EE 213 ELECTRIC CIRCUITS II

Chapters 6 and 9
Mutual Inductance and Transformers

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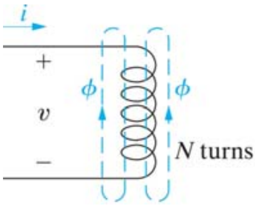
Magnetically Coupled Circuits

- The circuits studied so far can be considered as **conductively coupled**, since loops affect each other by current conduction.
- When two loops with or without contact affect each other through magnetic fields, they are said to be **magnetically coupled**.
- The **transformer** is a device designed based on the concept of magnetic coupling.
- In preparation for the study of transformers, we will first make a brief recap of **self inductance** and then discuss the concept of **mutual inductance**.

Faraday's Law



$$\underbrace{\lambda}_{\text{Flux linkage}} = \underbrace{\mathcal{N}}_{\text{\# of turns}} \times \underbrace{\phi}_{\text{Flux [Weber]}}$$

- Consider a coil of $\mathcal{N}$ turns, through which a current i is flowing.
- **Faraday's Law**: The voltage induced in the coil is given by the rate-of-change of the flux linkage:

$$v(t) = \frac{d\lambda(t)}{dt} = \mathcal{N}\frac{d\phi(t)}{dt}$$

Self Inductance

- The magnitude of the flux is given by:

$\phi = \mathcal{P} \times \mathcal{N} \times i$; $\mathcal{P}$: permeance of the field occupied by the flux

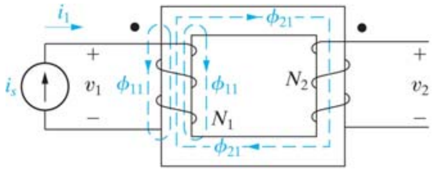
- The **permeance** is flux-dependent for magnetic materials (like iron, nickel, cobalt), whereas it is **constant** for **nonmagnetic** materials.
- When the core material of the coil is nonmagnetic:

$$v(t) = \mathcal{N}\frac{d\phi(t)}{dt} = \underbrace{\mathcal{N}^2\mathcal{P}}_{\mathcal{L}} \times \frac{di(t)}{dt}$$

- The proportionality constant is the **self-inductance**:

$$\mathcal{L} = \mathcal{N}^2\mathcal{P}$$

Magnetically-Coupled Coils



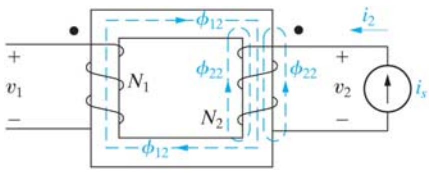
- Now consider two neighboring coils wound on a nonmagnetic core, with a current in the first one:

$$\underbrace{\phi_1}_{\text{total flux}} = \underbrace{\mathcal{P}_{11}\mathcal{N}_1 \times i_1}_{\phi_{11}\text{: flux linking coil-1}} + \underbrace{\mathcal{P}_{21}\mathcal{N}_1 \times i_1}_{\phi_{21}\text{: flux linking coil-2}} = \underbrace{\left(\mathcal{P}_{11} + \mathcal{P}_{21}\right)}_{\mathcal{P}_1}\mathcal{N}_1 \times i_1$$

- Using Faraday's law, we find:

$$v_1(t) = \mathcal{N}_1 \frac{d\phi_1(t)}{dt} = \underbrace{\mathcal{N}_1^2\mathcal{P}_1}_{\mathcal{L}_1}\frac{di_1(t)}{dt}; \quad v_2(t) = \mathcal{N}_2 \frac{d\phi_{21}(t)}{dt} = \underbrace{\mathcal{N}_2\mathcal{N}_1\mathcal{P}_{21}}_{\mathcal{M}_{21}}\frac{di_1(t)}{dt}$$

Mutual Inductance



- When the current is fed to the second coil:

$$v_2(t) = \mathcal{N}_2 \frac{d\phi_2(t)}{dt} = \underbrace{\mathcal{N}_2^2\mathcal{P}_2}_{\mathcal{L}_2}\frac{di_2(t)}{dt}; \quad v_1(t) = \mathcal{N}_2 \frac{d\phi_1(t)}{dt} = \underbrace{\mathcal{N}_1\mathcal{N}_2\mathcal{P}_{12}}_{\mathcal{M}_{12}}\frac{di_2(t)}{dt}$$

- For nonmagnetic core materials, we have:

$$\mathcal{P}_{12} = \mathcal{P}_{21} \Rightarrow \mathcal{M}_{12} = \mathcal{M}_{21} = \mathcal{M}$$

- $\mathcal{L}_1$ and $\mathcal{L}_2$ are the self inductances, whereas $\mathcal{M}$ is the **mutual inductance** between the coils.

The Coefficient of Coupling

- Recall the inductance expressions:

$$\mathcal{L}_1 = \mathcal{N}_1^2\mathcal{P}_1 \text{ and } \mathcal{L}_2 = \mathcal{N}_2^2\mathcal{P}_2 \Rightarrow \mathcal{L}_1\mathcal{L}_2 = \mathcal{N}_1^2\mathcal{N}_2^2\mathcal{P}_1\mathcal{P}_2$$

- For nonmagnetic core materials, we have:

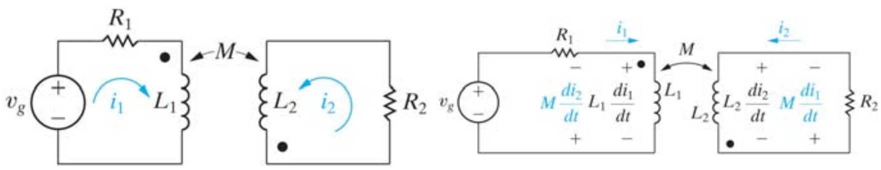
$$\mathcal{P}_{21} = \mathcal{P}_{12} \Rightarrow \mathcal{L}_1\mathcal{L}_2 = \underbrace{\left(\mathcal{N}_1\mathcal{N}_2\mathcal{P}_{12}\right)^2}_{\mathcal{M}^2}\underbrace{\left(1+\frac{\mathcal{P}_{11}}{\mathcal{P}_{12}}\right)\left(1+\frac{\mathcal{P}_{22}}{\mathcal{P}_{12}}\right)}_{1/k^2}$$

- Coupling is quantified by the **coefficient of coupling:**

$$k = \left[\left(1+\frac{\mathcal{P}_{11}}{\mathcal{P}_{12}}\right)\left(1+\frac{\mathcal{P}_{22}}{\mathcal{P}_{12}}\right)\right]^{-1/2} \in [0,1]; \quad \mathcal{M} = k\sqrt{\mathcal{L}_1\mathcal{L}_2}$$

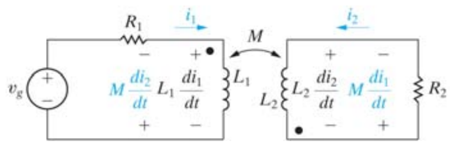
$k \in (0.5,1)$: tightly coupled; $k \in (0,0.5)$: loosely coupled

Voltage Polarity and the Dot Convention



- The polarity of self-induced voltage is identified from the direction of the current.
- The polarity of the mutually-induced voltage is identified based on the dot convention.
- **Dot convention:** When the current enters (leaves) the dotted terminal of a coil, the polarity of the voltage it induces in the other coil is positive (negative) at its dotted terminal.

Application of the Dot Convention

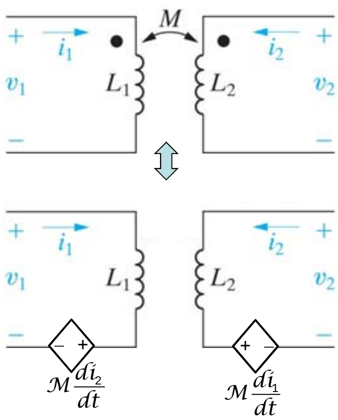


- The easiest way to analyze circuits containing mutual inductance is to use mesh currents.
- KVL is then applied with the addition of the mutually induced voltage with appropriate polarity:

$$v_g = R_1 i_1 + L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

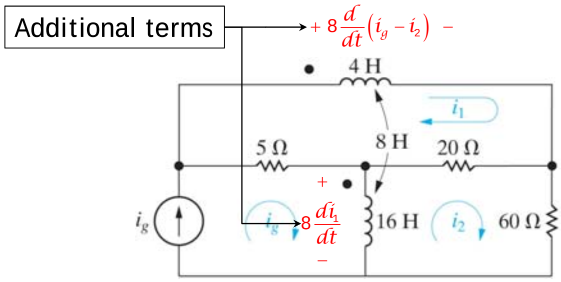
$$0 = R_2 i_2 + L_2\frac{di_2}{dt} - M\frac{di_1}{dt}$$

Equivalent Circuit for Mutual Inductance



- The dependent source voltages are determined by the derivatives of the currents. The polarities of the sources are identified from the dot convention.

Example: Mesh Current Equations

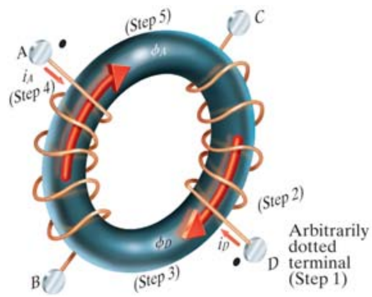


$$i_1 \text{ mesh} : \quad 4\frac{di_1}{dt} + 8\frac{d}{dt}\left(i_g - i_2\right) + 20\left(i_1 - i_2\right) + 5\left(i_1 - i_g\right) = 0$$

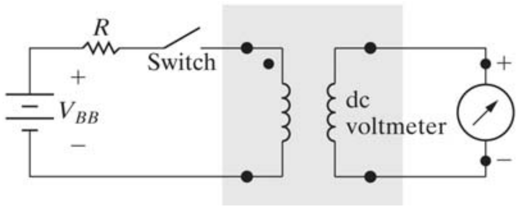
$$i_2 \text{ mesh} : \quad 20\left(i_2 - i_1\right) + 60 i_2 + 16\frac{d}{dt}\left(i_2 - i_g\right) - 8\frac{di_1}{dt} = 0$$

Procedure for Determining Dot Markings

- Arbitrarily select and mark a terminal, say D, with a dot.
- Assign a current i_D to it.
- Determine the direction of the induced magnetic flux, ϕ_D, based on the right-hand rule.
- Arbitrarily pick a terminal of the second coil, say A, and apply the same steps again.
- If the directions ϕ_A and ϕ_D are the same, then place a dot on A.
- If the directions are opposite, place a dot on the other terminal.



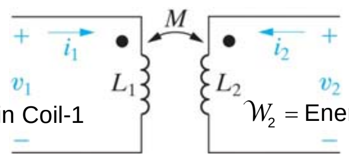
Experimental Setup for Dot Marking



- Put a dot on the terminal to which the resistor is connected.
- Observe the momentary deflection of the DC voltmeter when the switch is closed.
- If it is upscale/downscale, put a dot on the terminal connected to the positive/negative terminal of the voltmeter.

Energy Calculations

$\mathcal{W}_1$ = Energy Stored in Coil-1



$\mathcal{W}_2$ = Energy Stored in Coil-2

- Assume zero initial energy.
- Increase i_1 from zero to I_1:

$$\mathcal{W}_1^1 = \int_{i_1=0}^{i_1=I_1} i_1 \cdot L_1 \frac{di_1}{dt} dt = \int_0^{I_1} L_1 i_1 \, di_1 = \frac{1}{2} L_1 I_1^2; \quad \mathcal{W}_2^1 = 0$$

- Keep $i_1 = I_1$ constant; increase i_2 from zero to I_2:

$$\mathcal{W}_1^2 = \int_{i_2=0}^{i_2=I_2} I_1 \cdot M_{12} \frac{di_2}{dt} dt = I_1 I_2 M_{12}; \quad \mathcal{W}_2^2 = \int_{i_2=0}^{i_2=I_2} i_2 \cdot L_2 \frac{di_2}{dt} dt = \frac{1}{2} L_2 I_2^2$$

- Total energy stored in the coils:

$$\mathcal{W} = \mathcal{W}_1^1 + \mathcal{W}_1^2 + \mathcal{W}_2^1 + \mathcal{W}_2^2 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2$$

Total Energy Stored in the Coils

- Total energy stored in the coils at time t (with the dot marking not specified) :

$$\mathcal{W}(t)=\frac{1}{2}\mathcal{L}_1 i_1^2(t)+\frac{1}{2}\mathcal{L}_2 i_2^2(t)\pm\mathcal{M}_{12} i_1(t) i_2(t)$$

- When the order of the procedure is reversed:

$$\mathcal{W}(t)=\frac{1}{2}\mathcal{L}_1 i_1^2(t)+\frac{1}{2}\mathcal{L}_2 i_2^2(t)\pm\mathcal{M}_{21} i_1(t) i_2(t)$$

- For linear coupling media $\mathcal{M}_{12}$ = $\mathcal{M}_{21}$ = $\mathcal{M}$, which means that the total energies stored are the same.
- Determining the sign: If both currents are entering or leaving the dotted terminals, then the sign is positive; otherwise the sign is negative.

Positivity of the Total Energy

- Consider finding the i_2 that minimizes $\mathcal{W}$:

$$\mathcal{W}(i_2)=\frac{1}{2}\mathcal{L}_1 i_1^2+\frac{1}{2}\mathcal{L}_2 i_2^2\pm\mathcal{M} i_1 i_2$$

$$\Rightarrow \frac{d\mathcal{W}(i_2)}{di_2}=\mathcal{L}_2 i_2\pm\mathcal{M} i_1 \text{ and } \frac{d^2\mathcal{W}(i_2)}{di_2^2}=\mathcal{L}_2>0$$

- Equate the first derivative to zero find the minimum:

$$\mathcal{L}_2 i_2^\circ\pm\mathcal{M} i_1=0\Rightarrow i_2^\circ=\mp\frac{\mathcal{M}}{\mathcal{L}_2} i_1=\mp k\sqrt{\frac{\mathcal{L}_1}{\mathcal{L}_2}} i_1$$

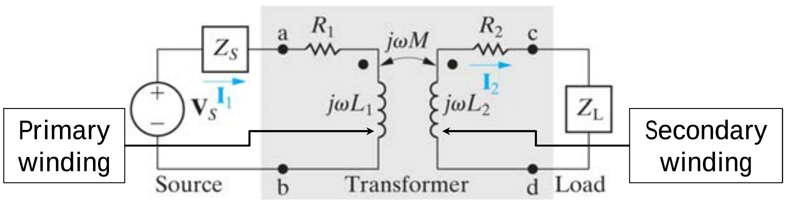
- Since the energy is ≥ its minimum value, we have:

$$\mathcal{W}(i_2)\ge\mathcal{W}(i_2^\circ)=\frac{1}{2}\mathcal{L}_1 i_1^2+\left(\frac{1}{2}\mathcal{L}_2 i_2^\circ\pm\mathcal{M} i_1\right)i_2^\circ=\frac{1}{2}\left(1-k^2\right)\mathcal{L}_1 i_1^2\ge 0$$

Transformers

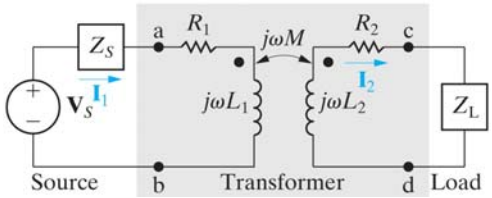
- Transformer is a device based on magnetic coupling.
- In communication circuits, transformers are used to match impedances and eliminate DC signals.
- In power circuits, transformers are used to establish AC voltage levels that facilitate the transmission, distribution and consumption of electrical power.
- We will first analyze the steady-state behavior of the **linear transformer**, which is common in communication systems.
- We will then study the **ideal transformer**, which models the ferromagnetic transformer used in power systems.

The Linear Transformer



R_1	Resistance of the secondary winding
R_2	Resistance of the secondary winding
L_1	Self-inductance of the primary winding
L_2	Self-inductance of the secondary winding
M	Mutual inductance
V_S	Source voltage
Z_S	Source impedance
Z_L	Load impedance
I_1	Primary current
I_2	Secondary current

Analysis of a Linear Transformer



- Phasor analysis is adapted easily: $dx/dt \leftrightarrow j\omega \times X$

$$\underbrace{(Z_s + R_1 + j\omega L_1)}_{Z_{11}} I_1 - j\omega M I_2 = V_s$$

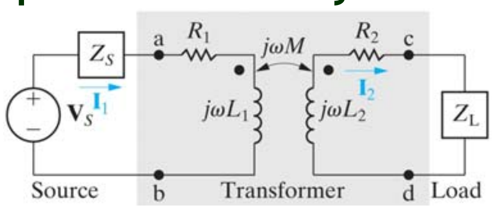
Z_{11} ← Self impedance of the primary

$$\underbrace{(Z_L + R_2 + j\omega L_2)}_{Z} I_2 - j\omega M I_1 = 0$$

Z ← Self impedance of the secondary

$$I_2 = \frac{j\omega M}{Z_{22}} I_1 \quad \text{and} \quad I_1 = \frac{Z_{22}}{Z_{11}Z_{22} + \omega^2 M^2} V_s$$

The Impedance Seen by the Source



- The impedance seen from the nodes a and b is:

$$Z_{ab} = \frac{V_s}{I_1} - Z_s = Z_{11} + \frac{\omega^2 M^2}{Z_{22}} - Z_s = R_1 + j\omega L_1 + \boxed{\frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}}$$

Z_r: Reflected impedance

- A key role of the transformer is thus revealed: Changing the impedance seen by the source:

$$Z_L \to Z_{ab} = R_1 + j\omega L_1 + Z_r(Z_L)$$

The Reflected Impedance

- Impedance of the second coil plus load is transmitted to the primary side via the mutual inductance:

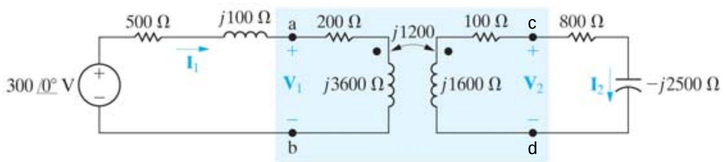
$$Z_r = \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}$$

- When the load impedance is $Z_L = R_L + jX_L$, we have:

$$Z_r = \underbrace{\boxed{\frac{\omega^2 M^2}{(R_2 + R_L)^2 + (\omega L_2 + X_L)^2}}}_{\text{Scaling Factor}} \left[(R_2 + R_L) - j(\omega L_2 + X_L) \right]$$

- The impedance is thus conjugated and then scaled during the reflection.

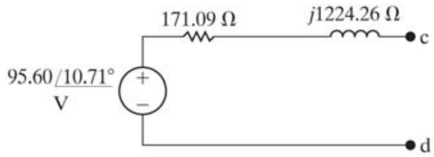
Example: Linear Transformer Circuit



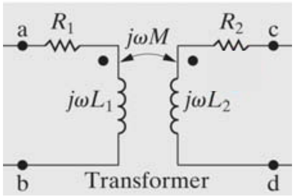
- Scaling factor: $= \frac{1200^2}{|800+100+1600j-2500j|^2} = \frac{1200^2}{|900+900j|^2} = \frac{1200^2}{900^2+900^2} = \frac{8}{9}$
- Reflected impedance: $= \frac{8}{9}(900+900j) = 800+800j\ \Omega$
- Impedance seen from a - b: $= 200+3600j+800+800j = 1000+4400j\ \Omega$
- Thevenin equivalent seen from c – d:

$$Z_{th} = 100 + 1600j + \frac{1200^2}{700^2+3700^2}(700-3700j)$$
$$= 171.09 + 1224.26j\ \Omega$$

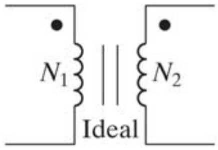
$$V_{th} = 1200j \times \frac{300}{500+200+(100+3600)j}$$
$$= 95.6\angle 10.71^\circ\ \mathrm{V}$$



The Ideal Transformer

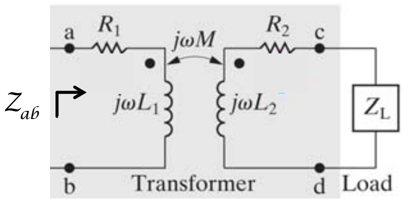


$$\begin{array}{c} k = 1 \\ \mathcal{L}_1 \to \infty,\ \mathcal{L}_2 \to \infty \\ \mathcal{R}_1 \to 0,\ \mathcal{R}_2 \to 0 \end{array} \Rightarrow$$



- An ideal transformer consists of two magnetically coupled coils having N_1 and N_2 turns respectively, and exhibiting the following three properties:
- ❑ The coefficient of coupling is unity.
- ❑ The self-inductance of each coil is infinite.
- ❑ The coil losses, due to parasitic resistances, are negligible.

Analysis of the Limit Values



$$\mathcal{Z}_{ab} = \underbrace{\mathcal{R}_1 + \frac{\omega^2 \mathcal{M}^2 (\mathcal{R}_2 + \mathcal{R}_L)}{(\mathcal{R}_2 + \mathcal{R}_L)^2 + (\omega \mathcal{L}_2 + \mathcal{X}_L)^2}}_{\mathcal{R}_{ab}} + j \underbrace{\left(\omega \mathcal{L}_1 - \frac{\omega^2 \mathcal{M}^2 (\omega \mathcal{L}_2 + \mathcal{X}_L)}{(\mathcal{R}_2 + \mathcal{R}_L)^2 + (\omega \mathcal{L}_2 + \mathcal{X}_L)^2} \right)}_{\mathcal{X}_{ab}}$$

- Setting $\mathcal{M}^2 = \mathcal{L}_1 \mathcal{L}_2$ and letting $\mathcal{L}_1 \to \infty$, $\mathcal{L}_2 \to \infty$, we find:

$$\mathcal{R}_{ab} = \mathcal{R}_1 + \left(\frac{\mathcal{L}_1}{\mathcal{L}_2} \right) \left(\frac{(\mathcal{R}_2 + \mathcal{R}_L)}{(\mathcal{R}_2 + \mathcal{R}_L)^2 / (\omega \mathcal{L}_2)^2 + (1 + \mathcal{X}_L / (\omega \mathcal{L}_2))^2} \right) \to \mathcal{R}_1 + \left(\frac{\mathcal{L}_1}{\mathcal{L}_2} \right) (\mathcal{R}_2 + \mathcal{R}_L)$$

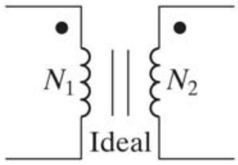
$$\mathcal{X}_{ab} = \left(\frac{\mathcal{L}_1}{\mathcal{L}_2} \right) \left(\frac{(\mathcal{R}_2 + \mathcal{R}_L)^2 / (\omega \mathcal{L}_2) + \mathcal{X}_L + \mathcal{X}_L^2 / (\omega \mathcal{L}_2)}{(\mathcal{R}_2 + \mathcal{R}_L)^2 / (\omega \mathcal{L}_2)^2 + (1 + \mathcal{X}_L / (\omega \mathcal{L}_2))^2} \right) \to \left(\frac{\mathcal{L}_1}{\mathcal{L}_2} \right) \mathcal{X}_L$$

- As $k \to 1$, the two permeances $\mathcal{P}_1$, $\mathcal{P}_2$ become equal:

$$\frac{\mathcal{L}_1}{\mathcal{L}_2} = \frac{\mathcal{N}_1^2 \mathcal{P}_1}{\mathcal{N}_2^2 \mathcal{P}_2} \to \left(\frac{\mathcal{N}_1}{\mathcal{N}_2} \right)^2 = a^2;$$

a: turns ratio

Ideal Transformer Relations



- The voltages of the primary and secondary windings of an ideal transformer are related as:

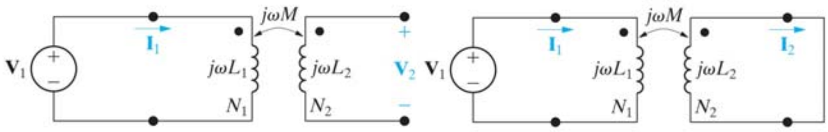
$$\left|\frac{V_1}{N_1}\right| = \left|\frac{V_2}{N_2}\right|$$

- The currents of the primary and secondary windings of an ideal transformer are related as:

$$|I_1 N_1| = |I_2 N_2|$$

- The polarities are determined via the dot convention.

Determining the Voltage-Current Ratios



- The voltage ratio is determined using the left circuit:

Since $I_2 = 0$, we have $V_2 = j\omega M I_1 = \dfrac{j\omega M}{j\omega L_1} V_1$.

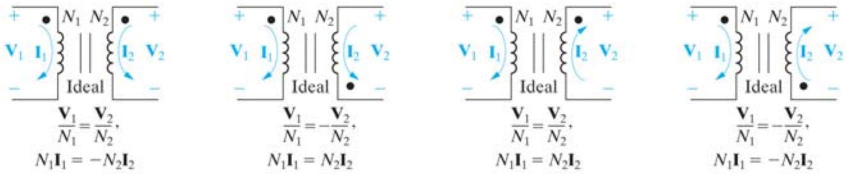
With $M = \sqrt{L_1 L_2}$, it follows that $\dfrac{V_1}{V_2} = \sqrt{\dfrac{L_1}{L_2}} = \sqrt{\dfrac{N_1^2 P}{N_2^2 P}} = \dfrac{N_1}{N_2}$.

- The current ratio is determined using the right circuit:

Since $V_2 = 0$, we have $j\omega M I_1 - j\omega L_2 I_2 = 0$.

With $M = \sqrt{L_1 L_2}$, it follows that $\dfrac{I_1}{I_2} = \sqrt{\dfrac{L_2}{L_1}} = \sqrt{\dfrac{N_2^2 P}{N_1^2 P}} = \dfrac{N_2}{N_1}$.

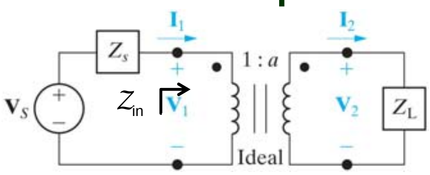
Determining the Polarities



- If the coil voltages and are both positive or negative at the dot-marked terminal, use a plus sign in the voltage ratio; otherwise, use a minus sign.

- If the coil currents are both directed into or out of the dot-marked terminal, use a minus sign in the current ration; otherwise, use a plus sign.

Ideal Transformer for Impedance Matching



- The ideal transformer relations are:

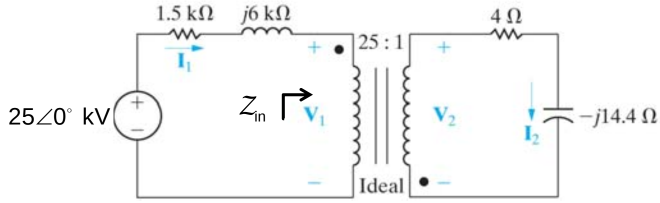
$$V_1 = \frac{V_2}{a} \quad \text{and} \quad I_1 = aI_2$$

- Then the impedance seen by the practical source is:

$$Z_{in} = \frac{V_1}{I_1} = \frac{1}{a^2}\frac{V_2}{I_2} = \frac{1}{a^2}Z_L$$

- The practical version of the ideal transformer is the ferromagnetic core transformer. It is used to match the magnitude of Z_L to the magnitude of Z_S.

Example: Ideal Transformer Circuit



- $\mathcal{Z}_{\text{in}}$: $= (25)^2 (4 + j14.4) = 2.5 - j9 \text{ k}\Omega$
- $\mathcal{V}_1$: $= \dfrac{2.5 - j9.0}{4.0 - j3.0} 25 \text{ kV} = 46\,703.85\angle -37.61^\circ$
- $\mathcal{V}_2$: $= -\mathcal{V}_1 / 25 = 1868.15\angle 142.39^\circ$
- $\mathcal{I}_2$: $= \dfrac{25 \text{ kV}}{4 - j3} \times 25 = 125\angle 36.87^\circ$