F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

$$\Leftrightarrow \quad f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

X & Y -- independent $\Rightarrow$

$$F_X(x|Y \le y) = \frac{P(X \le x, Y \le y)}{P(Y \le y)} = \frac{F_{X,Y}(x,y)}{F_Y(y)} = \frac{F_X(x)F_Y(y)}{F_Y(y)} = F_X(x)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y)$$

4.5 Statistical Independence

Ex 4.5-1: $\quad f_{X,Y}(x,y) = u(x)u(y)xe^{-x(y+1)}$

$$f_X(x) = u(x)e^{-x} \qquad f_Y(y) = \frac{u(y)}{(y+1)^2}$$

$f_{X,Y}(x,y) \ne f_X(x)f_Y(y) \quad \Rightarrow \quad X$ & Y -- NOT independent

Ex 4.5-2: $\quad f_{X,Y}(x,y) = \frac{1}{12}u(x)u(y)e^{-\frac{x}{4}-\frac{y}{3}}$

$$f_X(x) = \int_0^\infty \frac{1}{12}u(x)e^{-\frac{x}{4}}e^{-\frac{y}{3}}dy = \frac{1}{4}u(x)e^{-\frac{x}{4}}$$

$$f_Y(y) = \int_0^\infty \frac{1}{12}u(y)e^{-\frac{x}{4}}e^{-\frac{y}{3}}dx = \frac{1}{3}u(y)e^{-\frac{y}{3}}$$

$f_{X,Y}(x,y) = f_X(x)f_Y(y) \quad \Rightarrow$

X & Y -- independent

4.5 Statistical Independence

$X, Y,$ & Z -- independent $\quad \Leftrightarrow \quad f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z)$

$\Rightarrow \quad g(X)$ & $h(Y,Z)$ -- independent

4.6 Distribution and Density of a Sum of Random Variables

$W = X + Y$ $\quad$ X & Y -- independent

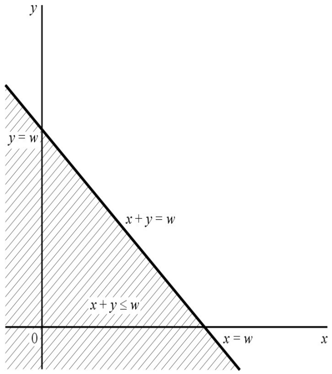
$f_W(w) = ?$

$$F_W(w) = P(W \le w) = P(X + Y \le w)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_{X,Y}(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_X(x)f_Y(y)dxdy$$

$$= \int_{-\infty}^{\infty} f_Y(y)\left(\int_{-\infty}^{w-y} f_X(x)dx\right)dy$$



$$f_W(w) = \frac{dF_W(w)}{dw} = \int_{-\infty}^{\infty} f_Y(y)f_X(w-y)dy = f_Y(w) * f_X(w)$$

4.6 Distribution and Density of a Sum of Random Variables

Ex 4.6-1: $\quad W = X + Y$ $\quad$ X & Y -- independent

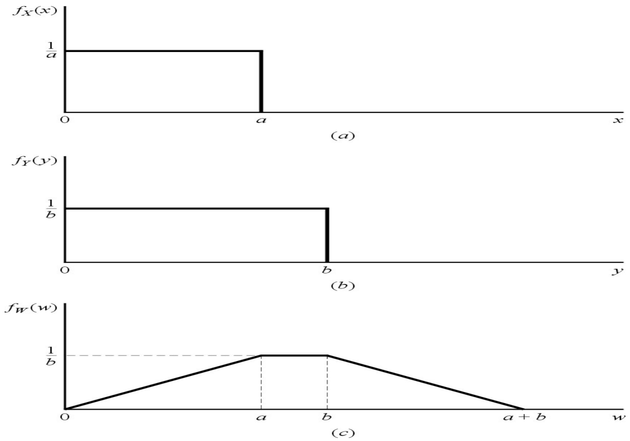
$$f_W(w) = f_X(w) * f_Y(w) = ? \quad \leftrightarrow \quad \frac{1-e^{-as}}{as}\frac{1-e^{-bs}}{bs} = \frac{1-e^{-as} - e^{-bs} + e^{-(a+b)s}}{abs^2}$$

$$f_X(x) = \frac{1}{a}[u(x) - u(x-a)] \quad \leftrightarrow \quad \frac{1-e^{-as}}{as}$$

$$f_Y(x) = \frac{1}{b}[u(y) - u(y-b)] \quad \leftrightarrow \quad \frac{1-e^{-bs}}{bs}$$

$$f_W(w) = \frac{1}{ab}[wu(w) - (w-a)u(w-a) - (w-b)u(w-b)$$
$$+(w-a-b)u(w-a-b)]$$

4.6 Distribution and Density of a Sum of Random Variables



4.6 Distribution and Density of a Sum of Random Variables

$$Y = X_1 + X_2 + X_3$$

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = f_{X_1}(x_1)f_{X_2}(x_2)f_{X_3}(x_3)$$

$$\Rightarrow \quad f_Y(y) = f_{X_1}(y) * f_{X_2}(y) * f_{X_3}(y)$$

4.7 Central Limit Theorem

$Y_N = X_1 + X_2 + \cdots + X_N$ independent

$\overline{X}_i = \overline{X}, \quad \forall i$

iid (indep. identically distributed)

$\sigma_{X_i}^2 = \sigma_X^2, \quad \forall i$

$$\Rightarrow \quad E[Y_N] = N\overline{X}$$

$$\sigma_{Y_N}^2 = E[\left(\sum_{n=1}^{N}(X_n - \overline{X})\right)^2] = E[\left(\sum_{n=1}^{N}\tilde{X}_n\right)^2] = E[\sum_{j=1}^{N}\sum_{i=1}^{N}\tilde{X}_i\tilde{X}_j]$$

$$= \sum_{j=1}^{N}\sum_{i=1}^{N}E[\tilde{X}_i\tilde{X}_j] = \sum_{i=1}^{N}E[\tilde{X}_i{}^2] + \sum\sum_{i \neq j}E[\tilde{X}_i]E[\tilde{X}_j] = \sum_{i=1}^{N}\sigma_{X_i}^2 = N\sigma_X^2$$

4.7 Central Limit Theorem

$$W_N = \frac{Y_N - E[Y_N]}{\sigma_{Y_N}} = \frac{\sum_{i=1}^{N}(X_i - \overline{X}_i)}{\left[\sum_{i=1}^{N}\sigma_{X_i}^2\right]^{\frac{1}{2}}} = \frac{1}{\sqrt{N}\sigma_X}\sum_{i=1}^{N}(X_i - \overline{X}_i)$$

$E[W_N] = 0 \qquad E[W_N^2] = 1$

Central Limit Theorem:

$\lim_{N \to \infty} W_N$ -- gaussian with zero mean and unit-variance

PF: $$\lim_{N \to \infty}\Phi_{W_N}(\omega) = \lim_{N \to \infty}E[e^{j\omega W_N}] = \cdots = e^{-\frac{\omega^2}{2}}$$

4.7 Central Limit Theorem

Ex 4.7-1: $W = X_1 + X_2$ iid

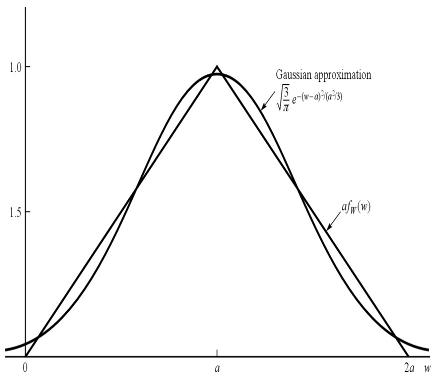
$$f_X(x) = \frac{1}{a}[u(x) - u(x-a)]$$

$$f_W(w) = \frac{1}{a}\mathrm{tri}(\frac{w}{a})$$

$$\overline{W} = 2E[X] = a$$

$$\sigma_W^2 = 2\sigma_X^2 = \frac{a^2}{6}$$

gaussian approx.= $$\frac{1}{\sqrt{\pi(a^2/3)}}e^{-\frac{(w-a)^2}{a^2/3}}$$



4.7 Central Limit Theorem

$Y_N = X_1 + X_2 + \cdots + X_N$

For not iid case, central limit theorem still holds under some conditions.

independent & not identically distributed $\Rightarrow$ (4.7 − 1)

not independent $\Rightarrow$ (Cramer, 1946, p.219)