EE 213 ELECTRIC CIRCUITS II

Lecture 7
Two-Port Circuits

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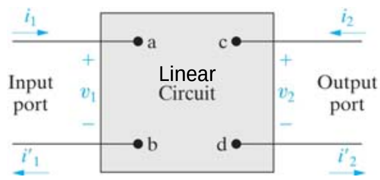
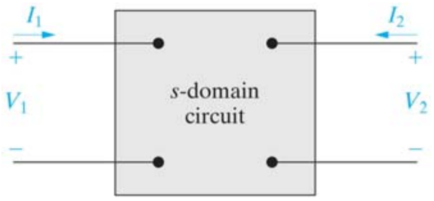
Two-Port Circuits



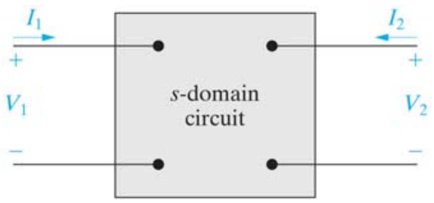
- Two-port model is used to describe a circuit in terms of its voltage & current at input & output terminals.
- As the points where the signals are fed or extracted, the terminals are also referred to as **ports**.
- The model is limited to circuits in which:
  - there are no independent sources inside the circuit between the ports
  - there is no energy stored inside the circuit between the ports
  - the current entering a port is equal to the current leaving that port
  - there is no external connection between the input and output ports

Description in the s-Domain



- We are mainly interested in relating the currents and voltages in both ports to each other.
- The most general description of a linear two-port is expressed in the s-domain with variables V_1, V_2, I_1, I_2.
- Two (out of four) variables are independent. Hence two equations are enough for describing the system.
- There are, however, six different ways for expressing the relation among the variables.

The Terminal Equations



- The six different equations are expressed in terms of six different groups of parameters: z, y, a, b, h, g

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \qquad \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} \qquad \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

The Impedance Parameters

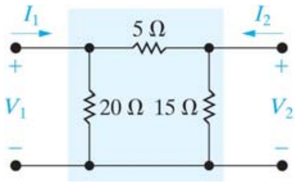
- The **impedance parameters** (z_{ij}) can be determined by computations or experiments as (in units of Ω):

$$
\begin{aligned}
\mathcal{V}_1 &= z_{11}I_1 + z_{12}I_2 \\
\mathcal{V}_2 &= z_{21}I_1 + z_{22}I_2
\end{aligned}
$$

$$
z_{11} = \left.\frac{\mathcal{V}_1}{I_1}\right|_{I_2=0},\quad z_{21} = \left.\frac{\mathcal{V}_2}{I_1}\right|_{I_2=0},\quad z_{12} = \left.\frac{\mathcal{V}_1}{I_2}\right|_{I_1=0},\quad z_{22} = \left.\frac{\mathcal{V}_2}{I_2}\right|_{I_1=0}
$$

- Also called **open circuit impedance parameters**:
- ❑ z_{11}: open circuit input impedance
- ❑ z_{22}: open circuit output impedance
- ❑ z_{21}: open circuit transfer impedance from port 2 to 1
- ❑ z_{12}: open circuit transfer impedance from port 1 to 2

Example: Two-Port z-Parameters



- z_{11}: $z_{11} = \left.\frac{\mathcal{V}_1}{I_1}\right|_{I_2=0} = \frac{20(5+15)}{20+5+15} = 10\ \Omega$
- z_{22}: $z_{22} = \left.\frac{\mathcal{V}_2}{I_2}\right|_{I_1=0} = \frac{15(5+20)}{15+5+20} = 9.375\ \Omega$
- z_{21}: $z_{21} = \left.\frac{\mathcal{V}_2}{I_1}\right|_{I_2=0} = \left.\frac{\mathcal{V}_2}{\mathcal{V}_1}\frac{\mathcal{V}_1}{I_1}\right|_{I_2=0} = z_{11}\left.\frac{\mathcal{V}_2}{\mathcal{V}_1}\right|_{I_2=0} = \frac{15}{15+5}10 = 7.5\ \Omega$
- z_{12}: $z_{12} = \left.\frac{\mathcal{V}_1}{I_2}\right|_{I_2=0} = \left.\frac{\mathcal{V}_1}{\mathcal{V}_2}\frac{\mathcal{V}_2}{I_2}\right|_{I_1=0} = z_{22}\left.\frac{\mathcal{V}_1}{\mathcal{V}_2}\right|_{I_1=0} = \frac{20}{20+5}9.375 = 7.5\ \Omega$

The Admittance Parameters

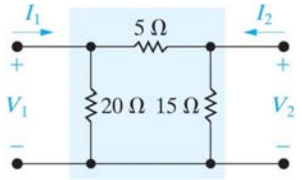
- The **admittance parameters** (y_{ij}) can be determined by computations or experiments as (in units of S):

$$\begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 \\ I_2 &= y_{21}V_1 + y_{22}V_2 \end{aligned}$$

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}, \quad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}, \quad y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}, \quad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

- Also called **short circuit admittance parameters**:
- ❑ y_{11}: short circuit input admittance
- ❑ y_{22}: short circuit output admittance
- ❑ y_{21}: short circuit transfer admittance from port 1 to 2
- ❑ y_{12}: short circuit transfer admittance from port 2 to 1
- z and y together are called **immitance parameters**.

Example: Two-Port y-Parameters



- In some cases it might be more convenient to obtain the two-port parameters directly from KVL/KCL:

$$\begin{aligned} I_1 &= \frac{V_1}{20} + \frac{V_1 - V_2}{5} \\ I_2 &= \frac{V_2}{15} + \frac{V_2 - V_1}{5} \end{aligned} \Rightarrow \begin{aligned} I_1 &= \underbrace{\left(\frac{1}{20}+\frac{1}{5}\right)}_{y_{11}}V_1 + \underbrace{\left(-\frac{1}{5}\right)}_{y_{12}}V_2 \\ I_2 &= \underbrace{\left(-\frac{1}{5}\right)}_{y_{21}}V_1 + \underbrace{\left(\frac{1}{15}+\frac{1}{5}\right)}_{y_{22}}V_2 \end{aligned}$$

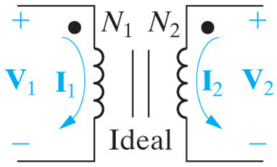
Relationship Among y and z Parameters

- Different two-port parameters might be obtained from each other.
- Relation between y and z parameters is derived as:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Rightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-1} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{y_{22}}{\Delta y} & -\frac{y_{12}}{\Delta y} \\ -\frac{y_{21}}{\Delta y} & \frac{y_{11}}{\Delta y} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \text{ where } \Delta y = \frac{1}{y_{11}y_{22} - y_{12}y_{21}}$$

Transmission and Hybrid Parameters



- Recall the equations of an ideal transformer:

$$\frac{V_1}{N_1} = \frac{V_2}{N_2}, \quad N_1 I_1 = -N_2 I_2$$

- Hence z and y parameters need not always exist !
- Hybrid and transmission parameters are therefore useful for describing some two-port circuits.
- These parameters do not have the same units. In fact some are just ratios and don't have any unit.

The Transmission Parameters

- The **transmission parameters** a_{ij} can be determined by computations or experiments as:

$$\begin{aligned} V_1 &= a_{11}V_2 - a_{12}I_2 \\ I_1 &= a_{21}V_2 - a_{22}I_2 \end{aligned}$$

$$a_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0}, \quad a_{21} = \left.\frac{I_1}{V_2}\right|_{I_2=0} \text{S}, \quad a_{12} = -\left.\frac{V_1}{I_2}\right|_{V_2=0} \Omega, \quad a_{22} = -\left.\frac{I_1}{I_2}\right|_{V_2=0}$$

- These parameters are referred to as follows:
- a_{11}: open-circuit reverse voltage ratio
- a_{22}: negative short-circuit reverse current ratio
- a_{21}: open circuit transfer admittance
- a_{12}: negative short-circuit transfer impedance

The Inverse Transmission Parameters

- The **inverse transmission parameters** b_{ij} can be determined by computations or experiments as:

$$\begin{aligned} V_2 &= b_{11}V_1 - b_{12}I_1 \\ I_2 &= b_{21}V_1 - b_{22}I_1 \end{aligned}$$

$$b_{11} = \left.\frac{V_2}{V_1}\right|_{I_1=0}, \quad b_{21} = \left.\frac{I_2}{V_1}\right|_{I_1=0} \text{S}, \quad b_{12} = -\left.\frac{V_2}{I_1}\right|_{V_1=0} \Omega, \quad b_{22} = -\left.\frac{I_2}{I_1}\right|_{V_1=0}$$

- These parameters are referred to as follows:
- b_{11}: open-circuit voltage gain
- b_{22}: negative short-circuit current gain
- b_{21}: open circuit transfer admittance
- b_{12}: negative short-circuit transfer impedance
- a & b parameters are used in transmission line analysis

Relationship Among a and z Parameters

- Relation between a and z parameters is derived as:

$$\begin{aligned} V_1 &= a_{11}V_2 - a_{12}I_2 \\ I_1 &= a_{21}V_2 - a_{22}I_2 \end{aligned} \quad \Rightarrow V_2 = \underbrace{\frac{1}{a_{21}}}_{z_{21}} I_1 + \underbrace{\frac{a_{22}}{a_{21}}}_{z_{22}} I_2$$

$$\Rightarrow V_1 = a_{11}\left(\frac{1}{a_{21}}I_1 + \frac{a_{22}}{a_{21}}I_2\right)V_2 - a_{12}I_2 = \underbrace{\frac{a_{11}}{a_{21}}}_{z_{11}} I_1 + \underbrace{\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{21}}}_{z_{12}} I_2$$

- One can also obtain an individual parameter by applying the associated method of computation:

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0} = \left.\frac{a_{11}V_2 - a_{12}I_2}{a_{21}V_2 - a_{22}I_2}\right|_{I_2 = 0} = \frac{a_{11}}{a_{21}}$$

The Hybrid Parameters

- The **hybrid parameters** h_{ij} can be determined by computations or experiments as:

$$\begin{aligned} V_1 &= h_{11}I_1 + h_{12}V_2 \\ I_2 &= h_{21}I_1 + h_{22}V_2 \end{aligned}$$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2 = 0} \Omega, \quad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2 = 0}, \quad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1 = 0}, \quad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1 = 0} \text{S}$$

- These parameters are referred to as follows:

- ❑ h_{11}: short-circuit input impedance
- ❑ h_{22}: open-circuit output admittance
- ❑ h_{21}: short-circuit forward current gain
- ❑ h_{12}: Open-circuit reverse voltage gain

The Inverse Hybrid Parameters

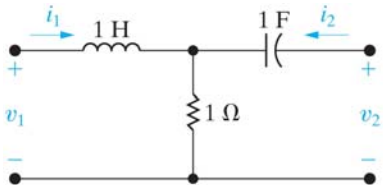
- The **inverse hybrid parameters** g_{ij} can be determined by computations or experiments as:

$$I_1 = g_{11}V_1 + g_{12}I_2$$
$$V_2 = g_{21}V_1 + g_{22}I_2$$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} \text{ S},\quad g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0},\quad g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0},\quad g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} \ \Omega$$

- These parameters are referred to as follows:
- ❑ g_{11}: open-circuit input admittance
- ❑ g_{22}: short-circuit output impedance
- ❑ g_{21}: open-circuit forward voltage gain
- ❑ g_{12}: short-circuit reverse current gain
- h & g parameters are used in transistor circuit analysis

Example: Two-Port h-Parameters

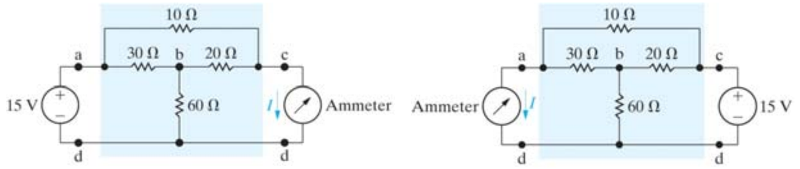


$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

- h_{11}: $h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = s + \frac{1/s}{1+1/s} = \frac{s^2+s+1}{s+1}$
- h_{21}: $h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \left.\frac{I_2}{V_1}\right|_{V_2=0} h_{11} = -\frac{1/(s+1)}{(1/(s+1)+s)1/s} h_{11} = -\frac{s}{s+1}$
- h_{12}: $h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{1}{1+1/s} = \frac{s}{s+1}$
- h_{22}: $h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1}{1+1/s} = \frac{s}{s+1}$

Reciprocal Two Port Circuits

- A two-port circuit is **reciprocal** if the interchange of an **ideal voltage source** at one port with an **ideal ammeter** at the other port produces the same ammeter reading.



- A circuit is also reciprocal if the interchange of an **ideal current source** at one port with an **ideal voltmeter** at the other port produces the same voltmeter reading.

Reciprocal Two Port Circuit Properties

- The impedance and admittance matrices of a reciprocal two port circuit are symmetric:

$$z_{12} = z_{21}, \quad y_{12} = y_{21}$$

- The hybrid parameters of a reciprocal circuit satisfy the following properties:

$$h_{12} = -h_{21}, \quad g_{12} = -g_{21}$$

- The transmission parameters of a reciprocal circuit satisfy the following properties:

$$\Delta a \triangleq a_{11}a_{22} - a_{12}a_{21} = 1, \quad \Delta b \triangleq b_{11}b_{22} - b_{12}b_{21} = 1$$

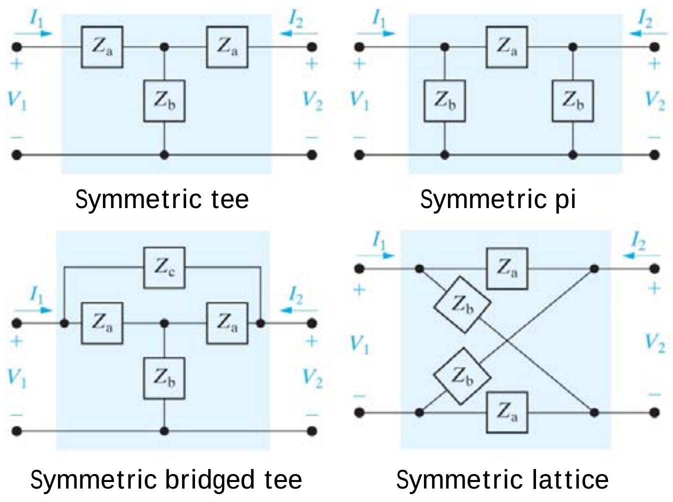
Symmetric Two Port Circuits

- A reciprocal two-port circuit is **symmetric**, if its ports can be interchanged without disturbing the values of the terminal currents and voltages.
- Parameters of a symmetric two-port circuit satisfy:

$$z_{11} = z_{22},\ y_{11} = y_{22},\ a_{11} = a_{22},\ b_{11} = b_{22}$$
$$\Delta h \triangleq h_{11}h_{22} - h_{12}h_{21} = 1,\ \ \Delta g \triangleq g_{11}g_{22} - g_{12}g_{21} = 1$$

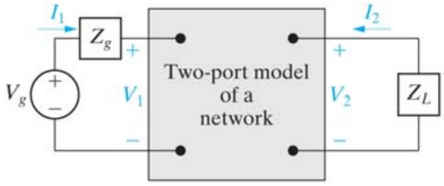
- For a reciprocal two-port, only three calculations or measurements are needed to determine a set of parameters.
- For a symmetric two-port, only two calculations or measurements are needed for the same purpose.

Examples of Symmetric Circuits



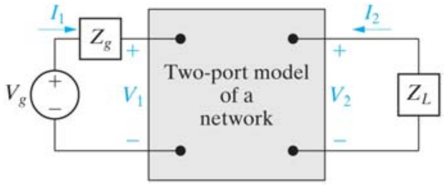
Symmetric tee
Symmetric pi
Symmetric bridged tee
Symmetric lattice

Analysis of Terminated Two-Port Circuits



- In the typical application of a two-port model, the circuit is driven at port-1 and loaded at port-2.
- The new ingredients in the above configuration are:
- ❑ Z_g : the internal impedance of the source
- ❑ V_g: the internal voltage of the source
- ❑ Z_L : the load impedance
- The solution of the circuit is obtained in terms of the two-port parameters, V_g, Z_g and Z_L.

Characteristics of Terminated Two-Ports



- Six characteristics of a terminated two-port circuit define its terminal behavior:
- ❑ The input impedance (admittance) $Z_{in}=V_1/I_1$ $(1/Z_{in})$
- ❑ The output current I_2
- ❑ Thevenin parameters (V_{th}, Z_{th}) with respect to port-2
- ❑ The current gain I_2/I_1
- ❑ The voltage gain V_2/V_1
- ❑ The voltage gain V_2/V_g

Characteristics in terms of z Parameters

- We derive the six characteristics in terms of z parameters.
- Recall the circuit relation for the z parameters:

$$\begin{aligned} \mathcal{V}_1 &= z_{11}\mathcal{I}_1 + z_{12}\mathcal{I}_2 \\ \mathcal{V}_2 &= z_{21}\mathcal{I}_1 + z_{22}\mathcal{I}_2 \end{aligned}$$

- Connection with the source and termination with the load imposes the following equations:

$$\begin{aligned} \mathcal{V}_1 &= \mathcal{V}_g - Z_g\mathcal{I}_1 \\ \mathcal{V}_2 &= -Z_L\mathcal{I}_2 \end{aligned}$$

- The characteristics are obtained by using the equations relevant for the computation.

The Input Impedance

- In order to obtain the input impedance, we first obtain port-2 current in terms of port-1 current:

$$\mathcal{V}_2 = z_{21}\mathcal{I}_1 + z_{22}\mathcal{I}_2 = -Z_L\mathcal{I}_2 \Rightarrow \mathcal{I}_2 = -\frac{z_{21}}{z_{22} + Z_L}\mathcal{I}_1$$

- We then use this in the equation for port-1 voltage:

$$\mathcal{V}_1 = z_{11}\mathcal{I}_1 + z_{12}\mathcal{I}_2 = \left(z_{11} - \frac{z_{12}z_{21}}{z_{22} + Z_L}\right)\mathcal{I}_1$$

- In this fashion, we obtain the input impedance as:

$$Z_{in} = z_{11} - \frac{z_{12}z_{21}}{z_{22} + Z_L}$$

The Terminal Current

- In order to find the terminal current, we first combine the equations in Slide-23 as:

$$\begin{aligned} \mathcal{V}_1 &= z_{11}I_1 + z_{12}I_2 = \mathcal{V}_g - Z_g I_1 \Rightarrow \left(z_{11} + Z_g\right) I_1 + z_{12}I_2 = \mathcal{V}_g \\ \mathcal{V}_2 &= z_{21}I_1 + z_{22}I_2 = -Z_L I_2 \qquad \Rightarrow z_{21}I_1 + \left(z_{22} + Z_L\right) I_2 = 0 \end{aligned}$$

- The solution for the port-2 current can directly be obtained from Cramer's rule as:

$$I_2 = \frac{\begin{vmatrix} z_{11} + Z_g & \mathcal{V}_g \\ z_{12} & 0 \end{vmatrix}}{\begin{vmatrix} z_{11} + Z_g & z_{12} \\ z_{21} & z_{22} + Z_L \end{vmatrix}} = -\frac{z_{12}}{\left(z_{11} + Z_g\right)\left(z_{22} + Z_L\right) - z_{12}z_{21}} \mathcal{V}_g$$

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Thevenin Parameters seen from Port-2

- Thevenin impedance is found by setting the source voltage to zero finding the ratio $\mathcal{V}_2 / I_2$:

$$\begin{aligned} \mathcal{V}_1 &= z_{11}I_1 + z_{12}I_2 = 0 - Z_g I_1 \Rightarrow \left(z_{11} + Z_g\right) I_1 + z_{12}I_2 = 0 \\ \mathcal{V}_2 &= z_{21}I_1 + z_{22}I_2 \qquad \Rightarrow z_{21}I_1 + \left(z_{22} + Z_L\right) I_2 = \mathcal{V}_2 \end{aligned}$$

- The Thevenin impedance is hence obtained as:

$$I_2 = \frac{\begin{vmatrix} z_{11} + Z_g & 0 \\ z_{21} & \mathcal{V}_2 \end{vmatrix}}{\begin{vmatrix} z_{11} + Z_g & z_{12} \\ z_{21} & z_{22} \end{vmatrix}} = \frac{z_{11} + Z_g}{\left(z_{11} + Z_g\right) z_{22} - z_{12}z_{21}} \mathcal{V}_2 \Rightarrow Z_g = z_{22} - \frac{z_{12}z_{21}}{z_{11} + Z_g}$$

- Thevenin voltage is found as:

$$\left.\begin{aligned} \mathcal{V}_1 &= z_{11}I_1 + z_{12} \times 0 = \mathcal{V}_g - Z_g I_1 \Rightarrow \left(z_{11} + Z_g\right) I_1 = \mathcal{V}_g \\ \mathcal{V}_2 &= z_{21}I_1 + z_{22} \times 0 \quad \Rightarrow \mathcal{V}_2 = z_{21}I_1 \end{aligned}\right\} \Rightarrow \mathcal{V}_{th} = \frac{z_{21}}{z_{11} + Z_g} \mathcal{V}_g$$

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The Current and Voltage Gains

- The current gain is easily found from:

$$\mathcal{V}_2 = z_{21}I_1 + z_{22}I_2 = -Z_L I_2 \Rightarrow z_{21}I_1 + \left(z_{22} + Z_L\right)I_2 = 0 \Rightarrow \frac{I_2}{I_1} = -\frac{z_{21}}{z_{22} + Z_L}$$

- To find the voltage gain, we recall that:

$$\mathcal{V}_1 = Z_{in}I_1 = \left(z_{11} - \frac{z_{12}z_{21}}{z_{22} + Z_L}\right)I_1 \quad \text{and} \quad \mathcal{V}_2 = -Z_L I_2$$

- Using the current gain derived above, we find:

$$\frac{\mathcal{V}_2}{\mathcal{V}_1} = \frac{-Z_L I_2}{Z_{in}I_1} = \frac{-Z_L}{z_{11} - \dfrac{z_{12}z_{21}}{z_{22} + Z_L}} \frac{-z_{21}}{z_{22} + Z_L} = \frac{z_{21}Z_L}{z_{11}\left(z_{22} + Z_L\right) - z_{12}z_{21}}$$

Source to Output Voltage Gain

- Recall the expression of the output current:

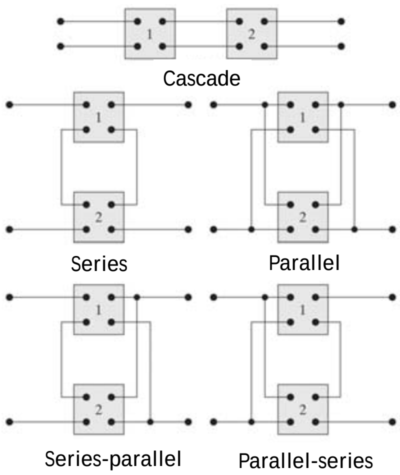
$$I_2 = -\frac{z_{12}}{\left(z_{11} + Z_g\right)\left(z_{22} + Z_L\right) - z_{12}z_{21}}\mathcal{V}_g$$

- Using the relation of the output voltage to output current, we obtain the source to output voltage gain:

$$\mathcal{V}_2 = -Z_L I_2 \Rightarrow \frac{\mathcal{V}_2}{\mathcal{V}_g} = \frac{z_{12}Z_L}{\left(z_{11} + Z_g\right)\left(z_{22} + Z_L\right) - z_{12}z_{21}}$$

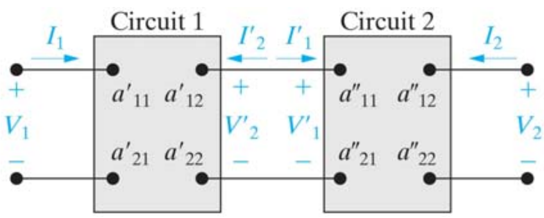
- Similar derivations can be performed for other two-port parameters (see Table 18.2 of the textbook).

Interconnected Two-Port Circuits



- Two-port circuits may be interconnected in five ways.

Cascade Connection



- The transmission parameters can be obtained by simple matrix multiplication:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix} \begin{bmatrix} V'_2 \\ -I'_2 \end{bmatrix} \qquad \begin{bmatrix} V'_1 \\ I'_1 \end{bmatrix} = \begin{bmatrix} a''_{11} & a''_{12} \\ a''_{21} & a''_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix} \begin{bmatrix} a''_{11} & a''_{12} \\ a''_{21} & a''_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$