Chapter 1. Probability

1. Set definitions
2. Set operations
3. Probability introduced through sets and relative frequency
4. Joint and conditional probability
5. Independent events
6. Combined experiments
7. Bernoulli trials
8. Summary

1.1 Set Definitions

Set A element

$a \in A$ $a \notin A$

Tabular method

$A = \{2, 4, 6, 8, 10, \cdots\}$

Rule method

$A = \{a \in N \mid a \text{ is even}\}$

N = the set of natural numbers

Z = the set of integers

Q = the set of rational numbers

R = the set of real numbers

1.1 Set Definitions

Def: Set A is finite, if A has a finite number of elements.
not finite = infinite

Def: Set A is countable, if there is an onto function $f : N \to A$.
not countable = uncountable

finite $\Rightarrow$ countable

$A = \{2, 4, 6\}$ is countable.

$N, Z,$ & Q are countable.

$R - Q$(= the set of irrational numbers) is uncountable.

R is uncountable.

1.1 Set Definitions

Def: Set A is a subset of set B, if $a \in A \Rightarrow a \in B$.

$$A \subset B$$

Def: (proper subset)

ϕ = null set

$A \cap B = \phi \Leftrightarrow A$ & B are disjoint.
(mutually exclusive)

1.1 Set Definitions

Ex 1.1-1: $A = \{1,3,5,7\}, \quad B = \{1,2,3,\cdots\}, \quad C = \{0.5 < c \le 8.5\}$

$D = \{0.0\}, \quad E = \{2,4,6,8,10,12,14\}, \quad F = \{-5.0 < f \le 12.0\}$

tabularly specified --

rule-specified --

finite --

infinite --

countable --

uncountable --

$A \subset B \qquad A \cap E = \phi \qquad D \ne \phi$

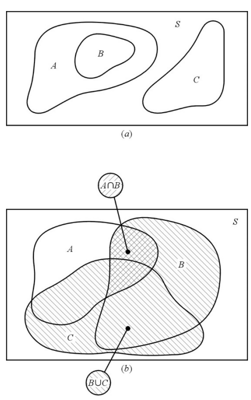
Universal set = S

rolling a die $\Rightarrow$ $S = \{1,2,3,4,5,6\}$

toss two coins $\Rightarrow$ $S = \{HH, HT, TH, TT\}$

1.2 Set Operations

Venn Diagram



1.2 Set Operations

$A \subset B \quad \& \quad B \subset A \quad \Leftrightarrow \quad A = B$

$A - B = \{x : \ x \in A \ \& \ x \notin B\} = A \cap \overline{B}$

$\overline{B} = S - B$ complement

$\overline{S} = \phi \qquad \overline{\overline{A}} = A$

$A_1 \cup A_2 \cup \cdots \cup A_N = \bigcup_{n=1}^{N} A_n \qquad A_1 \cap A_2 \cap \cdots \cap A_N = \bigcap_{n=1}^{N} A_n$

$A \cup B = B \cup A \qquad A \cap B = B \cap A$ commutative

$(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$ associative

1.2 Set Operations

$(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

distributive

$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

$\overline{(A \cap B)} = \overline{B} \cup \overline{A}$

De Morgan's Laws

Duality $\quad \cup \leftrightarrow \cap$

$\phi \leftrightarrow S$

1.3 Probability Introduced Through Sets and Relative Frequency

Definition of Probability
- set theory and axioms
- relative frequency

Experiment — Rolling a die.

- sample space S — $S = \{1,2,3,4,5,6\}$
- event (σ-algebra Ω) — $A = \{1,3,5\}$ $B = \{2,4,6\}$ $C = \{1,2,3,4\}$
- probability P — $P(A) = P(B) = 0.5$ $P(C) = \frac{2}{3}$

(S, Ω, P) -- probability space

1.3 Probability Introduced Through Sets and Relative Frequency

Sample space S

set of all possible outcomes in the experiment

Rolling a die $\Rightarrow$ $S = \{1,2,3,4,5,6\}$

Toss a coin $\Rightarrow$ $S = \{H, T\}$

Toss 2 coins $\Rightarrow$ $S = \{HH, HT, TH, TT\}$

Choose a number in [0,1] $\Rightarrow$ $S = \{s : 0 \le s \le 1\}$

Choose a number in N $\Rightarrow$ $S = N = \{1,2,3,\cdots\}$

discrete & continuous

finite & infinite

countable & uncountable

1.3 Probability Introduced Through Sets and Relative Frequency

Event = a subset of S

σ-algebra Ω = the set of all events

Toss a coin $\Rightarrow$ $S = \{H, T\}$ $\Rightarrow$ $\Omega = \{\phi, \{H\}, \{T\}, S\}$

Toss 2 coins $\Rightarrow$ $S = \{HH, HT, TH, TT\}$

$\Rightarrow$ $\Omega = \{\phi, \{HH\}, \{HT\}, \cdots, \{HH, HT, TH\}, S\}$

Choose a number in [0,1] $\Rightarrow$ $S = \{s : 0 \le s \le 1\}$

$A = \{a : 0 < a < 0.5\}$

Choose a number in N $\Rightarrow$ $S = N = \{1, 2, 3, \cdots\}$

$A = \{1, 3, 5, \cdots\}$

1.3 Probability Introduced Through Sets and Relative Frequency

Probability P = a function from Ω to [0,1]

Toss a coin $\Rightarrow$ $S = \{H, T\}$

$\Rightarrow$ $P(\phi) = 0,\ P(\{H\}) = 0.6,\ P(\{T\}) = 0.4,\ P(S) = 1$

Probability Axioms

1. $P(A) \ge 0$
2. $P(S) = 1$
3. $A \cap B = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$

3'. mutually exclusive $\{A_n : n = 1, 2, \cdots\}$ $\Rightarrow$ $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$

1.3 Probability Introduced Through Sets and Relative Frequency

$S = N = \{1, 2, 3, \cdots\}$ $\quad P(\{n\}) = \frac{1}{2^n}$

$P(\{n\}) = \frac{3}{4^n}$

$S = \{s : 0 \le s \le 1\}$ $\quad 0 \le a < b \le 1 \quad \Rightarrow \quad P(\{x : a < x < b\}) = b - a$

$P(\{0.5\}) = 0, \quad P((0.3, 0.7]) = 0.4$

Properties:

$P(\overline{A}) = 1 - P(A)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1.3 Probability Introduced Through Sets and Relative Frequency

Ex 1.3-2:

Probability as a relative frequency

coin toss $\quad P(\{H\}) = \lim_{n \to \infty} \frac{n_H}{n}$

Ex 1.3-3:

1.4 Joint and Conditional Probability

Joint probability $P(A \cap B)$

Conditional probability of event *A*, given event *B*

$$P(B) \neq 0 \quad \Rightarrow \quad P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Properties:

1. $P(A|B) \geq 0$ 2. $P(S|B) = 1$

$P(\bullet|B)$

3. $A \cap C = \phi \quad \Rightarrow \quad P(A \cup C|B) = P(A|B) + P(C|B)$

4. $A \cap B = \phi \quad \Rightarrow \quad P(A|B) = 0$

1.4 Joint and Conditional Probability

Ex 1.4-1:

A: draw a 47Ω resistor B: draw a 5% tolerance resistor

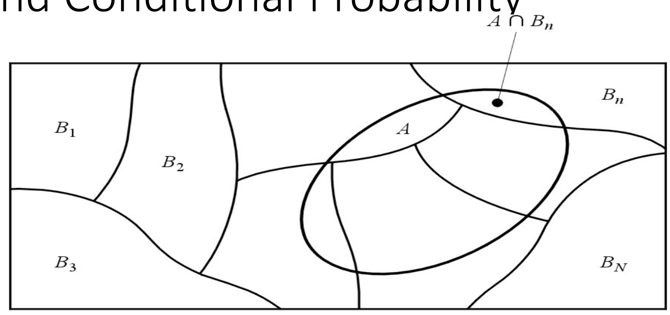
C: draw a 100Ω resistor

$$P(A) = \frac{44}{100}, \quad P(B) = \frac{62}{100}, \quad P(C) = \frac{32}{100}$$

$$P(A \cap B) = \frac{28}{100} \qquad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.28}{0.62} = \frac{28}{62}$$

$$P(A \cap C) = 0 \qquad P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0}{0.32} = 0$$

1.4 Joint and Conditional Probability



Total probability

$$A = A \cap S = A \cap (\bigcup_{n=1}^{N} B_n) = \bigcup_{n=1}^{N} (A \cap B_n)$$

mutually exclusive $\Rightarrow$ $P(A) = P[\bigcup_{n=1}^{N} (A \cap B_n)] = \sum_{n=1}^{N} P(A \cap B_n)$

1.4 Joint and Conditional Probability

Bayes' Theorem $\quad P(B_n|A) = \dfrac{P(B_n \cap A)}{P(A)} = \dfrac{P(A|B_n)P(B_n)}{P(A)}$

$$P(B_n|A) = \frac{P(A|B_n)P(B_n)}{P(A|B_1)P(B_1) + \cdots + P(A|B_N)P(B_N)}$$

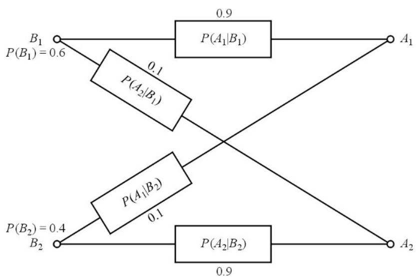
Ex 1.4-2:

B_1=1 before the channel

B_2=0 before the channel

A_1=1 after the channel

A_2=0 after the channel



1.4 Joint and Conditional Probability

$P(B_1|A_1) = ?$

$$P(A_1) = P(A_1|B_1)P(B_1) + P(A_1|B_2)P(B_2) = 0.9\times 0.6 + 0.1\times 0.4 = 0.58$$

$$P(B_1|A_1) = \frac{P(B_1 \cap A_1)}{P(A_1)} = \frac{P(A_1|B_1)P(B_1)}{P(A_1)} = \frac{0.9\times 0.6}{0.58} = \frac{54}{58}$$

$P(B_2|A_1) = ? = 1 - P(B_1|A_1)$

$P(B_1|A_2) = ?$ $\qquad$ $P(B_2|A_2) = ?$

1.5 Independent Events

Def: Two events A & B are (statistically) independent if

$$P(A \cap B) = P(A)P(B), \quad P(A) \neq 0, \quad P(B) \neq 0$$

A & B independent $\Leftrightarrow$ $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)P(A)}{P(A)} = P(B)$

$$B = B \cap S = B \cap (A \cup \overline{A}) = (B \cap A) \cup (B \cap \overline{A})$$

$$P(B) = P(B \cap A) + P(B \cap \overline{A})$$

A & B independent $\Rightarrow$ $P(B \cap \overline{A}) = P(B) - P(B \cap A) = P(B) - P(B)P(A)$

$= P(B)[1 - P(A)] = P(B)P(\overline{A})$

$\overline{A}$ & B independent

1.5 Independent Events

Def: 3 events $A_1, A_2, \& A_3$ independent $\Leftrightarrow$

$$P(A_i) \neq 0 \qquad P(A_1 \cap A_2) = P(A_1)P(A_2) \qquad P(A_2 \cap A_3) = P(A_2)P(A_3)$$

$$P(A_1 \cap A_3) = P(A_1)P(A_3) \qquad P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

Ex: $S = \{1,2,3,4\}$ $\quad A_1 = \{1,2\}, \quad A_2 = \{2,3\}, \quad A_3 = \{1,3\}$

$P(A_i \cap A_j) = P(A_i)P(A_j), \quad i \neq j$ $\quad A_1, A_2, \& A_3$ pairwise independent

$P(A_1 \cap A_2 \cap A_3) \neq P(A_1)P(A_2)P(A_3)$ $\quad A_1, A_2, \& A_3$ NOT independent

Fact: $A_1, A_2, \& A_3$ independent $\Rightarrow$ A_1 & $(A_2 \cap A_3)$ independent

Fact: $A_1, A_2, \& A_3$ independent $\Rightarrow$ A_1 & $(A_2 \cup A_3)$ independent

1.6 Combined Experiments

2 independent experiments $\quad (S_1, \Omega_1, P_1)$ & (S_2, Ω_2, P_2)

Can define a combined probability space $\quad (S, \Omega, P)$

$$S = S_1 \times S_2 = \{(s_1, s_2) : s_1 \in S_1, s_2 \in S_2\}$$

Ex 1.6-1: Ex 1.6-2:

$$\Omega = \Omega_1 \times \Omega_2 = \{A_1 \times A_2 : A_1 \in \Omega_1, A_2 \in \Omega_2\}$$

Ex 1.6-3:

$$P(A_1 \times A_2) = P_1(A_1)P_2(A_2), \quad A_1 \in \Omega_1, A_2 \in \Omega_2$$

$$P(A_1 \times S_2) = P_1(A_1)P_2(S_2) = P_1(A_1)$$

1.6 Combined Experiments

3 independent experiments $(S_i, \Omega_i, P_i), \quad i = 1,2,3$

Can define a combined probability space (S, Ω, P)

$$S = S_1 \times S_2 \times S_3 \qquad \Omega = \Omega_1 \times \Omega_2 \times \Omega_3$$

$$P(A_1 \times A_2 \times A_3) = P_1(A_1)P_2(A_2)P_3(A_3), \quad A_i \in \Omega_i$$

Permutation $P_r^n = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$

Combination $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

binomial expansion $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$

1.7 Bernoulli Trials

Basic experiment - 2 possible outcomes (A or $\overline{A}$)

$$P(A) = p \qquad P(\overline{A}) = 1 - p$$

Bernoulli Trials - repeat the basic experiment N times

(Assume that elementary events are independent for every trial.)

$$P(\{A \text{ occurs exactly } k \text{ times}\}) = \binom{N}{k} p^k (1-p)^{N-k}$$

Ex 1.7-1: $P(A) = 0.4 \qquad N = 3$

$$P(2 \text{ hits}) = \binom{3}{2} 0.4^2 (1-0.4)^1 = 0.288$$

1.7 Bernoulli Trials

$$P(3\text{ hits}) = \binom{3}{3} 0.4^3 (1-0.4)^0 = 0.064$$

$$P(\{\text{carrier sunk}\}) = P(2\text{ hits}) + P(3\text{ hits}) = 0.352$$

Ex 1.7-3: $P(A) = 0.4 \quad N = 120$

$$P(50\text{ hits}) = \binom{120}{50} 0.4^{50} (1-0.4)^{70} = ? \qquad 120! = ?$$

large $N \Rightarrow$ De Moivre-Laplace approximation

Poisson approximation